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On the biparametric quantum deformation of $GL(2) \otimes GL(1)$

by

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Abstract

We study the biparametric quantum deformation of $GL(2) \otimes GL(1)$ and exhibit its cross-product structure. We derive explictly the associated dual algebra, i.e., the quantised universal enveloping algebra employing the R-matrix procedure. This facilitates construction of a bicovariant differential calculus which is also shown to have a cross-product structure. Finally, a Jordanian analogue of the deformation is presented as a cross-product algebra.

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I Introduction

The biparametric quantum deformation of $GL(2) \otimes GL(1)$ was introduced in [1] as a novel Hopf algebra involving five generators $\{a, b, c, d, f\}$ and two deformation parameters $\{r, s\}$. From among the five generators, four $\{a, b, c, d\}$ correspond to $GL(2)$ and the fifth one f is related to $GL(1)$. These can be arranged in the matrix of generators

$$\mathcal{T} = \begin{pmatrix} f & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \tag{1}$$

with the labelling $0, 1, 2$. The associated solution of the quantum Yang-Baxter equation is

$$R = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \mathsf{S}^{-1} & 0 & 0 \\ 0 & \Lambda & \mathsf{S} & 0 \\ 0 & 0 & 0 & R_r \end{pmatrix} \tag{2}$$

in block form, i.e., in the order (00), (01), (02), (10), (20), (11), (12), (21), (22) (which is chosen in conjunction with the block form of the $\mathcal{T}$-matrix) where

$$R_r = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}; \qquad \mathsf{S} = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}; \qquad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \qquad \lambda = r - r^{-1}$$

The $R\mathcal{T}\mathcal{T}$ relations

$$R\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1 R \tag{3}$$

(where $\mathcal{T}_1 = \mathcal{T} \otimes \mathbf{1}$ and $\mathcal{T}_2 = \mathbf{1} \otimes \mathcal{T}$) give the commutation relations between the generators a,b,c,d and f

$$\begin{aligned} ab &= r^{-1}ba, & bd &= r^{-1}db \\ ac &= r^{-1}ca, & cd &= r^{-1}dc \\ bc &= cb, & [a, d] &= (r^{-1} - r)bc \end{aligned} \tag{4}$$

and

$$\begin{aligned} af &= fa, & cf &= sfc \\ bf &= s^{-1}fb, & df &= fd \end{aligned} \tag{5}$$

Note that the first set of these relations is exactly the q-deformation of $GL(2)$ with deformation parameter r while the second set involves the fifth generator f and the second deformation parameter s. This results in a biparametric q-deformation of $GL(2) \otimes GL(1)$, say, $\mathcal{A}_{r,s}$. The coproduct and counit is given as

$$\begin{aligned} \Delta(\mathcal{T}) &= \mathcal{T} \dot{\otimes} \mathcal{T} \\ \varepsilon(\mathcal{T}) &= \mathbf{1} \end{aligned} \tag{6}$$

The Casimir operator $\delta = ad - r^{-1}bc$ is invertible and determines the antipode

$$S(f) = f^{-1}, \quad S(a) = \delta^{-1}d, \quad S(b) = -\delta^{-1}rb, \quad S(c) = -\delta^{-1}r^{-1}c, \quad S(d) = \delta^{-1}a \tag{7}$$

The quantum determinant $\mathcal{D} = \delta f$ is group-like but not central. Some of the interesting features of the above quantum deformation are the following:

- If we write the set of generators $\{a, b, c, d, f\}$ as $\{f^N a, f^N b, f^N c, f^N d\}$ (N being a fixed nonzero integer), i.e., reducing the five-dimensional set to the four-dimensional set, then we obtain an exact realisation of the biparametric (p, q)-deformation of $GL(2)$, i.e., $GL_{p,q}(2)$ subject to the relations

$$p = r^{-1}s^N \quad \text{and} \quad q = r^{-1}s^{-N} \tag{8}$$

  This realisation also reproduces the full Hopf algebraic structure underlying $GL_{p,q}(2)$.

- Another interesting feature of the $\mathcal{A}_{r,s}$ deformation is that it can be contracted (by means of the contraction procedure [2] based on the concept of singular limit of a similarity transformation) to yield the corresponding biparametric Jordanian deformation of $GL(2) \otimes GL(1)$, which in turn provides a complete realisation of the biparametric (h, h')-deformation of $GL(2)$, i.e., $GL_{h,h'}(2)$ in a manner similar to that for the q-deformed case [3].

- Both the biparametric quantum and Jordanian deformations of $GL(2) \otimes GL(1)$ admit coloured extensions [3] which also commute with the contraction procedure.

- The physical interest in studying $\mathcal{A}_{r,s}$ lies in the observation that when endowed with a $*$-structure, this specialises to its compact form, i.e., provides a biparametric q-deformation of $SU(2) \otimes U(1)$ which is precisely the gauge group for the theory of electroweak interactions.

Another deformation similar to $\mathcal{A}_{r,s}$ has also been recently given in [4], though in a different context. In the present article, we give an explicit description of the algebra dual to $\mathcal{A}_{r,s}$ as a starting point in further investigation of this quantum group structure. Motivated by the relation of this deformation with gauge theory, we also construct a bicovariant differential calculus since gauge theories have an obvious differential geometric description. This would then provide insights into possible scenarios for constructing q-gauge theories based on this deformation. In pursuing our aim, we follow the convenient R-matrix approach [5, 6]. In Sec. II, we give the cross-product structure and go over to the R-matrix duality in Sec. III. The constructive calculus is presented in Sec. IV, while Sec. V is a brief description of the Jordanian analogue. The results are discussed in Sec. VI.

