SECOND QUANTIZED FROBENIUS ALGEBRAS

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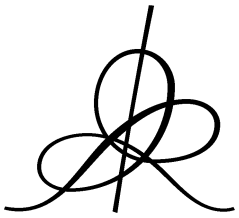

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Juin 2002

IHES/M/02/46

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ABSTRACT. We show that given a Frobenius algebra there is a unique notion of its second quantization, which is the sum over all symmetric group quotients of n–th tensor powers, where the quotients are given by symmetric group twisted Frobenius algebras. To this end, we consider the setting of Frobenius algebras given by functors from geometric categories whose objects are endowed with geometric group actions and prove structural results, which in turn yield a constructive realization in the case of n–th tensor powers and the natural permutation action. We also show that naturally graded symmetric group twisted Frobenius algebras have a unique algebra structure already determined by their underlying additive data together with a choice of super–grading. Furthermore we discuss several notions of discrete torsion and show that indeed a non–trivial discrete torsion leads to a non–trivial super structure on the second quantization.

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INTRODUCTION

In "stringy" geometry evaluating a functor from a geometric to a linear category on a group quotient is generally a two step process. The first is to evaluate the functor not only on the object, but to form the direct sum of the evaluations on all of the fixed point sets. The new summands corresponding to group elements which

[1]Partially supported by NSF grant #0070681

are not the identity are usually named twisted sectors. The second step is to find a suitable group action on the twisted sectors and take group invariants.

If the objects in the linear category also have an algebra structure there is an additional step, i.e. to find a new algebra structure that is not the diagonal one which is canonically present, but a group graded one.

If there is also a natural pairing such that the original functor have values in Frobenius algebra, then the result of the "stringy" extension of this functor should have values in G–twisted Frobenius algebras which were introduced for this purpose in [K2].

In particular, the question of importance is the step of finding the suitable multiplication. The theory of G–twisted Frobenius algebras is exactly tailored to classify the possible multiplicative structures.

We address this matter in the present paper once more in the general case of intersection Frobenius algebras and in the special case of symmetric group quotients which are naturally intersection Frobenius algebras.

The class of intersection Frobenius algebras incorporates the fact that all geometric construction of Frobenius algebras via functors from geometric categories with geometric group actions actually have a much richer structure which can be used to provide further constraints on the nature of the twisted multiplication.

We apply these general results to the case of symmetric group quotients of powers of Frobenius algebras.

The main result here is that *there is a unique multiplicative structure that makes the canonical extension of the n–th tensor power of a Frobenius algebra into a symmetric group twisted Frobenius algebra.* There are only two possible group actions compatible with this structure and they are given by a choice of parity, i.e. a group homomorphism of the symmetric group into $\mathbf{Z}/2\mathbf{Z}$. The choices are just all even or super–graded.

This result should be read as the statement that there is a well defined notion of second quantized Frobenius algebra. Recall that second quantization in an monoidal category with a notion of symmetric quotients is given by:

Second quantization of X $= \exp(X) = \sum_n X^n/\mathbf{S}_n$,

where $\mathbf{S}_n$ is action by permutations on the factors and the sum may be formal or contain a bookkeeping variable (e.g. q^n). From our result, we expect that one can also easily derive a definition of second quantized motives. All the objects are powers of the original object and the morphisms are given by structural morphisms. It would be interesting to explicitly see the multiplication in terms of correspondences.

Furthermore we discuss several notions of discrete torsion and show that indeed a non–trivial discrete torsion leads to a non–trivial super structure on the second quantization.

The paper is organized as follows. In §1 we review our definitions of [K2, K3] of G–twisted and special G–twisted Frobenius algebras. In the latter the multiplication and group action can be described by group cocycles and non–abelian group cocycles, respectively. Besides fixing and recalling the notation and definitions, we add several useful practical Lemmas as well as a new description of the non–abelian cocycles in terms of ordinary group one–cocycles with values in tori. The second paragraph contains the functorial setup of the general question posed in the introduction, i.e. to identify the underlying additive data and the possible extensions of this data by "stringy" product to the right type of group quotient algebra.

In §2 we also introduce the notion of intersection categories, which reflect the geometrical setups with geometrical group actions which are used for the known construction of Frobenius algebras such as cohomology, quantum cohomology, singularity theory, etc.. This setup is carried over to the Frobenius side in §3 where we prove general results about the structure of the cocycles in the special G–twisted Frobenius algebra case. These results are also the key to understanding the second quantization. Furthermore we introduce the notion of algebraic discrete torsion, which generalizes the case of discrete torsion for Jacobian algebras of [K3] and provides the discrete torsion that is linked to the super–structure of second quantization.

In order to give a clearer view of the geometry involved in the second quantization, it is useful to also consider the case of Jacobian Frobenius algebras and their second quantization. The relevant notions of Jacobian Frobenius algebras are recalled in §4.

We then start our consideration of $\mathbf{S}_n$–twisted Frobenius algebras. §5 contains general results about these structures. The main results of this section are the classification of possible non–abelian group cocycles and the uniqueness (up to normalization) of "stringy" products given a group grading compatible with the natural grading on $\mathbf{S}_n$. Before applying these results to general symmetric powers, we work out all the details in the case of the n–th tensor power of a Frobenius algebra in §6 and also show the existence of the natural $\mathbf{S}_n$–twisted Frobenius algebra based on the n–th tensor power. Here we also recover the known discrete torsion corresponding to the non–trivial Schur multiplier.

Using the geometric insight of the previous paragraph we turn to the general case of the n–th tensor power of a Frobenius algebra in §7 and show that there is a unique (up to a choice of parity for the group action) natural extension of n–th tensor power to a $\mathbf{S}_n$–twisted Frobenius algebra, establishing the existence of second quantized Frobenius algebras. There are two versions a purely even one a and super symmetric one. Passing from one to the other can be viewed as turning on a natural algebraic discrete torsion. Lastly, we relate our results to the ones of [LS].

There are also two appendices. The first contains a key result on the possible form of non–abelian $\mathbf{S}_n$ cocycles and the second contains the detailed version of the proof of normalizability of §5.

Acknowledgements

I would like to thank the IHÉS for its kind hospitality. My visits in 2001 and 2002 mark the conceptual origin and the finishing phase of the paper. I also greatfully acknowledge the support from the NSF. It is a pleasure to thank L. Borisov who sparked my interested in symmetric products, Y. Ruan and A. Adem for discussions and the wonderful conference in Madison and B. Guralnick for discussions on Schur multipliers and pointing out the reference [Ka].

Notation

We denote by $\bar{n} := \{1, \ldots, n\}$. Furthermore, we fix k to be a field of characteristic 0. The reader can think of $\mathbf{C}$ if he or she wishes. The theory is the same if k is a super-commutative $\mathbf{Q}$ algebra and (super-)vector spaces and dimensions are

replaced by free modules and ranks. Finally, if we fix a group G then all remains true for a field of a characteristic prime to $|G|$.

1. Orbifold Frobenius algebras

Recall the following definitions first presented in [K2] and contained in [K3].
We fix a finite group G and denote its unit element by e.

1.1. **Definition.** A *G–twisted Frobenius algebra* (or G–Frobenius algebra for short) over a field k of characteristic 0 is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where

- G finite group
- A finite dim G-graded k–vector space
  $A = \oplus_{g\in G} A_g$
  A_e is called the untwisted sector and
  the A_g for $g \neq e$ are called the twisted sectors.
- $\circ$ a multiplication on A which respects the grading:
  $\circ : A_g \otimes A_h \to A_{gh}$
- 1 a fixed element in A_e–the unit
- η non-degenerate bilinear form
  which respects grading i.e. $g|_{A_g\otimes A_h} = 0$ unless $gh = e$.
- φ an action by algebra automorphisms of G on A,
  $\varphi \in \mathrm{Hom}_{k-alg}(G, A)$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$
- χ a character $\chi \in \mathrm{Hom}(G, k^*)$

satisfying the following axioms:

Notation. We use a subscript on an element of A to signify that it has homogeneous group degree –e.g. a_g means $a_g \in A_g$–, and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$. We also drop the subscript if $a \in A_e$.

a) *Associativity*
$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$

b) *Twisted commutativity*
$a_g \circ a_h = \varphi_g(a_h) \circ a_g$

c) *G Invariant Unit*:
$1 \circ a_g = a_g \circ 1 = a_g$
and
$\varphi_g(1) = 1$

d) *Invariance of the metric*:
$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$

i) *Projective self–invariance of the twisted sectors*
$\varphi_g|A_g = \chi_g^{-1} id$

ii) *G–Invariance of the multiplication*
$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$

iii) *Projective G–invariance of the metric*
$\varphi_g^*(\eta) = \chi_g^{-2}\eta$

iv) *Projective trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

An alternate choice of data is given by a one–form ϵ, the co–unit with $\epsilon \in A_e^*$ and a three–tensor $\langle \cdot, \cdot, \cdot \rangle \in A^* \otimes A^* \otimes A^*$ which is of group degree e, i.e. $\langle \cdot, \cdot, \cdot \rangle|_{A_g\otimes A_h\otimes A_k} = 0$ unless $ghk = e$.

The relations between $\eta, \circ$ and ϵ, μ are given by dualization.

We denote by $\rho \in A_e$ the element dual to $\epsilon \in A_e^*$ and Poincaré dual to $1 \in A_e$.

In the graded case, we call the degree d of ρ the degree of A. This means that η is homogeneous of degree d.

1.2. **Definition.** A *special G–Frobenius algebra* is a G–twisted Frobenius algebra whose components A_g are cyclic A_e–modules, together with two collections of maps $(r_g), (i_g)$ indexed by G where

$r_g : A_e \to A_g$ is the map of A_e–modules induced by multiplication. In particular, if one sets $1_g := r_g(1)$, we obtain $r_g(a) = a1_g$. Notice that it is equivalent to specify the map r_g or generators 1_g of A_g as cyclic A_e–modules.

$i_g : A_g \mapsto A_e$ are collections of maps s.t. each i_g is an injection which splits the exact sequence of A_e-modules.

$$0 \longrightarrow I_g \longrightarrow A_e \underset{i_g}{\overset{r_g}{\rightleftarrows}} A_g \longrightarrow 0 \tag{1.1}$$

Here $I_g \subset A_e$ is the annihilator of 1_g and thus of A_g. We denote the concatenation of r_g and i_g by π_g

$$\pi_g = i_g \circ r_g : A_e \to A_e$$

and we take the statement that i_g is a section of A_e modules to mean

$$i_g(a_e b_g) = \pi_g(a_e) i_g(b_g) \tag{1.2}$$

Furthermore the following should hold:

$$\forall g \in G : \varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}$$

with $\varphi_{g,h} \in K$ and $\varphi_{g,1} = 1$.

1.3. **Remarks.** Notice that

$$\begin{aligned} \pi_g(ab) &= \pi_g(a)\pi_g(b) \\ r_g(ab) &= \pi_g(a) r_g(b) \end{aligned}$$

In particular, multiplication by 1_g acts as projection: $a1_g = \pi_g(a)1_g$ and $\pi_g(1)$ acts as identity on A_g.

1.4. **Frobenius algebra structure on the twisted sectors.** Recall that the A_g are Frobenius algebras by the multiplication

$$a_g \circ_g b_g = i_g(a_g) i_g(b_g) 1_g \tag{1.3}$$

and metric

$$\eta_g(a_g, b_g) := \eta(i_g(a_g)1_g, i_g(b_g)1_g^{-1}) \tag{1.4}$$

1.5. **Notation.** We denote by $\rho_g \in A_g$ the element defining η_g and by $d_g := \deg(\rho_g)$ the degree of A_g and $s_g := \deg(1_g)$ will be called the degree shift. We also set

$$s_g^+ := \frac{1}{2}(s(g) + s(g^{-1})) \quad s^- := \frac{1}{2}(s(g) - s(g^{-1}))$$

the degree defect.

Notice that $d = d_e$ if d denotes the degree of A given by η.

By considering $\eta|_{A_g \otimes A_{g^{-1}}}$ we find:

1.5.1. **Lemma.** [K3]

$$s_g^+ = d - d_g$$

Notice there is no restriction (except anti–symmetry) on s^-.

1.6. **Special super G–Frobenius algebra.** The super version of special G–Frobenius algebras is straightforward. Notice that since each A_g is a cyclic A_e–algebra its parity is fixed to be $(-1)^{\tilde{g}} := \widetilde{1_g}$ times that of A_e. I.e. $a_g = i_g(a_g)1_g$ and thus $\widetilde{a_g} = \widetilde{i_g(a_g)}\widetilde{1_g}$. In particular if A_e is purely even A_g is purely of degree $\tilde{g}$.

We also proved the following Reconstruction Theorem:

1.7. **Reconstruction.** Given a Frobenius algebra (A_e, η_e) and a G-action on A:$\varphi : G \times A \to A$ together with the following data

- Frobenius algebras $(A_g, \eta_g | g \in G \setminus \{1\})$.
- Injective algebra homomorphisms $i_g : A_g \to A_e$ s.t. $i_g(A_g) = i_{g^{-1}}(A_{g^{-1}})$.
- A graded G 2–cocycle γ with values in A_e and a compatible non–abelian G 2–cocycle φ with values in K^*
- A group homomorphism $\chi : G \to K^*$

such that

i) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2}\eta_e(a, b)$
ii) $\eta_g(a_g, b_g) = \eta(i_g(a_g)i_g(b_g))\gamma_{gg^{-1}}, 1)$
iii) The projective trace axiom $\forall c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h}) \tag{1.5}$$

hold, then there is a unique extension of this set of data to a special G–twisted Frobenius algebra, i.e. there is a unique special G–twisted Frobenius algebra with these underlying data.

1.7.1. **The multiplication.** We recall (cf. [K3]) that a graded cocycle γ is a map $G \times G \to A$, s.t. $\gamma(g, h) =: \gamma_{g,h} \in i_{gh}(A_{gh})$ with the property that

$$(I_g + I_h)\gamma_{g,h} \subset I_{gh}{}^1 \tag{1.6}$$

and

$$r_{ghk}(\gamma_{g,h}\gamma_{gh,k}) = r_{ghk}(\gamma_{h,k}\gamma_{g,hk}) \tag{1.7}$$

The multiplication is given by

$$1_g 1_h = \gamma_{g,h} 1_{g,h}$$

REMARK. One can drop the choice of i_g from the data and just postulate the existence of such a section.

1.7.2. **The G–action on the twisted sectors.** The cocycles φ are maps $G \times G \to k^*$ s.t.

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k} \tag{1.8}$$

where we used the notation $\varphi_{g,h} = \varphi(g, h)$

The G–action is given by

$$\varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}} \tag{1.9}$$

[1] In [K3] we chose a slightly different condition (called "associative" in [K3]) which is actually equivalent to the current one (called "section independent" in [K3]).

1.7.3. **The compatibility equations.** There are two compatibility equations:

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \tag{1.10}$$

and

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{1.11}$$

1.8. **Definition.** We call the data in the Proposition 1.7 *special Reconstruction datum.*

1.9. **Rescaling.** Given a special G–Frobenius algebra, we can rescale the cyclic generators by λ_g, i.e. we take the same underlying G–Frobenius algebra, but rescale the maps r_g to $\tilde{r}_g$ with $\tilde{1}_g = \tilde{r}_g(1) = \lambda_g 1_g$. We also fix $\lambda_e = 1$ to preserve the identity.

This yields an action of $\mathrm{Map}_{\text{pointed spaces}}(G, k^*)$ on the cocycles γ and φ preserving the underlying G–Frobenius algebra structure.

The action is given by:

$$\begin{aligned} \gamma_{g,h} &\mapsto \tilde{\gamma}_{g,h} = \frac{\lambda_g\lambda_h}{\lambda_{gh}}\gamma_{g,h} \\ \varphi_{g,h} &\mapsto \tilde{\varphi}_{g,h} = \frac{\lambda_h}{\lambda_{ghg^{-1}}}\varphi_{g,h} \end{aligned} \tag{1.12}$$

1.9.1. **Remark.** We can introduce the groups associated with the classes under this scaling and see that the classes of γ correspond to classes in $H^2(G, A)$. We can also identify the non–abelian cocycles φ with one–group cocycles with values in $k^*[G]$ where we treat $k^*[G]$ as an abelian group with diagonal multiplicative composition

$$(\sum_g \lambda_g g)\cdot(\sum_h \mu_h h) := \sum_g \lambda_g\mu_g g \tag{1.13}$$

and G–action given by conjugation:

$$s(g)(\sum_h \lambda_h h) = \sum_h \lambda_h ghg^{-1} \tag{1.14}$$

This is done as follows:

We view the collection $\varphi_{g,\cdot}$ as an element of $k^*[G]$ via

$$\varphi_g := \sum_h \varphi_{g,h} ghg^{-1} \tag{1.15}$$

then

$$\varphi_{gh} = s(g)\varphi_h \cdot \varphi_g$$

Indeed

$$\begin{aligned} s(g)\varphi_h\cdot\varphi_g &= s(g)(\sum_k \varphi_{h,k}\, hkh^{-1})\cdot\sum_k \varphi_{g,k}\, gkg^{-1} \\ &= \sum_k \varphi_{h,k}\, ghkh^{-1}g^{-1}\cdot\sum_k \varphi_{g,k}\, gkg^{-1} \\ &= \sum_k \varphi_{h,k}\varphi_{g,hkh^{-1}}\, ghkh^{-1}g^{-1} \\ &= \sum_k \varphi_{gh,k}\, (gh)k(gh)^{-1} \end{aligned}$$

In this identification, equivalence under scaling corresponds to taking cohomology classes.

The trivial cocycles are of the form $s(g)a \cdot a^{-1}$ with $a = \sum \mu_g\, g$

$$s(g)a \cdot a^{-1} = \sum_h \mu_h ghg^{-1} \cdot \sum_h \mu_h^{-1} h = \sum_h \frac{\mu_h}{\mu_{ghg^{-1}}} h \tag{1.16}$$

and

$$\tilde{\varphi}_g = \sum_h \tilde{\varphi}_{g,h} ghg^{-1} = \sum_h \varphi_{g,h} ghg^{-1} \sum \frac{\lambda_h}{\lambda_{ghg^{-1}}} ghg^{-1} = \varphi_g \cdot (s(g)a \cdot a^{-1}) \tag{1.17}$$

with $a = \sum_h \lambda_h h$.

It is clear that we could also take logarithms of the φ and then we would get cocycles with values in $k[G]$, but there is the problem of choosing a cut as it manifests itself in the setting of special G–Frobenius algebras in the definition of the degree shifts.

1.10. **pt/G and discrete torsion for G.** Elementary non–trivial examples of special G–Frobenius algebras are given by twisted group algebras $k^\gamma[G]$ where γ is a factor set. The multiplication is given by $g \cdot h = \gamma_{g,h} gh$. For all other structures of G–Frobenius algebra see 4.3.6. For such a $\gamma \in Z^2(G, k^*)$ one defines discrete torsion for $g, h \in G, [g, h] = e$ as

$$\epsilon(g, h) := \frac{\gamma_{g,h}}{\gamma_{h,g}} \tag{1.18}$$

Notice that a discrete torsion of G is invariant under scaling and thus passes to the cohomology classes or Schur multipliers. We also point out that due to the compatibility equations

$$\epsilon(g, h) = \varphi_{g,h} \tag{1.19}$$

1.11. **Lemma.** Given graded special reconstruction data, then for section independent cocycles the $\gamma_{g,g^{-1}}$ are uniquely determined as $\check{r_g}(1_g)$.

Proof. First, we prove the uniqueness. By section independence

$$I_g \gamma_{g,g^{-1}} = 0$$

Thus

$$\gamma_{g,g^{-1}} \in (i_g(A_g)^*)^{d - s_g^+}$$

where * is the dual w.r.t. the form η and we use the splitting induced by the sections i (N.B. if η is also positive definite, we could use an orthogonal splitting)

$$A^k = I_g^k \oplus (i_g(A_g))^k \tag{1.20}$$

and superscripts denote fixed degree. Furthermore

$$\dim((i_g(A_g)^*)^{d_g}) = \dim(i_g(A_g)^{d_g}) = \dim(A^{d_g}) - \dim(I_g) =$$
$$\dim(A^{d_g}) - \dim(\mathrm{Ker}(r_g)|_{A^{d_g}}) = \dim(\mathrm{Im}(r_g)|_{A^{d_g}}) = \dim(A_g^{d_g}) = 1$$

where we used the non–shifted grading on A_g. Thus $\gamma_{g,g^{-1}}$ is fixed up to a constant. This is fixed by the equation

$$\eta(1_g, \rho_{g^{-1}}) = \epsilon(\gamma_{g,g^{-1}} i_{g^{-1}}(\rho_{g^{-1}})) = 1$$

For the last statement, we notice that

$$\eta(\gamma_{g,g^{-1}}, a) = \eta(a1_g, 1_{g^{-1}}) = \eta(r_g(a), 1_{g^{-1}}) = \eta_g(1_g, r_g(a))$$

So that $\gamma_{g,g^{-1}} = \check{r}_g(1_g)$. This formula also proves the uniqueness.

1.12. **Lemma.** If $a = i_g(a_g) \in i_g(A_g)$ then $a\gamma_{g,g^{-1}} = \check{r}_g(a_g)$ and furthermore $i_g(A_g)^* = \gamma_{g,g^{-1}} i_g(A_g)$ where * is the Poincaré dual w.r.t. η and the splitting (1.20). Moreover if $aI_g = 0$ then $a = \tilde{a}\gamma_{g,g^{-1}}$ for some $\tilde{a} \in i_g(A_g)$.

Proof. For the first statement notice that:

$$\eta(i_g(a_g)\gamma_{g,g^{-1}}, b) = \eta_g(a_g, r_g(b))$$

the second and third statement follow from this using the non–degenerate nature of η, η_g and splitting (1.20). N.B. The statement is actually independent of the choice of splitting.

1.13. **Proposition.** If $\gamma_{g,h} = 0$ then $\pi_h(\gamma_{g,g^{-1}}) = 0$ and $\pi_g(\gamma_{h,h^{-1}}) = 0$

Proof.

If $\gamma_{g,h} = 0$ then

$0 = \pi_h(\gamma_{g^{-1},gh}\gamma_{g,h}) = \pi_h(\gamma_{g^{-1},g}\gamma_{e,h}) = \pi_h(\gamma_{g^{-1},g}) = \pi_h(\gamma_{g,g^{-1}})$ and also

$0 = \pi_g(\gamma_{g,h}\gamma_{gh,h^{-1}}) = \pi_g(\gamma_{g,e}\gamma_{h^{-1},h}) = \pi_g(\gamma_{h,h}^{-1})$

1.14. **Definition.** We call A_g and A_h transversal if $s_g + s_h = s_{gh}$ and $s_{g^{-1}} + s_{h^{-1}} = s_{(gh)^{-1}}$.