II Cross-product structure

The biparametric q-deformation $\mathcal{A}_{r,s}$ can also be considered as the semidirect or cross-product $GL_r(2) \underset{s}{\rtimes} \mathbb{C}[f, f^{-1}]$ built on the vector space $GL_r(2) \otimes \mathbb{C}[f, f^{-1}]$ where $GL_r(2) = \mathbb{C}[a, b, c, d]$ modulo the relations (4) and $\mathbb{C}[f, f^{-1}]$ has the cross relations (5). Then, $\mathcal{A}_{r,s}$ can also be interpreted as a skew Laurent polynomial ring $GL_r[f, f^{-1}; \sigma]$ where σ is the automorphism given by the action of element f on $GL_r(2)$. Knowing properties of cross-product algebras (general theory given in [7, 8]), we already know that the algebra dual to $\mathcal{A}_{r,s}$ would be the cross-coproduct coalgebra $\mathcal{U}_{r,s} = U_r(gl(2)) \underset{s}{\rtimes} \mathbb{C}[[\phi]]$ with ϕ as an element dual to f. If we let $A = GL_r(2)$ and $H = \mathbb{C}[f, f^{-1}]$, then A is a left H-module algebra and the action of f on $GL_r(2)$ is given by

$$f \triangleright a = a, \qquad f \triangleright b = sb, \qquad f \triangleright c = s^{-1}c, \qquad f \triangleright d = d \tag{9}$$

As a vector space, the dual is $\mathcal{U}_{r,s} = U_r(gl(2)) \otimes U(u(1))$. Now, the duality relation between $\langle GL_r(2), U_r(gl(2)) \rangle$ is already well-known [9], while that between $\langle \mathbb{C}[f, f^{-1}], U(u(1)) \rangle$ is given by $\langle f, \phi \rangle = 1$, i.e., $U(u(1)) = \mathbb{C}[[\phi]]$. More precisely, we work algebraically with $\mathbb{C}[s^{\phi}, s^{-\phi}]$ where $\langle f, s^{\phi} \rangle = s$ (this is a standard notational convention which we adopt). This induces duality on the vector space tensor products and the left action dualises to the left coaction. This results in the dual algebra being a cross-coproduct $\mathcal{U}_{r,s} = U_r(gl(2)) \underset{s}{\rtimes} \mathbb{C}[[\phi]]$. Let us recall [9] that $U_r(gl(2))$, the algebra dual to $GL_r(2)$, is isomorphic to the tensor product $U_r(sl(2)) \otimes \tilde{U}(u(1))$ where $U_r(sl(2))$ has the usual generators $\{H, X_{\pm}\}$ and $\tilde{U}(u(1)) = \mathbb{C}[[\xi]] = \mathbb{C}[r^{\xi}, r^{-\xi}]$ with ξ central. Therefore, $\mathcal{U}_{r,s}$ is nothing but $U_r(sl(2))$ and

two central generators ξ and ϕ, where ξ is the generating element of $\tilde{U}(u(1))$ and ϕ is the generating element of $U(u(1))$. Also note that s^{ϕ} (s being the second deformation parameter) is dually paired with the element f of $\mathcal{A}_{r,s}$. Defining the left coaction $U_r(gl(2)) \longrightarrow U(u(1)) \otimes U_r(gl(2))$ we have

$$X_+ \longrightarrow s^{\phi} \otimes X_+, \qquad X_- \longrightarrow s^{-\phi} \otimes X_-, \qquad H \longrightarrow 1 \otimes H, \qquad \xi \longrightarrow 1 \otimes \xi \tag{10}$$

It can be checked that this gives the correct duality pairings. For example, we have for X_+

$$\begin{aligned}
&\langle \Delta_L(X_+), 1 \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \langle s^{\phi} \otimes X_+, 1 \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \langle s^{\phi}, 1 \rangle \langle X_+, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\
&\langle \Delta_L(X_+), f \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \langle s^{\phi} \otimes X_+, f \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \langle s^{\phi}, f \rangle \langle X_+, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = s \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\
&\langle X_+, f \triangleright \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \langle X_+, \left(\begin{smallmatrix} a & sb \\ s^{-1}c & d \end{smallmatrix}\right) \rangle = s \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)
\end{aligned} \tag{11}$$

Therefore, the coalgebra structure of $\mathcal{U}_{r,s}$ is given as

$$\Delta(X_+) = X_+ \otimes r^{\frac{H}{2}} + r^{-\frac{H}{2}} s^{\phi} \otimes X_+ \tag{12}$$

$$\Delta(X_-) = X_- \otimes r^{-\frac{H}{2}} + r^{\frac{H}{2}} s^{-\phi} \otimes X_+ \tag{13}$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H \tag{14}$$

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi \tag{15}$$

$$\Delta(\phi) = \phi \otimes 1 + 1 \otimes \phi \tag{16}$$

In this way, we have obtained the Drinfeld-Jimbo form of the dual algebra $\mathcal{U}_{r,s}$ using the cross-product construction. Given other approaches to the problem of duality for quantum groups, we also construct explicitly the dual algebra using the R-matrix procedure.