From the section independence, we obtain:

1.14.1. **Lemma.** If A_g and A_h are transversal and $\gamma_{g,h} \neq 0$ then

$$I_g + I_h = I_{gh}$$

1.15. **Proposition.** The converse of 1.13 it true if A_g and A_h are transversal.

Proof. If A_g and A_h are transversal then $\deg(\gamma_{g,h}) = 0$ and $\gamma_{g,h} \in k$. The same holds for $\gamma_{h^{-1},g^{-1}}$. By associativity:

$$1_g 1_h 1_{h^{-1}} 1_{g^{-1}} = \gamma_{h,h^{-1}}\gamma_{g,g^{-1}} = \gamma_{g,h}\gamma_{h^{-1},g^{-1}}\gamma_{(gh),(gh)^{-1}}$$

and since $\gamma_{(gh),(gh)^{-1}} \neq 0$, we see that if $\gamma_{g,h} \neq 0$ and $\gamma_{h^{-1},g^{-1}} \neq 0$ then $\gamma_{h,h^{-1}}\gamma_{g,g^{-1}} \neq 0$ so $\pi_h(\gamma_{g,g^{-1}}) \neq 0$ and $\pi_g(\gamma_{h,h^{-1}}) \neq 0$.

1.16. **Lemma.** If $[g,h] = e$

$$\varphi_{g,h} = \varphi_{kgk^{-1},khk^{-1}} \tag{1.21}$$

Proof. $\varphi_{kgk^{-1},khk^{-1}} = \varphi_{k,h}\varphi_{g,h}\varphi_{k^{-1},khk^{-1}} = \varphi_{k,h}\varphi_{g,h}\varphi_{k,h}^{-1} = \varphi_{g,h}$

2. Functorial setup

The functorial setup of orbifold Frobenius algebras and reconstruction is discussed in the following.

Let $\mathcal{FROB}$ be the category of Frobenius algebras, whose objects are Frobenius algebras and morphisms are morphisms which respect all the structures.

2.1. **Definitions.** A *covariant (contravariant) G–category* is an additive category $\mathcal{C}$ where for each object $X \in Ob(\mathcal{C})$ and each $g \in G$ there exists an object X^g and an injective (surjective) morphism $i_g \in \mathrm{Hom}(\mathrm{X^g}, \mathrm{X})$ ($r_g \in \mathrm{Hom}(\mathrm{X}, \mathrm{X^g})$) with $X^e = X$ and $i_e = id$. And there are isomorphisms $\psi_{g,g^{-1}} \in \mathrm{Hom}(\mathrm{X^g}, \mathrm{X^{g^{-1}}})$.

We call a category a *covariant (contravariant) G intersection category* if it is a G category and for each pair $(g,h) \in G \times G$ and object $X \in Ob(\mathcal{C})$ there are isomorphisms $\psi \in \mathrm{Hom}((\mathrm{X^g})^{\mathrm{h}}, (\mathrm{X^h})^{\mathrm{g}})$ and surjective (injective) morphisms $r^{gh}_{g,h} \in \mathrm{Hom}(\mathrm{X^{gh}}, (\mathrm{X^g})^{\mathrm{h}})$ ($i^{gh}_{g,h} \in \mathrm{Hom}((\mathrm{X^g})^{\mathrm{h}}, \mathrm{X^{gh}})$)

A *G–action* for a G–category is given by a collection of morphisms $\phi_g(X,h) \in \mathrm{Hom}(\mathrm{X^h}, \mathrm{X^{ghg^{-1}}})$ which are compatible with the structural homomorphisms and satisfy $\phi_g(X, g'hg'^{-1})\phi_{g'}(X,h) = \phi_{gg'}(X,h)$.

An example of a G–category with G–action are categories of spaces equipped with a G–action whose fixed point sets are in the same category.

Examples for G intersection categories are the category of singularities and as above categories of spaces equipped with a G action whose fixed point sets are in the same category. Here the $(X^g)^h$ are the structures on the intersection of the fixed point sets and the maps i are induced by the inclusion of the fixed point sets into the total space and the inclusion of the intersection of the fixed point sets of g and h into the fixed point set of gh etc..

2.2. **The reconstruction program.** Suppose we have a covariant (contravariant) G–category $\mathcal{C}$ and a contravariant (covariant) functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{FROB}$, then for each $X \in Ob(\mathcal{C})$ we naturally obtain the following collection of Frobenius algebras: $(\mathcal{F}(X^g) : g \in G)$ together with restriction maps $r_g : \mathcal{F}(X) \mapsto \mathcal{F}(X^g)$.

Examples of these functors are K-theory, Milnor ring and cohomology.

2.2.1. **The classical picture.** One possibility is to regard the direct sum of the Frobenius algebras $A_g := \mathcal{F}(X^g)$. The first obstacle would lie in the grading as it is well known that the direct sum of two graded Frobenius algebras is only well defined if their Euler dimensions (cf. e.g. [K3]) agree. This can, however, be fixed by using the shifts s^+ discussed in the previous paragraph.

The most important remark is the following. Each of the Frobenius algebras A_g has its own multiplication, so there is a diagonal multiplication in the direct sum of them.

Using the shift one obtains the *classical theory* if one is considering the diagonal algebra structure as was done in [AS], [T]. One can of course forget the algebra structure altogether and retain only the additive structure. This was done e.g. in [S] for V–manifolds (i.e. orbifolds). Concentrating only on the dimensions one arrives for instance at the notion of stringy "numbers" [BB].

2.2.2. **The "stringy" products.** The diagonal multiplication is however *not* the right object to study from the perspective of "stringy geometry". The multiplication should rather be G–graded, i.e. map $A_g \otimes A_h \to A_{gh}$.

Here the natural question is the following: given the additive structure of a G Frobenius algebra: what are the possible "stringy" products? This is the setting of our reconstruction program [K2, K3].

2.2.3. **The G–action.** One part of the structure of a G–Frobenius algebra is the G–action. If the G–category is already endowed with a G–action we can use it

to reconstruct the G–action on the G–Frobenius algebra, which in turn limits the choices of "stringy" products to those that are compatible.

2.2.4. **Examples.** Examples of such products are orbifold (quantum) cohomology [CR] and orbifold K-theory [AR] (on the invariants) or orbifold cohomology [LS, FG] with the full G–Frobenius algebra structure as already postulated in [K2]. For new developments on quantum deformations of the G–Frobenius algebras see [JKK].

2.2.5. **Special G–Frobenius algebras.** The special reconstruction data reflects this situation in the special case that the A_g algebras are cyclic A_e modules. This is a restriction which leads to an answer in terms of cocycles for a large class of examples. This class includes all Jacobian Frobenius algebras as well as e.g. symmetric products and special cases of geometric actions on manifolds.

The general idea can easily generalize to non–cyclic case although computations get more involved.

2.3. **Definition.** Given a G–category $\mathcal{C}$, we call the tuple $(X^g) : g \in G$ a G–collection.

The category of G–collections of a G–category is the category whose objects are G–collections and whose morphisms are collections of morphisms (f^g) s.t. the diagrams

$$\begin{array}{ccc} X^g & \stackrel{i_g}{\to} & X \\ \downarrow f^g & & \downarrow f \\ Y^g & \stackrel{i_g}{\to} & Y \end{array}$$

commute.

2.4. **Definition.** A G–Frobenius functor is a functor from the category of G–collections of a G–category to G–Frobenius algebras.

The main question of the reconstruction program is whether one can extend a functor from a G–category $\mathcal{C}$ to Frobenius algebras to a G–Frobenius functor, and if so how many ways are there to do this.

One can view this as the analogue of solving the associativity equations for general Frobenius algebras. Some of the solutions correspond to quantum cohomology, some to singularities, etc. and maybe others to other "string"–schemes. The structures of possible "stringy" products provide a common approach. The systematic consideration of all possible products confines the choices of string equivalents of classical concepts and allows to identify divers approaches.

The answer to the general question in the special case is given in the Reconstruction Theorem 1.7.

The consequences are sometimes quite striking as it is in the case of symmetric products, where there is only *one* possible "stringy" orbifold product.

The restrictions on the possible multiplicative structures are even stricter if one is considering data stemming from a G–intersection category.

This is the content of the next section.

3. Intersection G–Frobenius algebras

We will now concentrate on the situation of functors from G–intersection categories to Frobenius algebras.

Given a G–class in such a category a functor to Frobenius algebras will provide the following structure.

3.1. **Definition.** A G–intersection Frobenius data of level k is the following: For each collection $(g_1, \dots, g_n)$ with $n \leq k$ of elements of G a Frobenius algebra $A_{g_1,\dots,g_n}$ and the following maps:

Isomorphisms

$$\Psi_\sigma : A_{g_1,\dots,g_n} \to A_{g_{\sigma(1)},\dots,g_{\sigma(n)}}$$

for each $\sigma \in \mathbf{S}_n$ called *permutations.*

Isomorphisms

$$\Psi^{g_1,\dots,g_i,\dots,g_n}_{g_1,\dots,g_i^{-1},\dots,g_n} : A_{g_1,\dots,g_i,\dots,g_n} \to A_{g_1,\dots,g_i^{-1},\dots,g_n}$$

commuting with the permutations.

Surjective morphisms

$$r^{g_1,\dots,\hat{g_i},\dots,g_n}_{g_1,\dots,g_i,\dots g_n} : A_{g_1,\dots,\hat{g_i},\dots,g_n} \to A_{g_1,\dots,g_n}$$

commuting with the permutations. (Here as usual the symbol $\hat{}$ is used to denote omission.)

Isomorphisms

$$i^{g_1,\dots g,\dots,\hat{g},\dots,g_n}_{g_1,\dots,g,\dots,g,g_n} : A_{g_1,\dots,g,\dots,g,g_n} \to A_{g_1,\dots g,\dots,\hat{g},\dots,g_n}$$

commuting with the permutations.

And finally surjective morphisms:

$$r^{g_1,\dots,g_ig_{i+1},\dots,g_n}_{g_1,\dots,g_i,g_{i+1},\dots,g_n} : A_{g_1,\dots,g_ig_{i+1},\dots,g_n} \to A_{g_1,\dots,g_i,g_{i+1},\dots,g_n}$$

commuting with the permutations.

If this data exists for all k we call the data simply G–intersection Frobenius data.

Furthermore an intersection G–Frobenius algebra of level $k \geq 2$ is an intersection G–Frobenius data of level $k \geq 2$ together with a G–Frobenius algebra structure on $A := \bigoplus A_g$.

3.2. **Remarks.**

1) In order to reconstruct a suitable multiplication on $\bigoplus A_g$ we will only need the double and triple intersections (i.e. level 3). Where the latter appear to show associativity.
2) We can use the higher intersection to define G–Frobenius algebras based on each of the A_g i.e. on $\bigoplus_{h \in Z(g)} A_{g,h}$ for each fixed g–where $Z(g)$ denotes the centralizer of g. Of course we can iterate this process.

3.2.1. **Definition.** A *G–action* for an intersection G–Frobenius algebra of level k is given by a collection of morphisms

$$\phi_g(A_{g_1,\dots,g_n}, h) \in \mathrm{Hom}(\mathrm{A_{g_1,\dots,g_n,h}}, \mathrm{A_{g_1,\dots,g_n,ghg^{-1}}})$$

which are compatible with the structural homomorphisms and satisfy

$$\phi_g(A_{g_1,\dots,g_n}, g'hg'^{-1})\phi_{\gamma'}(A_{g_1,\dots,g_n}, h) = \phi_{gg'}(A_{g_1,\dots,g_n}, h)$$

3.3. **Notation.** We set $r_{g_1,\dots,g_n} := r^{g_1,\dots,g_n-1}_{g_1,\dots,g_n} \circ \cdots \circ r_{g_1}$ and we set $I_{g_1,\dots,g_n} := \mathrm{Ker}(r_{g_1,\dots,g_n})$. Notice that this definition of $I_{g_1,\dots,g_n}$ is independent of the order of g_i.

3.4. **Definition.** We call an intersection algebra *special intersection algebra data*, if all of the $A_{g_1,\dots,g_n}$ are cyclic $A := A_e$ algebras. Here the generators are given by $r_{g_1,\dots,g_n}(1)$ and the A_e module structure is given by $a \cdot b := r_{g_1,\dots,g_n}(a)b$.

3.5. **Proposition.** Given special intersection algebra data the following decomposition holds for section independent cocycles γ:

$$r_{gh}(\gamma_{g,h}) = \check{r}^{gh}_{g,h}(\tilde{g}_{g,h}) = i^{gh}_{g,h}(\tilde{\gamma}_{g,h})\check{r}^{gh}_{g,h}(1_{g,h}) =: \gamma^t_{g,h}\gamma^{\perp}_{g,h} \tag{3.1}$$

for some $\tilde{\gamma}_{g,h} \in i^{gh}_{g,h}(A_{g,h})^e$ where $e = s_g + s_h - s_{gh} - s^+_{g,h} + s^+_{gh}$ with $s^+_{g,h} := d - d_{g,h}$ and $d_{g,h} = \deg(\rho_{g,h})$ and we again used the unshifted degrees. (In particular if the $s^- = 0$ then $e = \frac{1}{2}(s^+_g + s^+_h + s^+_{gh}) - s^+_{g,h} = \frac{1}{2}(d - d_g - d_h - d_{gh}) + d_{g,h}$)

Proof. We notice that $I_g + I_h = I_{g,h}$ and $(I_g + I_h)\gamma_{g,h} \subset I_{gh}$, and set $J := r_{gh}I_{g,h}$. Choosing some section $i^{gh}_{g,h}$ of $r^{gh}_{g,h}$, we can define the splitting

$$A^k_{gh} = i^{gh}_{g,h}(A_{g,h}) \oplus J \tag{3.2}$$

where again ^{k} means the homogeneous component of degree k. Now

$$\gamma_{g,h} \in (i^{gh}_{g,h}(A_{gh})^*)^e$$

where * is the dual w.r.t. the form η_{gh} and the splitting (3.2) and $e = s_g + s_h - s_{gh} + s^+_{gh} - s^+_{g,h}$.

From which the claim follows by an argument completely analogous to the proof of Lemmas 1.11 and 1.12.

Also generalizing the fact that

$$I_g\gamma_g = I_g\check{r}_g(1_g) = 0 \tag{3.3}$$

we obtain

3.6. **Lemma.**

$$(I_g + I_h)\gamma^{\perp}_{g,h} \subset I_{g,h} \tag{3.4}$$

3.7. **Multiplication.** From the section independence of γ, we see that the multiplication $A_g \otimes A_h \to A_{gh}$ can be factored through $A_{g,h}$. To be more precise, we have the following commutative diagram.

$$\begin{array}{ccc} A_g \otimes A_h & \xrightarrow{\mu} & A_{gh} \\ \downarrow r^g_{g,h} \otimes r^h_{g,h} & & \uparrow \check{r}^{g,h}_{gh} \circ l_{\tilde{\gamma}_{g,h}} \\ A_{g,h} \otimes A_{g,h} & \xrightarrow{\mu} & A_{g,h} \end{array}$$

where $l_{\tilde{\gamma}_{g,h}}$ is the left multiplication with $\tilde{\gamma}_{g,h}$.

3.8. **Associativity equations.** Furthermore the associativity equations can be factored through $A_{g,h,k}$. More precisely, we have the following commutative diagram of restriction maps:

$$\begin{array}{ccccccccc} & & & & A_{ghk} & & & & \\ & & & \swarrow & & \searrow & & & \\ A_{gh} & \to & A_{gh,k} & & \downarrow & & A_{g,hk} & \leftarrow & A_{hk} \\ \downarrow & & & \searrow & & \swarrow & & & \downarrow \\ A_{g,h} & & \to & & A_{g,h,k} & & \leftarrow & & A_{h,k} \end{array} \tag{3.5}$$

More technically: Using the associativity equations for the γ, we set

$$r_{ghk}(\gamma_{g,h}\gamma_{gh,k}) := \gamma_{g,h,k} \tag{3.6}$$

and associativity says that also

$$r_{ghk}(\gamma_{h,k}\gamma_{g,hk}) = \gamma_{g,h,k} \tag{3.7}$$

By analogous arguments as utilised above one finds

$$\gamma_{g,h,k} = i^{ghk}_{g,h,k}(\tilde{\gamma}_{g,h,k})\check{r}^{ghk}_{g,h,k}(1_{g,h,k}) = \check{r}^{ghk}_{g,h,k}(\tilde{\gamma}_{g,h,k}) \tag{3.8}$$

for some $\tilde{\gamma}_{g,h,k} \in i^{ghk}_{g,h,k}(A_{g,h,k})$. So vice–versa to show associativity one needs to show that

$$\check{r}^{ghk}_{gh,k}(r^{gh}_{gh,k}(\check{r}^{gh}_{g,h}(\tilde{\gamma}_{g,h}))\tilde{\gamma}_{gh,k}) = \check{r}^{ghk}_{g,h,k}(\tilde{\gamma}_{g,h,k}) \tag{3.9}$$

for some $\tilde{\gamma}_{g,h,k}$ which is a symmetric expression in the indices.

3.9. **The Sign.**

3.9.1. **Remark.** Given a special G–Frobenius algebra A we denote the Eigenvalue of ρ w.r.t. φ_g by λ_g and furthemore denote the Eigenvalue of φ^h_g on $i_h(\rho_h)$ by λ^h_g i.e. $\varphi_g(\rho) = \lambda_g\rho$ and $\varphi^h_g(i_h(\rho_h)) = \lambda^h_g i_h(\rho_h)$. By the projective G–invariance of the metic

$$\lambda_h = \chi_h^{-2} \tag{3.10}$$

and we can regard the ensembles λ_g and λ^g_h as characters.

3.9.2. **Definition.** We define a sign $sign$ to be an element of $\mathrm{Hom}(\mathrm{G}, \mathrm{k}^*)$. Fixing an element $sign \in \mathrm{Hom}(\mathrm{G}, \mathbf{Z}/2\mathbf{Z})$ we can define the associated character ψ by

$$\psi(g) := (-1)^{sign(g)}\chi_g \tag{3.11}$$

Vice–versa given a character $\psi \in \mathrm{Hom}(\mathrm{G}, \mathrm{k}^*)$ with the property that $\psi^2 = \chi^2$ we define the *sign given by* ψ to be

$$(-1)^{sign(g)} := \chi_g\psi(g)^{-1} \tag{3.12}$$

Finally, any choice of root of λ defines a sign.

Given $sign$ and $sign^g$ for A and A^g for all $g, h \in G, [g, h] = e$ we set

$$\nu(g, h) \equiv sign(g) + sign^g(h) + \tilde{h}^g + \tilde{g}\ (2) \tag{3.13}$$

$sign$ and $sign^g$ are said to be compatible if for all $h \in g$

$$\nu(g, h) = \nu(gh, h) = \nu(h, g) = \nu(g^{-1}, h) \tag{3.14}$$

3.10. **Algebraic Discrete Torsion.** Denote the centralizer of an element $g \in G$ by $Z(g)$ and fix a sign of A. We will consider G–intersection Frobenius data of level 2.

3.10.1. **The induced $Z(g)$–Frobenius algebra structure.** If we are in an intersection Frobenius algebra of level $k \geq 2$, given A_g we can consider
THE UNDERLYING ADDITIVE STRUCTURE.

$$\hat{A}_g = \bigoplus_{h \in Z(g)} (A_g)_h = \bigoplus_{h \in Z(g)} A_{g,h} \tag{3.15}$$

Notice that if $h \in Z(g)$, $\varphi_h : A_g \to A_g$ and φ descends to a $Z(g)$ action on A_g. However, we have that $\varphi_h(1_g) = \varphi_{h,g} 1_g$, but 1_g should be invariant under the $Z(g)$–action as the new identity. Therefore we set
THE $Z(g)$–ACTION.

$$\varphi_h^g := \varphi_{h,g}^{-1} \varphi_h \tag{3.16}$$

With this definition $\varphi_h^g(1_g) = \varphi_{h,g}^{-1} \varphi_{h,g} 1_g = 1_g$.

THE CHARACTER. Given a G–action on the level 2 G–intersection algebra, we can augment the picture with a character χ_h^g, which will be determined by the trace axiom.

SUPERGRADING. We fix the super–degree of $A_{g,h}$ in $\hat{A}_g$ and denote it by $\tilde{h}^g$.

3.10.2. **Definition.** An intersection Frobenius algebra of level $k \geq 2$ is said to satisfy *the discrete torsion condition*, if the above data satisfy the projective trace axiom and for all $g, h \in G$ there are isomorphisms between $A_{ghh,h} \simeq A_{g,h}$.

3.10.3. **Proposition.** In an intersection Frobenius algebra A of level $k \geq 2$ that satisfies the discrete torsion condition, the following equality holds for all $g, h \in G, [g,h] = e$:

$$\chi_g \mathrm{STr}(\varphi_g|_{A_h}) = \varphi_{g,h} \chi_g (\chi_g^h)^{-1} (-1)^{\tilde{g}} (-1)^{\tilde{h}^g} \dim(A_{g,h}) \tag{3.17}$$

or given roots ψ, ψ^g of λ, λ^g

$$\chi_g \mathrm{STr}(\varphi_g|_{A_h}) = \varphi_{g,h} \psi_g (\phi_g^h)^{-1} (-1)^{sign(g) + sign^h(g)} (-1)^{\tilde{g}} (-1)^{\tilde{h}^g} \dim(A_{g,h}) \tag{3.18}$$

Proof. From the discrete torsion condition we obtain

$$(-1)^{\tilde{g}^h} \dim(A_{g,h}) = \chi_h^g \mathrm{STr}(\varphi_g^h|_{A_{h,e}})$$

and furthermore

$$\mathrm{STr}(\varphi_g|_{A_h}) = (-1)^{\tilde{g}} \varphi_{g,h}^{-1} \mathrm{STr}(\varphi_g^h|_{A_{h,e}})$$

3.10.4. **Corollary.** If ψ and ψ^g are compatible then

$$\chi_g \mathrm{STr}(\varphi_g|_{A_h}) = \varphi_{g,h} \psi_g (\phi_g^h)^{-1} (-1)^{sign(g) + sign(h)} (-1)^{\nu(g,h)} \dim(A_{g,h}) \tag{3.19}$$

3.10.5. **Definition.** If $sign$ and the $sign^g$ are compatible, we set for $g, h \in G, [g,h] = e$

$$T(h,g) = (-1)^{sign(g) sign(h)} (-1)^{sign(g) + sign(h)} (-1)^{|\nu_{g,h}|} \dim(A_{g,h}) \tag{3.20}$$

it satisfies for $g, h \in G, [g,h] = e$

$$T(g,h) = T(h,g) = T(gh,h) = T(g^{-1},h) \tag{3.21}$$

$$\epsilon(h,g) = \varphi_{g,h} (-1)^{sign(g) sign(h)} \psi_g (\psi_g^h)^{-1} \tag{3.22}$$

Due to the projective trace axiom and by definition ϵ viewed as a function from $G \times G \to k^*$ satisfies the conditions of discrete torsion which are defined by:

$$\epsilon(g,h) = \epsilon(h^{-1},g) \quad \epsilon(g,g) = 1 \quad \epsilon(g_1g_2,h) = \epsilon(g_1,h)\epsilon(g_2,h) \tag{3.23}$$

4. Jacobian Frobenius Algebras

We first recall the main definitions and statements about Jacobian Frobenius algebras from [K2, K3].