III R-matrix duality

The biparametric (r,s)-deformation, $\mathcal{A}_{r,s}$, of $GL(2) \otimes GL(1)$ has been defined in the previous section at the group level, i.e., as the q-deformation of algebra of functions on $GL(2) \otimes GL(1)$. In this section, we derive explicitly the corresponding quantised universal enveloping algebra, i.e., its dual within the framework of the R-matrix formulation. We first construct functionals (matrices) $\mathcal{L}^+$ and $\mathcal{L}^-$ which are dual to the matrix of generators in the fundamental representation. The linear functionals $(\mathcal{L}^{\pm})^a_b$ (following the method of [5, 7]) are defined by their value on the elements of the matrix of generators $\mathcal{T}$

$$\langle (\mathcal{L}^{\pm})^a_b, \mathcal{T}^c_d \rangle = (R^{\pm})^{ac}_{bd} \tag{17}$$

where

$$(R^+)^{ac}_{bd} = c^+(R)^{ca}_{db} \tag{18}$$

$$(R^-)^{ac}_{bd} = c^-(R^{-1})^{ac}_{bd} \tag{19}$$

and c^+ , c^- are free parameters. Matrices $(\mathcal{L}^\pm)^a_b$ satisfy

$$\langle (\mathcal{L}^\pm)^a_b, uv\rangle = \langle (\mathcal{L}^\pm)^a_c \otimes (\mathcal{L}^\pm)^c_d, u\otimes v\rangle = (\mathcal{L}^\pm)^a_c(u)(\mathcal{L}^\pm)^c_d(v) \tag{20}$$

$$\text{i.e.} \quad \Delta(\mathcal{L}^\pm)^a_b = (\mathcal{L}^\pm)^a_c \otimes (\mathcal{L}^\pm)^c_b$$

For $\mathcal{A}_{r,s}$, the (R^+) and (R^-) matrices read

$$(R^+) = c^+ \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \mathsf{S} & \Lambda & 0 \\ 0 & 0 & \mathsf{S}^{-1} & 0 \\ 0 & 0 & 0 & R_r^T \end{pmatrix}; \quad (R^-) = c^- \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & \mathsf{S} & 0 & 0 \\ 0 & -\Lambda & \mathsf{S}^{-1} & 0 \\ 0 & 0 & 0 & R_r^{-1} \end{pmatrix} \tag{21}$$

where R_r, Λ and S are the same as before and $R_r^{-1} = R_{r^{-1}}$. Before proceeding further, it is pertinent to make the following remark about the $\mathcal{L}^\pm$ functionals. Let $A(R)$ be a bialgebra or a Hopf algebra underlying a 3×3 quantum matrix and let $\tilde{U}(R)$ be a similar matrix bialgebra with two full matrices $\mathcal{L}^\pm$ of generators. These may be viewed as functionals $A(R) \longrightarrow \mathbb{C}$ via (17), but duality pairing at this level may be degenerate. So, we look at appropriate quotients of these such that the pairing is non-degenerate. In our case, upon quotienting $A(R)$ would descend to $\mathcal{A}_{r,s}$, and likewise $\tilde{U}(R)$ to the dual of $\mathcal{A}_{r,s}$. The quotient on $A(R)$ is obtained by setting certain entries of the T-matrix to zero. The most general 3×3 quantum matrix has nine elements

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}^0_0 & \mathcal{T}^0_1 & \mathcal{T}^0_2 \\ \mathcal{T}^1_0 & \mathcal{T}^1_1 & \mathcal{T}^1_2 \\ \mathcal{T}^2_0 & \mathcal{T}^2_1 & \mathcal{T}^2_2 \end{pmatrix} \tag{22}$$

Now, let $\mathcal{T}^0_1 = 0 = \mathcal{T}^0_2$ and $\mathcal{T}^1_0 = 0 = \mathcal{T}^2_0$. Checking the coideal property (via coproduct of $\mathcal{T}$), we have

$$\begin{aligned} \Delta(\mathcal{T}^0_1) &= \mathcal{T}^0_0 \otimes \mathcal{T}^0_1 + \mathcal{T}^0_1 \otimes \mathcal{T}^1_1 + \mathcal{T}^0_2 \otimes \mathcal{T}^2_1 \\ \Delta(\mathcal{T}^0_2) &= \mathcal{T}^0_0 \otimes \mathcal{T}^0_2 + \mathcal{T}^0_1 \otimes \mathcal{T}^1_2 + \mathcal{T}^0_2 \otimes \mathcal{T}^2_2 \\ \Delta(\mathcal{T}^1_0) &= \mathcal{T}^1_0 \otimes \mathcal{T}^0_0 + \mathcal{T}^1_1 \otimes \mathcal{T}^1_0 + \mathcal{T}^1_2 \otimes \mathcal{T}^2_0 \\ \Delta(\mathcal{T}^2_0) &= \mathcal{T}^2_0 \otimes \mathcal{T}^0_0 + \mathcal{T}^2_1 \otimes \mathcal{T}^1_0 + \mathcal{T}^2_2 \otimes \mathcal{T}^2_0 \end{aligned} \tag{23}$$

These generate biideals. Therefore, setting them to zero gives the quotient of $A(R)$

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_0^0 & 0 & 0 \\ 0 & \mathcal{T}_1^1 & \mathcal{T}_2^1 \\ 0 & \mathcal{T}_1^2 & \mathcal{T}_2^2 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} = \mathcal{T}(\mathcal{A}_{r,s}) \tag{24}$$