4.1. **Reminder.** A Frobenius algebra A is called *Jacobian* if it can be represented as the Milnor ring of a function f. I.e. if there is a function $f \in \mathcal{O}_{\mathbf{A}^n_k}$ s.t. $A = \mathcal{O}_{\mathbf{A}^n_k}/J_f$ where J_f is the Jacobian ideal of f. And the bilinear form is given by the residue pairing. This is the form given by the Hessian of $\rho = \mathrm{Hess}_f$.

If we write $\mathcal{O}_{\mathbf{A}^n_k} = k[x_1 \dots x_n]$, J_f is the ideal spanned by the $\frac{\partial f}{\partial x_i}$.

A *realization of a Jacobian Frobenius algebra* is a pair (A,f) of a Jacobian Frobenius algebra and a function f on some affine k space $\mathbf{A}^n_k$, i.e. $f \in \mathcal{O}_{\mathbf{A}^n_k} = k[x_1 \dots x_n]$ s.t. $A = k[x_1 \dots x_n]$ and $\rho := \det(\frac{\partial^2 f}{\partial x_i \partial x_j})$.

4.2. **Definition.** A *natural G action on a realization of a Jacobian Frobenius algebra* (A_e, f) is a linear G action on $\mathbf{A}^n_k$ which leaves f invariant. Given a natural G action on a realization of a Jacobian Frobenius algebra (A,f) set for each $g \in G$, $\mathcal{O}_g := \mathcal{O}_{\mathrm{Fix}_g(\mathbf{A}^n_k)}$.

We also write $V(g) := \mathrm{Fix}_g(\mathbf{A}^n_k)$.

This is the ring of functions of the fixed point set of g for the G action on $\mathbf{A}^n_k$. These are the functions fixed by g: $\mathcal{O}_g = k[x_1, \dots, x_n]^g$.

Denote by $J_g := J_{f|_{\mathrm{Fix}_g(\mathbf{A}^n_k)}}$ the Jacobian ideal of f restricted to the fixed point set of g.

Define

$$A_g := \mathcal{O}_g/J_g \tag{4.1}$$

The A_g will be called twisted sectors for $g \neq 1$. Notice that each A_g is a Jacobian Frobenius algebra with the natural realization given by $(A_g, f|_{\mathrm{Fix}_g})$. In particular, it comes equipped with an invariant bilinear form $\tilde{\eta}_g$ defined by the element $\mathrm{Hess}(f|_{\mathrm{Fix}_g})$.

For $g = 1$ the definition of A_e is just the realization of the original Frobenius algebra, which we also call the untwisted sector.

Notice there is a restriction morphism $r_g : A_e \to A_g$ given by $a \mapsto a \bmod J_g$.

Denote $r_g(1)$ by 1_g. This is a non–zero element of A_g since the action was linear. Furthermore it generates A_g as a cyclic A_e module.

The set $\mathrm{Fix}_g \mathbf{A}^n_k$ is a linear subspace. Let I_g be the vanishing ideal of this space.

We obtain a sequence

$$0 \to I_g \to A_e \stackrel{r_g}{\to} A_g \to 0$$

Let i_a be any splitting of this sequence induced by the inclusion: $\hat{i}_g : \mathcal{O}_g \to \mathcal{O}_e$ which descends due to the invariance of f.

In coordinates, we have the following description. Let $\mathrm{Fix}_g \mathbf{A}^n_k$ be given by equations $x_i = 0 : i \in N_g$ for some index set N_g.

Choosing complementary generators $x_j : j \in T_g$, we have $\mathcal{O}_g = k[x_j : j \in T_g]$ and $\mathcal{O}_e = k[x_j, x_i : j \in T_g, i \in N_g]$. Then $I_g = (x_i : i \in N_g)_{\mathcal{O}_e}$ is the ideal in $\mathcal{O}_e$ generated by the x_i and $\mathcal{O}_e = I_g \oplus i_g(A_g)$ using the splitting i_g coming from the

natural inclusion $\hat{i}_g : k[x_j : j \in T_g] \to k[x_j, x_i : j \in T_g, i \in N_g]$. We also define the projections

$$\pi_g : A_e \to A_e; \pi_g = i_g \circ r_g$$

which in coordinates are given by $f \mapsto f|_{x_j=0:j\in N_g}$ Let

$$A := \bigoplus_{g\in G} A_g$$

where the sum is a sum of A_e modules.

Some of the conditions of the reconstruction program are automatic for Jacobian Frobenius algebras. The conditions and freedoms of choice of compatible data to the above special reconstruction data are given by the following:

4.3. **Theorem (Reconstruction for Jacobian algebras).** Given a natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) with a quasi–homogeneous function f with $d_g = 0$ iff $g = e$ together with a natural choice of splittings i_g the possible structures of naturally graded special G twisted Frobenius algebra on the A_e module $A := \bigoplus_{g\in G} A_g$ are in 1–1 correspondence with the set of G graded cocycles γ together with a choice of parity $\tilde{} \in Hom(G, \mathbf{Z}/2\mathbf{Z})$ and a compatible non–abelian two cocycle φ with values in k^*, which satisfy the condition of discrete torsion

$$\forall g, h \text{ s.t. } [g,h] = e : \varphi_{g,h}\varphi_{h,g} \det(g|_{N_h}) \det(h|_{N_g}) = 1 \tag{4.2}$$

This means in particular that the trace condition is replaced by (4.2). Also notice that if $\gamma_{g,h} \neq 0$ then the factor $\varphi_{g,h}\varphi_{h,g} = 1$ in (4.2) by the compatibility equations so that (4.2) reads

$$\det(g|_{N_h}) \det(h|_{N_g}) = 1 \tag{4.3}$$

NOTATION. If $[g,h] \neq 0$ then $\deg(g|_{N_h})$ is taken as and abbreviation for $\deg(g) \det^{-1}(g|_{T_h})$.

4.3.1. **Character and Sign.** The character and sign are fixed by a choice of parity and are given by:

$$\chi_g = (-1)^{\tilde{g}}(-1)^{|N_g|}\det(g) \tag{4.4}$$

The sign is defined by

$$\chi_g = (-1)^{sign(g)} \det(g)$$

i.e. we choose $\psi_g = \det(g)$

and satisfies

$$sign(g) := \tilde{g} + |N_g| \mod 2 \tag{4.5}$$

The sign is necessarily also a group homomorphism. This means that

$$|N_g| + |N_h| \equiv |N_{gh}| \ (2) \tag{4.6}$$

4.3.2. **Bilinear form on the twisted sectors.** If the character χ in non–trivial, we have to shift the natural bilinear forms η_g on A_g by

$$((-1)^{\tilde{g}}\chi_g)^{1/2}\eta_g \tag{4.7}$$

where we choose to cut the plane along the negative real axis. For more comments on this procedure see [K3] and the following remarks.

4.3.3. **Remarks about the normalization.** We would like to point out that the setup of reconstruction data already includes the forms η_g. This is the reason for the above shift. Indeed there is always a pencil of metrics for any given irreducible Frobenius algebra. The overall normalization is fixed by $\gamma_{g,g^{-1}}$. More precisely, we always have the equation:

$$\gamma_{g,g^{-1}} i_g(\rho_g) = \rho \tag{4.8}$$

Notice that since $\gamma_{g,g^{-1}} I_g = 0$ this equation determines ρ_g uniquely at least in the graded irreducible case since ρ_g is of necessarily of top degree in A_g. So if, we were not to include the η_g into the data, the only conditions on the $\gamma_{g,g^{-1}}$ would be that they do not vanish, live in the right degree and satisfy the compatibility but there would be no need for rescaling.

Another way to avoid the shift is to include it in the restriction data by setting

$$A_g := \mathcal{O}_{f_g} \text{ with } f_g = ((-1)^{\tilde{g}}\chi_g)^{1/2}\eta_g f|_{\mathrm{Fix}(g)} \tag{4.9}$$

4.3.4. **Natural discrete Torsion for Jacobian Frobenius algebras.** We can write

$$\chi_h \mathrm{STr}(\varphi_h|_{A_g}) = \epsilon(h,g)T(h,g)$$

where

$$\begin{aligned} T(h,g) &= (-1)^{sign(g)sign(h)}(-1)^{sign(g)+sign(h)}(-1)^{|T_g\cap T_h|+N} \\ &\quad \dim(i_g(A_g)\cap i_h(A_h)) \\ &= (-1)^{sign(g)sign(h)}(-1)^{sign(g)+sign(h)}(-1)^{|N_{g,h}|}\dim(A_{g,h}) \end{aligned} \tag{4.10}$$

where we introduced the notation $|N_{g,h}|$ for $\dim \mathrm{Fix}(g)\cap\mathrm{Fix}(h)$ and $A_{g,h}$ for $\mathcal{O}_{f|_{\mathrm{Fix}(g)\cap\mathrm{Fix}(h)}}$

$$\epsilon(h,g) = \varphi_{g,h}(-1)^{sign(g)sign(h)}\det(g|_{N_h}) \tag{4.11}$$

The projective trace axiom is satisfied in the graded case if ϵ satisfies the equations of discrete torsion

$$\epsilon(g,h) = \epsilon(h^{-1},g) \quad \epsilon(g,g) = 1 \quad \epsilon(g_1g_2,h) = \epsilon(g_1,h)\epsilon(g_2,h) \tag{4.12}$$

which in terms of the φ is equivalent to the condition (4.2).

4.3.5. **Remark.** This definition of discrete torsion agrees with the more general one of 3.10 if we set $\psi = \det(g)$ and $\psi^g(h) = \det(h)|_{T_g}$. Indeed we find $sign^g(h) \equiv \tilde{h}^g + |N^g_{g,h}|$ with $|N^g_{g,h}| = \mathrm{codim}_{\mathrm{Fix_g}}(\mathrm{Fix_g}\cap\mathrm{Fix_h})$ and thus

$$\begin{aligned} \nu(g,h) &\equiv sign(g) + sign^g(h) + \tilde{h}^g + \tilde{g}\ (2) \\ &\equiv sign(g) + \mathrm{codim}(\mathrm{Fix_g}\cap\mathrm{Fix_h}) + |\mathrm{N_g}| + \tilde{\mathrm{g}} \equiv |\mathrm{N_{g,h}}|\ (2) \end{aligned} \tag{4.13}$$

4.3.6. **Examples.**

1) (pt/G). Recall (cf. [K3]) that given a linear representation $\rho G \to O(n,k)$, we obtain the G–twisted Frobenius algebra pt/G from the Morse function $f = z_n^1 + \ldots + z_n^2$.

All sectors are isomorphic to k:

$$A = \bigoplus_{g \in G} k$$

all the $d_g = 0$ and all the $r_g = id$. In particular, we have that $\gamma_{g,g^{-1}} = \check{r}_g(1) = 1$ and $\pi_g(\gamma_{h,h^{-1}}) = 1 \neq 0$, so we see that the $\gamma_{g,h} \in k^*$ and are given (up to rescaling) by group cocycles $\gamma \in H^2(G, k^*)$ and since the $g_{g,h} \neq 0$ the φ and hence the discrete torsion are fixed by the compatibility $\gamma_{g,h} = \varphi_{g,h}\gamma_{ghg^{-1},g}$. Vice–versa the choice of discrete torsion or cocycles φ limits the possible cocycles severely.

Explicitly: Fix a parity $\tilde{} \in \mathrm{Hom}(\mathrm{G}, \mathbf{Z}/2\mathbf{Z})$.

The sign and character are given by

$$sign(g) \equiv \tilde{g} \quad \chi_g = (-1)^{sign(g)} = (-1)^{\tilde{g}} \tag{4.14}$$

The discrete torsion is given by

$$\epsilon(g,h) = \varphi_{g,h}(-1)^{s(g)s(h)} = \varphi_{g,h}(-1)^{\tilde{g}\tilde{h}} \tag{4.15}$$

Thus the discrete torsion condition (4.2) is satisfied.

But all $d_g = 0$ and thus we have to check the trace condition for non–commuting elements. It reads

$$(-1)^{\tilde{h}}(-1)^{\tilde{g}}\varphi_{h,g}\gamma_{[g,h],hgh^{-1}} = (-1)^{\tilde{g}}(-1)^{\tilde{h}}\varphi_{g^{-1},ghg^{-1}}\gamma_{[g,h],h}$$

we rewrite the l.h.s. as

$$\begin{aligned} (-1^{\tilde{h}})\varphi_{h,g}\gamma_{[g,h],hgh^{-1}}1_{gh} &= \varphi_{h,g}1_{[g,h]}1_{hgh^{-1}}1_h\gamma_{g,h}^{-1} \\ = \varphi_{h,g}\gamma_{[g,h],hg}\gamma_{hgh^{-1},h}\gamma_{g,h}^{-1}1_{gh} &= \gamma_{[g,h],hg}\gamma_{h,g}\gamma_{g,h}^{-1}1_{gh} \end{aligned}$$

and the r.h.s. can be rewritten as

$$\begin{aligned} \varphi_{g^{-1},ghg^{-1}}\gamma_{[g,h],h}1_{gh} &= \varphi_{g^{-1},ghg^{-1}}1_{[g,h]}1_h1_g\gamma_{ghg^{-1},g}^{-1} \\ = \varphi_{g^{-1},ghg^{-1}}\gamma_{[g,h],hg}\gamma_{h,g}\gamma_{ghg^{-1},g}^{-1}1_{gh} &= \gamma_{[g,h],hg}\gamma_{h,g}\gamma_{g,h}^{-1}1_{gh} \end{aligned}$$

which coincides with the calculation above.

This is of course all clear if $[g,h] = e$, but there is no restriction that the group be commutative.

2) Another example to keep in mind is A_n which is the Frobenius algebra associated to z^{n+1} together with the $\mathbf{Z}/(n+1)\mathbf{Z}$ action $z \mapsto \zeta_n z$ where $\zeta_n^n = 1$ [cf. K3].

3) $A^{\otimes n}$ together with the permutation action. We will consider this example in depth in §6 and §7. This example has appeared many times in different guises in [DHVV,D,LS,U,WZ]. Our treatment is the completely general and subsumes all these cases. Also, there is an ambiguity of signs which is explained by our treatment.

4.4. **Theorem.** Jacobian algebras naturally give intersection algebras.

Proof. This is straight–forward. We set

$$A_{g_1,\dots,g_k} := \mathcal{O}_{f_{g_1,\dots,g_k}} \text{ with } f_{g_1,\dots,g_k} := f|_{\bigcap_{i=1}^k \mathrm{Fix}(g_i)} \tag{4.16}$$

and use the obvious restriction maps. Here again the remarks of 4.3.3 apply.

5. Special $\mathbf{S}_n$–twisted Frobenius algebras

5.1. **Notation.** Given a permutation $\sigma \in \mathbf{S}_n$, we associate to it its cycle decomposition $c(\sigma)$ and its index type $I(\sigma) := \{I_1, \dots I_k\}$ where the I_j are the independent sets in the cycle decomposition of σ. Notice that the $I(\sigma)$ can also be written as $\langle\sigma\rangle\backslash\bar{n}$ where this is the quotient set of $\bar{n}$ w.r.t. group action of the group generated by σ.

The length of a cycle decomposition $|c(\sigma)|$ is defined to be the number of independent cycles in the decomposition. The partition gives rise to its norm $(n_1, \dots, n_k)$ of n where $n_i := |I_i|$. And the type of a cycle is defined to be $(N_1(\sigma), N_2(\sigma), \dots)$ where $N_i = \#$ of $n_j = i$ in $(n_1, \dots, n_k)$, i.e. N_i the number of cycles of length i in the cyclce decomposition of σ.

We define the degree of $\sigma \in \mathbf{S}_n$ to be $|\sigma| :=$ the minimal length of σ as a word in transpositions $= n - |c(\sigma)|$.

Recall the relations in $\mathbf{S}_n$ are

$$\tau^2 = 1 \tag{5.1}$$

$$\tau\tau' = \tau'\tau'' \text{ where } \tau = (ij), \tau' = (jk), \tau'' = (kl). \tag{5.2}$$

5.2. **Definition.** We call two elements $\sigma, \sigma' \in \mathbf{S}_n$ *transversal*, if $|\sigma\sigma'| = |\sigma| + |\sigma'|$.

5.3. **The linear subspace arrangement.** A good deal of the theory of $\mathbf{S}_n$ Frobenius algebras is governed by the canonical permutation representation of $\mathbf{S}_n$ on k^n given by $\rho(\sigma)(e_i) = e_{\sigma(i)}$ for the canonical basis (e_i) of k^n.

We set $V_\sigma := \mathrm{Fix}(\sigma)$

and $V_{\sigma_1,\dots,\sigma_n} := \bigcap_{i=1}^n V_{\sigma_i}$. Notice that

$$l(\sigma) = \dim(V_\sigma) = |\langle\sigma\rangle\backslash\bar{n}| \tag{5.3}$$

and

$$|\sigma| = \mathrm{codim}(\mathrm{V}_\sigma) \tag{5.4}$$

In the same spirit, we define

$$\begin{aligned} l(\sigma_1, \dots \sigma_n) &:= \dim(V_{\sigma_1,\dots,\sigma_n}) \\ |\sigma_1, \dots, \sigma_n| &:= \mathrm{codim}(\mathrm{V}_{\sigma_1,\dots,\sigma_n}) \end{aligned} \tag{5.5}$$

This explains the name transversal. Since if σ and σ' are transversal then

$$V_{\sigma,\sigma'} = V_\sigma \cap V_{\sigma'} = V_{\sigma\sigma'}$$

and the intersection is transversal.

Furthermore notice that

$$|\sigma_1, \dots, \sigma_n| = |\langle\sigma_1, \dots, \sigma_n\rangle\backslash\bar{n}|. \tag{5.6}$$

where again the last set is the quotient set of $\bar{n}$ by the action under the group generated by $\sigma_1, \dots, \sigma_n$.

5.4. **Definition.** We call a cocycle $\gamma : \mathbf{S}_n \times \mathbf{S}_n \to A$ *normalizable* if for all *transversal* pairs $\tau, \sigma \in \mathbf{S}_n, |\tau| = 1 : \deg(\gamma_{\sigma,\tau}) = 0$ and $\gamma_{\sigma,\tau} \neq 0$ and *normalized* if it is normalizable and
for all *transversal* $\tau, \sigma \in \mathbf{S}_n, |\tau| = 1 : \gamma_{\sigma,\tau} = 1$.

5.4.1. **Lemma.** If a cocycle is normalized then for any *transversal* $\sigma, \sigma' \in \mathbf{S}_n$: $\gamma_{\sigma,\sigma'} = (-1)^{\tilde{\sigma}\tilde{\sigma}'}$.

Proof. We write $\sigma' = \tau'_1 \cdots \tau'_k$ with $k = |\sigma'|$ where all τ_i are transpositions.
Thus by associativity:

$$\sigma\sigma' = (((\ldots(\sigma\tau'_1)\tau'_2)\cdots)\tau'_k),$$

so

$$\gamma_{\sigma\sigma'} = \pi_{\sigma\sigma'}(\gamma_{\sigma,\sigma'}) = \pi_{\sigma\sigma'}(\prod_{i=1}^{k} \gamma_{\sigma \prod_{j=1}^{i-1}(\tau_j),\tau_i}) = \pi_{\sigma\sigma'}(\prod_{i=1}^{k} 1) = 1$$

5.4.2. **Remark.** Recall by Lemma 1.11 $\gamma_{\tau,\tau} = \check{r}_g(1_\tau)$ for a transposition τ.

5.4.3. **Lemma.** Let $\sigma \in \mathbf{S}_n$. If γ is a normalized cocycle, then for any decomposition into transpositions $\sigma = \tau_1 \cdots \tau_{|\sigma|} : \gamma_{\sigma,\sigma^{-1}} \prod_{i=1}^{|\sigma|} \gamma_{\tau_i,\tau_i}$

Proof. Let $k = |\sigma|$. Thus by associativity:

$$\sigma\sigma^{-1} = (\tau_1(\tau_2(\cdots(\tau_k\tau_k \cdots \tau_2\tau_1)\cdots))),$$

and if τ and σ' are transversal

$$\pi_{\sigma'}(\gamma_{\tau,\tau\sigma'}) = \pi_{\sigma'}(\gamma_{\tau,\tau\sigma'}\gamma_{\tau,\sigma'}) = \pi_{\sigma'}(\gamma_{\tau,\tau}\gamma_{e,\sigma'}) = \pi_{\sigma'}(\gamma_{\tau,\tau}).$$

So $\gamma_{\sigma,\sigma^{-1}} = \prod_{i=1}^{k} \gamma_{\tau_i,\tau_i}$.

5.5. **Theorem.** Given special S_n reconstruction data, a choice of normalized cocycle $\gamma : \mathbf{S}_n \times \mathbf{S}_n \to A$ is unique. Furthermore a choice of normalizable cocycle is fixed by a choice of the $\gamma_{\tau,\sigma}$ with τ and σ transversal.

Proof. By Lemma 1.11, we have that the $\gamma_{\sigma,\sigma^{-1}}$ are given by $\gamma_{\sigma,\sigma^{-1}} = \check{r}_\sigma(1_\sigma)$ and thus fixed after the normalization which fixes the r_σ. Again choosing any minimal decomposition $\sigma' = \tau'_1 \cdots \tau'_{|\sigma|}$ and by using the normalization associativity repeatedly, we obtain that

$$\gamma_{\sigma,\sigma'} = \pi_{\sigma\sigma'}(\gamma_{\sigma,\sigma'} \prod_{i=1}^{|\sigma'|} \gamma_{\tau'_{i+1},\prod_{j=1}^{i}\tau'_j}) = \pi_{\sigma,\sigma'}(\prod_{i=1}^{|\sigma'|} \gamma_{\sigma\prod_{j=1}^{i}\tau'_{i-1},\tau'_i})$$
$$= \pi_{\sigma\sigma'}(\prod_{i\in I} \gamma_{\tau'_i,\tau'_i})$$

where $I := \{i : |\sigma(\prod_{j=1}^{i-1}\tau'_j)\tau'_i| = |\sigma\prod_{j=1}^{i-1}\tau'_j| - 2\}$.

Thereby the $\gamma_{\sigma,\sigma'}$ are already determined by the $\gamma_{\tau,\tau}$ which are in turn given by $\check{r}_\tau(1_\tau)$.

If the cocycles are only normalizable, we obtain the result in a similar fashion.