Similarly, the quotient on $\tilde{U}(R)$ is obtained by setting certain entries of $\mathcal{L}^{\pm}$ matrices to zero. Starting with

$$\mathcal{L}^+ = \begin{pmatrix} \mathcal{L}^{+0}_{\ 0} & \mathcal{L}^{+0}_{\ 1} & \mathcal{L}^{+0}_{\ 2} \\ \mathcal{L}^{+1}_{\ 0} & \mathcal{L}^{+1}_{\ 1} & \mathcal{L}^{+1}_{\ 2} \\ \mathcal{L}^{+2}_{\ 0} & \mathcal{L}^{+2}_{\ 1} & \mathcal{L}^{+2}_{\ 2} \end{pmatrix}, \qquad \mathcal{L}^- = \begin{pmatrix} \mathcal{L}^{-0}_{\ 0} & \mathcal{L}^{-0}_{\ 1} & \mathcal{L}^{-0}_{\ 2} \\ \mathcal{L}^{-1}_{\ 0} & \mathcal{L}^{-1}_{\ 1} & \mathcal{L}^{-1}_{\ 2} \\ \mathcal{L}^{-2}_{\ 0} & \mathcal{L}^{-2}_{\ 1} & \mathcal{L}^{-2}_{\ 2} \end{pmatrix} \tag{25}$$

we make the ansatz

$$\begin{aligned} &\mathcal{L}^{+2}_{\ 1} = 0 = \mathcal{L}^{-1}_{\ 2} \\ &\mathcal{L}^{+0}_{\ 1} = \mathcal{L}^{+0}_{\ 2} = \mathcal{L}^{+1}_{\ 0} = \mathcal{L}^{+2}_{\ 0} = 0 \\ &\mathcal{L}^{-0}_{\ 1} = \mathcal{L}^{-0}_{\ 2} = \mathcal{L}^{-1}_{\ 0} = \mathcal{L}^{-2}_{\ 0} = 0 \end{aligned} \tag{26}$$

and, similar to the above for $A(R)$, check the coideal property. We also verify explicitly [7] that this ansatz is compatible with the duality pairing

$$\begin{aligned} &\langle \mathcal{L}^{+2}_{\ 1}, \mathcal{T}^i_j \rangle = R^{+2i}_{\ \ 1j} = R^{i2}_{j1} = 0 \\ &\langle \mathcal{L}^{-1}_{\ 2}, \mathcal{T}^i_j \rangle = R^{-1i}_{\ \ 2j} = (R^{-1})^{1i}_{2j} = 0 \end{aligned} \tag{27}$$

and so on for their pairing with products of the $\mathcal{T}^i_j$. Therefore, setting these elements to zero yields a quotient bialgebra $U(R)$ of $\tilde{U}(R)$

$$\mathcal{L}^+ = \begin{pmatrix} \mathcal{L}^{+0}_{\ 0} & 0 & 0 \\ 0 & \mathcal{L}^{+1}_{\ 1} & \mathcal{L}^{+1}_{\ 2} \\ 0 & 0 & \mathcal{L}^{+2}_{\ 2} \end{pmatrix}, \qquad \mathcal{L}^- = \begin{pmatrix} \mathcal{L}^{-0}_{\ 0} & 0 & 0 \\ 0 & \mathcal{L}^{-1}_{\ 1} & 0 \\ 0 & \mathcal{L}^{-2}_{\ 1} & \mathcal{L}^{-2}_{\ 2} \end{pmatrix} \tag{28}$$

Therefore, the initial pairing $\langle A(R), \tilde{U}(R) \rangle$ descends to $\langle \mathcal{A}_{r,s}, U(R) \rangle$. So, for $\mathcal{U}_{r,s}$ (or $U(R)$) we make the following ansatz for the $\mathcal{L}^{\pm}$ matrices:

$$\mathcal{L}^+ = c^+ r \begin{pmatrix} s^{-\frac{1}{2}(\tilde{F}-H_2-1)} r^{\frac{1}{2}(\tilde{F}-H_1-1)} & 0 & 0 \\ 0 & s^{-\frac{1}{2}(\tilde{F}-H_1+1)} r^{\frac{1}{2}(-\tilde{F}+H_2-1)} & r^{-1}\lambda \tilde{C} \\ 0 & 0 & s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{\frac{1}{2}(-\tilde{F}-H_2-1)} \end{pmatrix}$$

$$\mathcal{L}^- = c^- r^{-1} \begin{pmatrix} s^{-\frac{1}{2}(\tilde{F}-H_2-1)} r^{-\frac{1}{2}(\tilde{F}-H_1-1)} & 0 & 0 \\ 0 & s^{-\frac{1}{2}(\tilde{F}-H_1+1)} r^{-\frac{1}{2}(-\tilde{F}+H_2-1)} & 0 \\ 0 & -r\lambda \tilde{B} & s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{-\frac{1}{2}(-\tilde{F}-H_2-1)} \end{pmatrix}$$

where $H_1 = \tilde{A} + \tilde{D}$, $H_2 = \tilde{A} - \tilde{D}$, and $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{F}\}$ is the set of generating elements of the dual algebra. This is consistent with the action on the generators of $\mathcal{A}_{r,s}$ and gives the correct duality pairings. More conveniently,

$$\mathcal{L}^+ = \begin{pmatrix} J & 0 & 0 \\ 0 & M & P \\ 0 & 0 & N \end{pmatrix} \quad \text{and} \quad \mathcal{L}^- = \begin{pmatrix} J' & 0 & 0 \\ 0 & M' & 0 \\ 0 & Q & N' \end{pmatrix} \tag{29}$$

where

$$\begin{aligned} J &= s^{-\frac{1}{2}(\tilde{F}-H_2-1)} r^{\frac{1}{2}(\tilde{F}-H_1+1)} \\ M &= s^{-\frac{1}{2}(\tilde{F}-H_1+1)} r^{\frac{1}{2}(-\tilde{F}+H_2+1)} \\ N &= s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{\frac{1}{2}(-\tilde{F}-H_2+1)} \\ J' &= s^{-\frac{1}{2}(\tilde{F}-H_2-1)} r^{-\frac{1}{2}(\tilde{F}-H_1+1)} \\ M' &= s^{-\frac{1}{2}(\tilde{F}-H_1+1)} r^{-\frac{1}{2}(-\tilde{F}+H_2+1)} \\ N' &= s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{-\frac{1}{2}(-\tilde{F}-H_2+1)} \end{aligned} \tag{30}$$

and

$$\begin{aligned} P &= \lambda \tilde{C} \\ Q &= -\lambda \tilde{B} \end{aligned} \tag{31}$$