5.6. **Supergrading and Parity p.** Since $\mathbf{S}_n$ is generated by transpositions which all lie in the same conjugacy class, we see that the choices of $\mathbf{Z}/2\mathbf{Z}$–grading $\tilde{} \in \mathrm{Hom}(\mathbf{S}_n, \mathbf{Z}/2\mathbf{Z})$ are given by

i) pure even $\forall\sigma : \tilde{\sigma} = 1$. We call this the even case and set the parity $p = 0$.
ii) The sign representation $\tilde{\sigma} \equiv |\sigma|$ (2). We call this the odd case and set the parity $p = 1$.

5.7. **The non–abelian cocycles φ.**

5.7.1. **Remark.** Due to the relation (1.8) we see that φ is determined by the $\varphi_{\tau,\sigma}$ with $|\tau| = 1$.

5.7.2. **Lemma.** $\varphi_{\tau,\tau} = (-1)^{\tilde{\tau}\tilde{\tau}}$

Proof. $\gamma_{\tau,\tau} = \varphi_{\tau,\tau}(-1)^{\tilde{\tau}\tilde{\tau}}\gamma_{\tau,\tau}$ and $\gamma_{\tau,\tau} \neq 0$.

5.7.3. **Remark.** The relation (5.2) is automatically satisfied: i.e. using it twice

$$\begin{aligned}
\gamma_{\tau,\tau'} &= \varphi_{\tau,\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\gamma_{\tau\tau'\tau,\tau} = \varphi_{\tau,\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\gamma_{\tau'',\tau} \\
&= \varphi_{\tau,\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\varphi_{\tau'',\tau}(-1)^{\tilde{\tau}''\tilde{\tau}}\gamma_{\tau',\tau''} \\
&= \varphi_{\tau,\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\varphi_{\tau'',\tau}(-1)^{\tilde{\tau}''\tilde{\tau}}\varphi_{\tau',\tau''}(-1)^{\tilde{\tau}'\tilde{\tau}'}\gamma_{\tau,\tau'} \\
&= \varphi_{\tau''\tau'\tau,\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\gamma_{\tau,\tau'} = \varphi_{\tau',\tau'}(-1)^{\tilde{\tau}\tilde{\tau}'}\gamma_{\tau,\tau'} = \gamma_{\tau,\tau'}
\end{aligned}$$

5.8. **Definition.** We call a non–abelian cocycle φ normalized if $\varphi_{\sigma,\tau} = (-1)^{\tilde{\sigma}} = (-1)^{p|\sigma|}$.

The Theorem A.1 contained in the appendix A implies that all non–abelian cocycles φ can be rescaled to a normalized cocycle.

5.9. **Theorem.** Any normalizable $\mathbf{S}_n$ cocycle with normalized φ can be normalized by a rescaling $1_\sigma \mapsto \lambda_\sigma 1_\sigma$.

And vice–versa given any normalized $\mathbf{S}_n$ cocycle and a choice of parity $p \in \{0,1\}$ there is only one compatible non–abelian cocycle φ given by

$$\varphi_{\sigma,\sigma'} = (-1)^{p|\sigma||\sigma'|} \tag{5.7}$$

Proof. First notice that by assumption of normalizability the $\gamma_{\sigma,\tau} \in k^*$ for transversal τ, σ we define the rescaling inductively on $|\sigma|$ by $\lambda_\tau := 1$ and $\lambda_\sigma := \lambda_{\sigma'}\gamma_{\sigma',\tau'}$ where $\sigma = \sigma'\tau'$ and τ and σ are transversal.

More precisely: let $\sigma = \sigma'\tau'$ with $|\tau| = 1, |\sigma'| = |\sigma| - 1$ with induction on $|\sigma|$ we define

$$\lambda_\sigma := \lambda_{\sigma'}\gamma_{\sigma',\tau'} \tag{5.8}$$

Then after scaling we obtain:

$$\tilde{\gamma}_{\sigma',\tau'} = \frac{\lambda_{\tau'}\lambda_{\sigma'}}{\lambda_{\sigma'\tau'}}\gamma_{\sigma',\tau'} = \lambda_{\tau'} = 1$$

We have to show that (5.8) is well defined i.e. is independent of the decomposition. This can again be seen by induction.

First notice that if $|\sigma| = 1$, $\lambda_\sigma = 1$ poses no problems. If $|\sigma| = 2$ either there is a unique decomposition into two disjoint transpositions or

$$\sigma = \tau\tau' = \tau'\tau'' \tag{5.9}$$

where $\tau = (ij), \tau' = (jk), \tau'' = (kl)$. The first case again poses no problem. For the second one notice that $\lambda_\tau = \lambda_{\tau'} = 1$ and $\tau'\tau''\tau' = \tau$ thus

$$\gamma_{\tau',\tau''} = \varphi_{\tau',\tau''}(-1)^{\tilde{\tau}'\tilde{\tau}''}\gamma_{\tau'\tau''\tau',\tau'} = \varphi_{\tau',\tau''}(-1)^{\tilde{\tau}'\tilde{\tau}''}\gamma_{\tau,\tau'} = \gamma_{\tau,\tau'} \tag{5.10}$$

Assume the λ_σ are well defined for $|\sigma| < k$. Fix σ with $|\sigma| = k$ and decompose $\sigma = \sigma'\tau' = \sigma''\tau''$ in two different ways. Then we have to show that

$$\lambda_{\sigma'}\gamma_{\sigma',\tau'} = \lambda_{\sigma'}\gamma_{\sigma'',\tau''}$$

where by induction $\lambda_{\sigma'} = \prod_{i=1}^{|\sigma'|} \gamma_{\prod_{j=1}^{i-1}\tau'_j,\tau'_i}$ and $\sigma' = \prod_{i=1}^{|\sigma'|}\tau'_i$ is any minimal representation. We observe that in $\mathbf{S}_n$ we can obtain $\sigma'\tau'$ from $\sigma''\tau''$ by using the relation (5.9) repeatedly. Thus by using associativity and (5.10) we obtain:

$$\lambda_{\sigma''}\gamma_{\sigma'',\tau''} = (\prod_{i=1}^{|\sigma''|}\gamma_{\prod_{j=1}^{i-1}\tau''_j,\tau''_i})\gamma_{\sigma'',\tau''} = (\prod_{i=1}^{|\sigma'|}\gamma_{\prod_{j=1}^{i-1}\tau'_j,\tau'_i})\gamma_{\sigma',\tau'} = \lambda_{\sigma'}\gamma_{\sigma',\tau'}$$

The fastidious reader can find the explicit case study in Appendix B.

For the second statement notice that by Lemma 5.4.1 given a normalized γ we have for *all* transversal $\sigma, \sigma' : \gamma_{\sigma,\sigma'} = 1$.

Thus for transversal τ, σ

$$1 = \gamma_{\tau,\sigma} = \varphi_{\tau,\sigma}(-1)^{\tilde{\tau}\tilde{\sigma}}\gamma_{\tau\sigma\tau,\tau} = \varphi_{\tau,\sigma}(-1)^{\tilde{\tau}\tilde{\sigma}}$$

since $\tau\sigma\tau$ and τ are transversal $|\tau\sigma\tau| = |\sigma|, |\tau\sigma\tau\tau| = |\tau\sigma| = |\tau| + |\sigma|$.

And if σ, τ are not transversal, then $\sigma = \tau\sigma'$ with $|\sigma'| = |\sigma - 1|$ and σ' and τ transversal.

$$\gamma_{\tau,\tau} = \gamma_{\tau,\sigma} = \varphi_{\tau,\sigma}(-1)^{|\sigma|}\gamma_{\sigma'\tau,\tau} = (-1)^{|\sigma|}\gamma_{\tau,\tau}$$

and since $\gamma_{\tau,\tau} \neq 0$, we find

$$\varphi_{\tau,\sigma} = \varphi_{\sigma,\tau} = (-1)^p. \tag{5.11}$$

And finally if $\sigma = \prod_{i=1}^{|\sigma|}\tau_i$

$$\varphi_{\sigma,\sigma'} = \prod_{i=1}^{|\sigma|}\varphi_{\tau_i,\tilde{\sigma}'_i} = (-1)^{p|\sigma||\sigma'|}$$

by using (5.11) with $\tilde{\sigma}_i = (\prod_{j=i+1}^{|\sigma|}\tau_i)\sigma'(\prod_{j=i+1}^{|\sigma|}\tau_i)^{-1}$, $|\tilde{\sigma}_i| = |\sigma'|$

6. Symmetric powers of Jacobian Frobenius algebras

In this paragraph, we study $\mathbf{S}_n$ orbifolds of $A^{\otimes n}$ where A is a Jacobian Frobenius algebra. We also fix the degree d of A to be the degree of ρ — the element defining η.

The most important result for Jacobian Frobenius algebras (or manifolds) is that $A_f \otimes A_g = A_{f+g}$ [K1]. Therefore

$$A^{\otimes n}_{f(z)} = A_{f(z_1)+\cdots+f(z_n)}$$

where z is actually a multi-variable $z = (z^1, \ldots, z^m)$.

6.1. **Remark.** In the above notation, we should keep it mind that for functions $g_1, \ldots g_n$, we have that

$$g_1 \otimes \cdots \otimes g_n = g_1(z_1)\cdots g_n(z_n)$$

6.2. **$\mathbf{S}_n$–action.** In this situation there is a natural action ρ of $\mathbf{S}_n$ by permuting the z_i i.e. for $\sigma \in \mathbf{S}_n$

$$\rho(\sigma)(z^k_i) = z^k_{\sigma(i)}$$

It is clear that the function $f_n := f(z_1) + \cdots + f(z_n)$ is invariant under this action, so that we can apply the theory of [K2, K3]. We see that the representation ρ is just the dim A–fold sum of the standard representation of $\mathbf{S}_n$ on k^n.

6.3. **The twisted sectors.** To analyse the twisted sectors, we have to diagonalize the given representation. To this end, we regard the cycle decomposition and realize that for each cycle with index set I_l there is a m–dimensional Eigenspace generated by

$$\frac{1}{n_i}\sum_{i\in I_l} z_i^l \text{ for } l=1,\dots,m$$

The other Eigenvectors being given by

$$\frac{1}{n_l}\sum_{i\in I_l} \zeta_{n_l}^j f(i) z_i^l$$

with Eigenvalue $\zeta_{n_l}^j$ where $f: I_l \to \{1,\dots,n_l\}$ is a bijective map respecting the cycle order.

Restricting f_n to the space where all the variables with Eigenvalue different from one vanish see that

$$f_\sigma = f(z_i = z_j = u_k) \text{ if } i,j \in I_k$$

Using the variables u_k it is obvious that

$$A_\sigma = A_{f_\sigma} \simeq A^{\otimes|\sigma|}$$

6.4. **Restriction maps.** With the above choice of u_k as variables and using Remark 6.1, we find that the restriction maps are given as follows:

$$r_\sigma(g_1\otimes\cdots\otimes g_n) := \bigotimes_{i=1}^{k}(\prod_{j\in I_i} g_i) \in A^{\otimes|\sigma|}$$

Thus these maps are just contractions by multiplication.