These can also be arranged in terms of smaller L^+ and L^- matrices

$$\begin{aligned} \mathcal{L}^+ &= c^+ \begin{pmatrix} J & 0 \\ 0 & L^+ \end{pmatrix} \qquad \text{where} \qquad L^+ = \begin{pmatrix} M & P \\ 0 & N \end{pmatrix} \\ \mathcal{L}^- &= c^- \begin{pmatrix} J' & 0 \\ 0 & L^- \end{pmatrix} \qquad \text{where} \qquad L^- = \begin{pmatrix} M' & 0 \\ Q & N' \end{pmatrix} \end{aligned} \tag{32}$$

Commutation relations of the dual

The dual algebra is generated by $\mathcal{L}^\pm$ functionals which satisfy the q-commutation relations (the so-called $R\mathcal{L}\mathcal{L}$ relations)

$$R_{12}\mathcal{L}_2^\pm \mathcal{L}_1^\pm = \mathcal{L}_1^\pm \mathcal{L}_2^\pm R_{12} \tag{33}$$

$$R_{12}\mathcal{L}_2^+ \mathcal{L}_1^- = \mathcal{L}_1^- \mathcal{L}_2^+ R_{12} \tag{34}$$

where $\mathcal{L}_1^\pm = \mathcal{L}^\pm \otimes \mathbf{1}$ and $\mathcal{L}_2^\pm = \mathbf{1} \otimes \mathcal{L}^\pm$. Since $\mathcal{A}_{r,s}$ is a quotient Hopf algebra, it is necessary to amend the R-matrix to eliminate relations that are inconsistent with the quotient structure.

Consequently, the R-matrix for the $R\mathcal{L}\mathcal{L}$ relations is different from the one used in the $R\mathcal{T}\mathcal{T}$ relations. The $R\mathcal{L}\mathcal{L}$ relations are constructed with the R-matrix:

$$R_{12} = c^{-}\langle \mathcal{L}^{-}, \mathcal{T}\rangle^{-1} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \mathsf{S}^{-1} & 0 & 0 \\ 0 & 0 & \mathsf{S} & 0 \\ 0 & 0 & 0 & R_r \end{pmatrix} \tag{35}$$

Evaluating $\mathcal{L}_1^{\pm}$, $\mathcal{L}_2^{\pm}$ matrices and substituting in the above $R\mathcal{L}\mathcal{L}$- relations yields the dual algebra commutation relations. From $R_{12}\mathcal{L}_2^{-}\mathcal{L}_1^{-} = \mathcal{L}_1^{-}\mathcal{L}_2^{-}R_{12}$ and $R_{12}\mathcal{L}_2^{+}\mathcal{L}_1^{+} = \mathcal{L}_1^{+}\mathcal{L}_2^{+}R_{12}$ we obtain

$$R_r L_2^{-} L_1^{-} = L_1^{-} L_2^{-} R_r \tag{36}$$

$$R_r L_2^{+} L_1^{+} = L_1^{+} L_2^{+} R_r \tag{37}$$

$$\begin{aligned} MJ &= JM & M'J' &= J'M' \\ NJ &= JN & N'J' &= J'N' \\ PJ &= sJP & J'Q &= sQJ' \end{aligned} \tag{38}$$

where

$$\begin{aligned} R_r L_2^{-} L_1^{-} = L_1^{-} L_2^{-} R_r \implies & QM' = rM'Q, \quad N'Q = rQN' \quad \text{and} \quad N'M' = M'N' \\ R_r L_2^{+} L_1^{+} = L_1^{+} L_2^{+} R_r \implies & PM = rMP, \quad NP = rPN \quad \text{and} \quad NM = MN \end{aligned} \tag{39}$$

In addition, the cross relation $R_{12}\mathcal{L}_2^{+}\mathcal{L}_1^{-} = \mathcal{L}_1^{-}\mathcal{L}_2^{+}R_{12}$ yields

$$\begin{aligned} NJ' &= J'N & MJ' &= J'M & PJ' &= sJ'P \\ N'J &= JN' & M'J &= JM' & JQ &= sQJ \end{aligned} \tag{40}$$

and $R_r L_2^{+} L_1^{-} = L_1^{-} L_2^{+} R_r$ which further implies

$$QP - PQ = -\lambda(N'M - NM') \tag{41}$$

Simplifying the above, we get the following commutation relations

$$\begin{aligned} [\tilde{A}, \tilde{B}] &= \tilde{B}, & [\tilde{A}, \tilde{C}] &= -\tilde{C} \\ [\tilde{D}, \tilde{B}] &= -\tilde{B}, & [\tilde{D}, \tilde{C}] &= \tilde{C} \\ [\tilde{A}, \tilde{D}] &= 0, & [\tilde{F}, \bullet] &= 0 \end{aligned} \tag{42}$$

and

$$[\tilde{B},\tilde{C}] = \frac{r^{\tilde{A}-\tilde{D}}s^{-\tilde{F}} - r^{-(\tilde{A}-\tilde{D})}s^{-\tilde{F}}}{r-r^{-1}} = \frac{r^{\gamma\tilde{F}}}{r-r^{-1}}[r^{\tilde{A}-\tilde{D}} - r^{-(\tilde{A}-\tilde{D})}] \tag{43}$$

where $\gamma = \frac{\ln s}{\ln r}$. So, we obtain a single-parameter deformation of $U(gl(2)) \otimes U(u(1))$ as an algebra. Including the coproduct, we again obtain a semidirect product $U_r(gl(2)) \underset{s}{\rtimes} U(u(1))$, as expected.

IV Constructive calculus

In order to investigate the differential geometric structure of the (r,s)-deformation, $\mathcal{A}_{r,s}$, of $GL(2)\otimes GL(1)$, we use Jurčo's constructive procedure [6] based on the R-matrix formulation. This method has so far been applied only to full matrix quantum groups but we demonstrate here that it works equally well for appropriate quotients of these. For $\mathcal{A}_{r,s}$, we obtain a first order bicovariant differential calculus employing the ansatz for $\mathcal{L}^{\pm}$ introduced in Sec. III.

A One-forms

Let $\{\omega\}$ be the basis of all left-invariant quantum one-forms. So, we have

$$\Delta_L(\omega) = \mathbf{1} \otimes \omega \tag{44}$$

This defines the left action on the bimodule Γ (space of quantum one-forms). The bimodule Γ is further characterised by the commutation relations between ω and $a \in \mathcal{A}$ ($\equiv \mathcal{A}_{r,s}$),

$$\omega a = (f * a)\omega \tag{45}$$

The left convolution product is

$$f * a = (\mathbf{1} \otimes f)\Delta(a) \tag{46}$$

where $f \in \mathcal{A}'(= \mathrm{Hom}(\mathcal{A}, \mathbf{C}))$ belongs to the dual. This means

$$\omega a = (\mathbf{1} \otimes f)\Delta(a)\omega \tag{47}$$

Now the linear functional f is defined in terms of the $\mathcal{L}^{\pm}$ matrices as

$$f = S(\mathcal{L}^{+})\mathcal{L}^{-} \tag{48}$$

Thus we have

$$\omega a = [(\mathbf{1} \otimes S(\mathcal{L}^+)\mathcal{L}^-)\Delta(a)]\omega \tag{49}$$

In terms of components,

$$\omega_{ij} a = [(\mathbf{1} \otimes S(l^+_{ki}) l^-_{jl})\Delta(a)]\omega_{kl} \tag{50}$$

using the expressions $\mathcal{L}^{\pm} = l^{\pm}_{ij}$ and $\omega = \omega_{ij}$ where $i, j = 1..3$. For Γ to be a bicovariant bimodule, the right coaction is given by

$$\Delta_R(\omega) = \omega \otimes M \tag{51}$$

where functionals M are defined in terms of the matrix of generators $\mathcal{T}$,

$$M = \mathcal{T} S(\mathcal{T}) \tag{52}$$

Again, in component form, we can write

$$\Delta_R(\omega_{ij}) = \omega_{kl} \otimes t_{ki} S(t_{jl}) \tag{53}$$

Using the above formulas, we obtain the commutation relations of all the left-invariant one forms with the generating elements $\{a, b, c, d, f\}$ of $\mathcal{A}_{r,s}$:

$$\begin{aligned}
&\omega^0 a = a\omega^0 && \omega^0 b = b\omega^0 \\
&\omega^1 a = r^{-2} a\omega^1 && \omega^1 b = b\omega^1 \\
&\omega^+ a = r^{-1} a\omega^+ && \omega^+ b = r^{-1} b\omega^+ - \lambda r^{-1} a\omega^1 \\
&\omega^- a = r^{-1} a\omega^- - \lambda r^{-1} b\omega^1 && \omega^- b = r^{-1} b\omega^- \\
&\omega^2 a = a\omega^2 - \lambda b\omega^+ && \omega^2 b = r^{-2} b\omega^2 - \lambda r^{-1} a\omega^- + \lambda^2 b\omega^1
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\omega^0 c = c\omega^0 && \omega^0 d = d\omega^0 \\
&\omega^1 c = r^{-2} c\omega^1 && \omega^1 d = d\omega^1 \\
&\omega^+ c = r^{-1} c\omega^+ && \omega^+ d = r^{-1} d\omega^+ - \lambda r^{-1} c\omega^1 \\
&\omega^- c = r^{-1} c\omega^- - \lambda r^{-1} d\omega^1 && \omega^- d = r^{-1} d\omega^- \\
&\omega^2 c = c\omega^2 - \lambda d\omega^+ && \omega^2 d = r^{-2} d\omega^2 - \lambda r^{-1} c\omega^- + \lambda^2 d\omega^1
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\omega^0 f = r^{-2} f\omega^0 \\
&\omega^1 f = f\omega^1 \\
&\omega^+ f = s f\omega^+ \\
&\omega^- f = s^{-1} f\omega^- \\
&\omega^2 f = f\omega^2
\end{aligned} \tag{56}$$

where $\omega^0 = \omega_{11}, \omega^1 = \omega_{22}, \omega^+ = \omega_{23}, \omega^- = \omega_{32}, \omega^2 = \omega_{33}$ and the components $\omega_{12}, \omega_{13}, \omega_{21}, \omega_{31}$ have null contribution, given the structure of the $\mathcal{T}$ matrix (i.e., $t_{12} = t_{13} = t_{21} = t_{31} = 0$).

B Vector fields

The linear space Γ (space of all left invariant one-forms) contains a bi-invariant element $\tau = \sum_i \omega_{ii}$ which can be used to define a derivative on $\mathcal{A}$. For $a \in \mathcal{A}$, one sets

$$\mathbf{d}a = \tau a - a\tau \tag{57}$$

Now

$$\omega_{ii} a = [(\mathbf{1} \otimes S(l^+_{ki}) l^-_{il}) \Delta(a)] \omega_{kl} \tag{58}$$

So

$$\mathbf{d}a = [(\mathbf{1} \otimes \chi_{kl}) \Delta(a)] \omega_{kl} \tag{59}$$

where $\chi_{kl} = S(l^+_{ki}) l^-_{il} - \delta_{kl}\varepsilon$, ε being the counit. Denote

$$\chi_{ij} = S(l^+_{ik}) l^-_{kj} - \delta_{ij}\varepsilon \tag{60}$$

or more compactly

$$\chi = S(\mathcal{L}^+)\mathcal{L}^- - \mathbf{1}\varepsilon \tag{61}$$

the matrix of left-invariant vector fields χ_{ij} on $\mathcal{A}$. The action of the vector fields on the generating elements is

$$\chi_{ij} a = (S(l^+_{ik}) l^-_{kj} - \delta_{ij}\varepsilon) a \tag{62}$$

$$\chi_{ij} a = \langle S(l^+_{ik}) l^-_{kj}, a \rangle - \delta_{ij}\varepsilon(a) \tag{63}$$

Explicitly, we obtain

$$\begin{array}{ll} \chi_0(a) = 0 & \chi_0(b) = 0 \\ \chi_1(a) = r^{-2} - 1 & \chi_1(b) = 0 \\ \chi_+(a) = 0 & \chi_+(b) = 0 \\ \chi_-(a) = 0 & \chi_-(b) = -(r - r^{-1}) \\ \chi_2(a) = 0 & \chi_2(b) = 0 \end{array} \tag{64}$$

$$\begin{aligned}
&\chi_0(c) = 0 && \chi_0(d) = 0\\
&\chi_1(c) = 0 && \chi_1(d) = (r - r^{-1})^2\\
&\chi_+(c) = -(r - r^{-1}) && \chi_+(d) = 0\\
&\chi_-(c) = 0 && \chi_-(d) = 0\\
&\chi_2(c) = 0 && \chi_2(d) = r^{-2} - 1
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\chi_0(f) = r^{-2} - 1\\
&\chi_1(f) = 0\\
&\chi_+(f) = 0\\
&\chi_-(f) = 0\\
&\chi_2(f) = 0
\end{aligned} \tag{66}$$

where $\chi_0 = \chi_{11}, \chi_1 = \chi_{22}, \chi_+ = \chi_{23}, \chi_- = \chi_{32}, \chi_2 = \chi_{33}$ and again (by previous argument) the components $\chi_{12}, \chi_{13}, \chi_{21}, \chi_{31}$ have null contribution. The left convolution products are given as

$$\begin{aligned}
&\chi_0 * a = 0 && \chi_0 * b = 0\\
&\chi_1 * a = (r^{-2} - 1)a && \chi_1 * b = ((r - r^{-1})^2)b\\
&\chi_+ * a = -(r - r^{-1})b && \chi_+ * b = 0\\
&\chi_- * a = 0 && \chi_- * b = -(r - r^{-1})a\\
&\chi_2 * a = 0 && \chi_2 * b = (r^{-2} - 1)b
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\chi_0 * c = 0 && \chi_0 * d = 0\\
&\chi_1 * c = (r^{-2} - 1)c && \chi_1 * d = ((r - r^{-1})^2)d\\
&\chi_+ * c = -(r - r^{-1})d && \chi_+ * d = 0\\
&\chi_- * c = 0 && \chi_- * d = -(r - r^{-1})c\\
&\chi_2 * c = 0 && \chi_2 * d = (r^{-2} - 1)d
\end{aligned} \tag{68}$$

$$\begin{aligned}
&\chi_0 * f = (r^{-2} - 1)f\\
&\chi_1 * f = 0\\
&\chi_+ * f = 0\\
&\chi_- * f = 0\\
&\chi_2 * f = 0
\end{aligned} \tag{69}$$

C Exterior derivatives

Using $\mathbf{d}a = \sum_i (\chi_i * a)\omega^i$ for $a \in \mathcal{A}$, we obtain the action of the exterior derivatives:

$$\mathbf{d}a = (r^{-2} - 1)a\omega^1 - \lambda b\omega^+ \tag{70}$$

$$\mathbf{d}b = \lambda^2 b\omega^1 - \lambda a\omega^- + (r^{-2} - 1)b\omega^2 \tag{71}$$

$$\mathbf{d}c = (r^{-2} - 1)c\omega^1 - \lambda d\omega^+ \tag{72}$$

$$\mathbf{d}d = \lambda^2 d\omega^1 - \lambda c\omega^- + (r^{-2} - 1)d\omega^2 \tag{73}$$

$$\mathbf{d}f = (r^{-2} - 1)f\omega^0 \tag{74}$$

where $\lambda = r - r^{-1}$. The exterior derivative $\mathbf{d} : \mathcal{A} \longrightarrow \Gamma$ satisfies the Leibniz rule and $\mathbf{d}\mathcal{A}$ generates Γ as a left $\mathcal{A}$-module. This then defines a first-order differential calclulus $(\Gamma, \mathbf{d})$ on $\mathcal{A}_{r,s}$. Furthermore, the calculus is bicovariant due to the coexistence of the left and the right actions

$$\Delta_L : \Gamma \longrightarrow \mathcal{A} \otimes \Gamma \tag{75}$$

$$\Delta_R : \Gamma \longrightarrow \Gamma \otimes \mathcal{A} \tag{76}$$

since $\mathbf{d}$ has the invariance property

$$\Delta_L \mathbf{d} = (\mathbf{1} \otimes \mathbf{d})\Delta \tag{77}$$

$$\Delta_R \mathbf{d} = (\mathbf{d} \otimes \mathbf{1})\Delta \tag{78}$$

The bicovariance holds also due to the existence of the bi-invariant element $\tau = \sum_i \omega_{ii}$ (eqn.(57)) of the linear space of left-invariant one-forms. If we rewrite the derivatives $\{\mathbf{d}a, \mathbf{d}b, \mathbf{d}c, \mathbf{d}d, \mathbf{d}f\}$ as $\{\mathbf{d}(f^N a), \mathbf{d}(f^N b), \mathbf{d}(f^N c), \mathbf{d}(f^N d)\}$, i.e., reducing from the five-dimensional to the four-dimensional algebra, then the latter set of exterior derivatives provides a realisation of the differential calculus on the biparametric (p, q)-deformation of $GL(2)$, i.e., $GL_{p,q}(2)$, with the defining relations between the two sets of deformation parameters (p, q) and (r, s) as before. Furthermore, the differential calculus also respects the cross-product structure of $\mathcal{A}_{rs}$. It can be checked (using the Leibniz rule) that

$$\mathbf{d}(af - fa) = 0, \quad \mathbf{d}(cf - sfc) = 0, \quad \mathbf{d}(bf - s^{-1}fb) = 0, \quad \mathbf{d}(df - fd) = 0, \tag{79}$$

which is consistent with the cross relations (5).

V Jordanian analogue

It was shown in [3] that the $\mathcal{A}_{r,s}$ deformation could be contracted (by means of singular limit of similarity transformations) to obtain a nonstandard or Jordanian analogue, say $\mathcal{A}_{m,k}$, with deformation parameters $\{m,k\}$ and the associated R-matrix is triangular. In analogy with $\mathcal{A}_{r,s}$, $\mathcal{A}_{m,k}$ can also be considered as the semidirect or cross-product $GL_m(2) \underset{k}{\rtimes} \mathbb{C}[f,f^{-1}]$ where $GL_m(2) = \mathbb{C}[a,b,c,d]$ modulo the relations

$$\begin{aligned} [c,d] &= -mc^2, \qquad & [c,b] &= -m(ac+cd) = -m(ca+dc) \\ [c,a] &= -mc^2, \qquad & [d,a] &= -m(d-a)c = -mc(d-a) \end{aligned} \tag{80}$$

$$\begin{aligned} [d,b] &= -m(d^2-\delta) \\ [b,a] &= -m(\delta-a^2) \end{aligned} \tag{81}$$

where $\delta = ad - bc + mac = ad - cb - mcd$, and $\mathbb{C}[f,f^{-1}]$ has the cross relations

$$\begin{aligned} [f,a] &= kcf, \qquad & [f,b] &= k(df - fa) \\ [f,c] &= 0, \qquad & [f,d] &= -kcf \end{aligned} \tag{82}$$

Thus, $\mathcal{A}_{m,k} \simeq GL_m(2) \underset{k}{\rtimes} \mathbb{C}[f,f^{-1}]$ can also be interpreted as a skew Laurent polynomial ring $GL_m[f,f^{-1};\sigma]$ where σ is the automorphism given by the action of element f on $GL_m(2)$. The (left) action is given by

$$f \triangleright a = a + kc, \qquad f \triangleright b = b + k(d-a) - k^2 c, \qquad f \triangleright c = c, \qquad f \triangleright d = d - kc \tag{83}$$

VI Discussion

In this article, we have investigated the algebro-geometric structure of the biparametric quantum deformation of $GL(2) \otimes GL(1)$, namely, $\mathcal{A}_{r,s}$. A particular feature of this deformation is that it has an interpretation as a semidirect or cross-product algebra. We exhibit this cross-product structure and establish a picture of duality in this setting. Using the R-matrix formalism, we have given an explicit derivation of the corresponding dual agebra, i.e., the quantised universal enveloping algebra and also constructed a bicovariant differential calculus. The dual algebra obtained via R-matrices is isomorphic to the dual algebra obtained by the cross-product construction. We note that the differential calculus satisfies the required axioms, contains the calculus on $GL_q(2)$ and our results match with those given in [10]. Besides, the calculus is also consistent with the cross-product structure of

$\mathcal{A}_{r,s}$. We expect that the calculus could as well be obtained by projection from the calculus on multiparameter q-deformed $GL(3)$. The differential calculus obtained on $\mathcal{A}_{r,s}$ enables us to investigate the associated gauge theory from a noncommutative perspective. It would be useful to repeat the analysis presented in this paper for the biparametric Jordanian deformation of $GL(2) \otimes GL(1)$ obtained in [3] and also to investigate corresponding hybrid (q,h)-deformations [11, 12]. Furthermore, it would indeed be interesting to generalise the setting to the case of coloured quantum and Jordanian deformations.

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