6.5. **Fixed point sets.** By the above, we see that

$$\mathrm{Fix}(\sigma) = \bigoplus_{i=1}^{m} V_\sigma \subset (k^n)^m \tag{6.1}$$

where we used the notation of 5.3. Notice that

$$\dim(V_\sigma) = ml(\sigma) \quad \mathrm{codim}(\mathrm{V}_\sigma) = |\mathrm{N}_\sigma| = \mathrm{m}|\sigma| \tag{6.2}$$

6.6. **Bilinear form on $A^{\otimes n}$.** We notice that if the bilinear form on A is given by the element $\rho = \mathrm{Hess}(f)$ then the bilinear form on $A^{\otimes n}$ is given by $\rho^{\otimes n} = \mathrm{Hess}(f_n)$ and it is invariant under the $\mathbf{S}_n$ action. Indeed $\det^2(\rho(\sigma)) = 1$. To be more precise, we have that

$$\det(\rho(\sigma)) = (-1)^{m|\sigma|}$$

(Here ρ is of course the representation, not the element defining the bilinear form.)

6.7. **The Character and Sign.** Notice that the character is either the alternating or the trivial one depending on the choice of the sign, which is determined by the choice of parity p and on the choice of the number of variables m. (We have to keep in mind that we can always stabilize the function f by adding squares of new variables).

Using the equation (4.5), we find however:

$$\chi_\sigma = (-1)^{\tilde{\sigma}}(-1)^{m|\sigma|}\det(\sigma) = (-1)^{\tilde{\sigma}} \tag{6.3}$$

and find the sign of σ to be

$$sign(\sigma) \equiv \tilde{\sigma} + m|\sigma| = (m+p)|\sigma| \tag{6.4}$$

Thus only the sign, but not the character depends on the number of variables!

6.8. **Bilinear form on the twisted sectors.** Since it is always the case that $(-1)^{\tilde{\sigma}}\chi_\sigma = 1$, we do not have to shift the natural bilinear forms on the twisted sectors. They are given by $\eta^{\otimes l(\sigma)}$ or equivalently by $\rho_\sigma = \rho^{\otimes l(\sigma)}$.

6.9. **Remark.** Notice also that since $\det(\rho(\sigma)) = \pm 1$ (i.e. the Schur–Frobenius indicator is 1) the form η will descend to the $\mathbf{S}_n$ invariants (see e.g. [K3]).

6.10. **Proposition.** Any compatible cocycle γ is normalizable.

Proof. We check that $\pi_\sigma(\gamma_{\tau,\tau}) \neq 0$ for τ and σ transversal. Then the claim follows from Proposition 1.13.

Suppose τ and σ are transversal and say $\tau = (ij)$, then i and j belong to different subsets of the partition $I(\sigma)$ (say $I(\sigma)_i$ and $I(\sigma)_j$). So since $\gamma_{\tau,\tau} \neq 0$ neither is $\pi_\sigma(\gamma_{\tau,\tau})$.

More explicitly:

$$\gamma_{\tau,\tau} = \check{r}_\tau(1_\tau) = \sum_k 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\underset{\downarrow}{a_k}} \otimes 1 \otimes \cdots \otimes 1 \otimes \overset{j}{\underset{\downarrow}{b_k}} \otimes 1 \otimes \cdots \otimes 1 \tag{6.5}$$

where $\sum_k a_k \otimes b_k = \Delta(1) \neq 0 \in A \otimes A$ and $\Delta := \check{\mu} : A \to A \otimes A$ is the natural co–multiplication on A. And

$$r_\sigma(\gamma_{\tau,\tau}) = \sum_k 1 \otimes \cdots \otimes 1 \otimes \overset{I(\sigma)_i}{\underset{\downarrow}{a_k}} \otimes 1 \otimes \cdots \otimes 1 \otimes \overset{I(\sigma)_j}{\underset{\downarrow}{b_k}} \otimes 1 \otimes \cdots \otimes 1 \tag{6.6}$$

Thus $\gamma_{\tau,\tau}$ is not in the kernel of the contraction r_σ and thus not in the kernel of π_σ.

6.11. **Discrete Torsion.** The choices of discrete torsion are given by the choices of cocycles φ and the sign. Since there is only one φ for a given choice of parity and fixing the parity the sign is determined by the number of variables m. We would expect that there might be four different ϵ, but there are overall only two choices of discrete torsion, as is well known (see e.g. [Ka]). This is due to the fact that the dependence on the parity cancels out.

Recall (4.11)

$$\epsilon(\sigma,\sigma') = \varphi_{\sigma,\sigma'}(-1)^{sign(\sigma)sign(\sigma')}\det(\sigma|_{N_{\sigma'}}) = (-1)^{m|\sigma||\sigma'|}\det(\sigma|_{N_{\sigma'}})$$

and

$$\begin{aligned} T(\sigma,\sigma') &= (-1)^{sign(\sigma)sign(\sigma')}(-1)^{sign(\sigma)+sign(\sigma')}(-1)^{m|\sigma,\sigma'|}\dim(A_{\sigma,s'}) \\ &= (-1)^{p(|\sigma|+|\sigma'|+|\sigma||\sigma'|)}(-1)^{m(|\sigma|+|\sigma'|+|\sigma||\sigma'|+|\sigma,\sigma'|)}\dim(A_{\sigma,s'}) \end{aligned}$$

6.12. **Reminder.** Recall that the centralizer of an element $\sigma \in \mathbf{S}_n$ is given by

$$Z(\sigma) \cong \prod_k \mathbf{S}_{N_k} \ltimes \mathbf{Z}/k\mathbf{Z}^{N_k}$$

where N_i the number of cycles of length i in the cycle decomposition of σ (cf. 5.1) This result can also be restated as: "discrete torsion can be undone by a choice of sign".

We note that $Z(\sigma)$ is generated by elements of the type τ_k and c_k where τ_k permutes two cycles of length k of σ and c_k is a cycle of length k of σ.

Also ϵ is a group homomorphism in both variables, so that by 3.23 ϵ is fixed by its value on elements of the above type.

6.12.1. **Proposition.** The discrete torsion is given by

$$\epsilon(\sigma',\sigma) = \begin{cases} (-1)^{mk|\sigma|}(-1)^{m(k-1)} & \text{if } \sigma' = \tau_k \\ (-1)^{m(k-1)(|\sigma|-1)} & \text{if } \sigma' = c_k \end{cases}$$

where τ_k and c_k are the generators of $Z(\sigma)$ described above.

Proof.

$$\det(\tau_k)|_{N_\sigma} = \det(\tau_k)\det^{-1}(\tau_k|_{T_\sigma}) = (-1)^{mk}(-1)^m$$

and

$$\det(c_k)|_{N_\sigma} = \det(c_k)\det^{-1}(c_k|_{T_\sigma}) = (-1)^{m(k-1)}$$

6.12.2. **Remark.** What this calculation shows is that we are dealing with the m–th power of the non–trivial cocycle which in the case $m = 1$ has been calculated in [D]. We again see the phenomenon that the addition of variables (stabilization) changes the sign and hence the discrete torsion — as is well known in singularity theory. Actually the whole trace i.e. the product of ϵ and T is constantly equal to $(-1)^{p(|\sigma||\sigma'|+|\sigma|+|\sigma'|)}\dim(A_{\sigma,\sigma'})$ which coincides with the general statement c.f. (7.13).

6.12.3. **Corollary.** The discrete torsion condition holds.

6.13. **Grading and shifts.**

6.13.1. **Proposition.**

$$s_\sigma^+ = d|\sigma|, \quad s_\sigma^- = 0 \tag{6.7}$$

$$s_\sigma = \frac{1}{2}(s_\sigma^+ + s_\sigma^-) = \frac{d}{2}|\sigma| \tag{6.8}$$

where s^+ and s^- are the standard shifts for Jacobian Frobenius algebras as defined in [K2,K3].

For the calculation of s^+, we fix some $\sigma \in \mathbf{S}_n$. Let $c(\sigma)$ be its cycle decomposition and $I(\sigma) := \{I_1, \dots I_k\}$ be its index decomposition. Then the shift s_σ^+ can be read off from the definition and the identification

$$A_\sigma \simeq \bigotimes_{i=1}^{|c(\sigma)|} A_{I_i} \simeq A^{n-|\sigma|}$$

with the degree of $A^{\otimes l}$ being dl, we obtain

$$s_\sigma^+ = nd - (n - |\sigma|)d = d|\sigma|$$

The shift s_σ^- is again calculated via the natural representation $\rho : \mathbf{S}_n \to GL(n,k)$. Recall (cf. [K3])

$$s_g^- := \frac{1}{2\pi i}\mathrm{Tr}(\log(g)) - \mathrm{Tr}(\log(g^{-1})) := \frac{1}{2\pi i}(\sum_i \lambda_i(g) - \sum_i \lambda_i(g^{-1}))$$
$$= \sum_{i:\lambda_i \neq 0} 2(\frac{1}{2\pi i}\lambda_i(g) - 1)$$

For a cycle c of length k, we have the eigenvalues $\zeta_k^i, i = 0, \ldots k-1$ where ζ_k is the k-th root of unity $\exp(2\pi i \frac{1}{k})$. So we get the shift

$$s_c^- = 2[\sum_{j=1}^{k-1}(\frac{j}{k} - \frac{1}{2})] = \frac{k(k-1)}{k} - (k-1) = 0$$

For an arbitrary σ, we regard its cycle decomposition and obtain the result.

6.14. **Theorem.** Given a Jacobian Frobenius algebra A there are exactly two natural $\mathbf{S}_n$ Frobenius algebra structures on $A^{\otimes n}$.

Proof. The uniqueness follows from §5. The existence result is deferred to §7 which can be carried over verbatim.

7. Second quantized Frobenius algebras

Given a Frobenius algebra A with multiplication $\mu : A \otimes A \to A$, we can regard its tensor powers $T^n A := A^{\otimes n}$. These are again Frobenius algebras with the natural tensor multiplication $\mu^{\otimes n} \in A^{\otimes 3n} = (A^{\otimes n})^{\otimes 3}$, tensor metric $\eta^{\otimes n}$ and unit $1^{\otimes n}$.

We can also form the symmetric powers $S^n A$ of A. The metric, multiplication and unit all descend to make $S^n A$ into a Frobenius algebra, but in terms of general theory [K] we should not regard this object alone, but rather look at the corresponding orbifold quotient $T^n A/\mathbf{S}_n$.

7.1. **Assumption.** We will assume from now on that A is irreducible and the degree of A is d.

7.2. **Notation.** We keep the notation of the previous paragraphs: $l(\sigma)$ is the number of cycles in the cycle decomposition of σ and $|\sigma| = n - l(\sigma)$ is the minimal number of transpositions.

7.2.1. **Lemma.** Let ρ be the permutation representation of $\mathbf{S}_n$ on $A^{\otimes n}$ permuting the tensor factors. Then the following equations hold

$$\mathrm{Tr}(\rho(\sigma)) = \dim(A) l(\sigma) \tag{7.1}$$

$$\det(\rho(\sigma)) = (-1)^{|\sigma|\binom{\dim(A)}{2}} \begin{cases} 1 & \dim(A) \equiv 0 \text{ or } 1(4) \\ (-1)^{|\sigma|} & \dim(A) \equiv 2 \text{ or } 3(4) \end{cases} \tag{7.2}$$

Proof For the first statement we use the fact that entries in the standard tensor basis of the matrix of $\rho(\sigma)$ are just 0 or 1. A diagonal entry is 1 if all of the basis elements whose index is in the same subset of $\bar{n}$ defined by the partition $c(\sigma)$ are equal. The number of such elements is precisely $\dim(A) l(\sigma)$.

For the second statement we notice that

$$\det(\rho(\sigma)) = \det(\rho(\tau))^{|\sigma|}$$

where τ is any transposition. For $\tau = (12)$ we decompose $A \otimes A = \bigoplus_{i=1}^{\dim A} e_i \otimes e_i \oplus (\bigoplus_{i,j \in \bar{n}, i \neq j} e_i \otimes e_j)$ for some basis e_i of A. Using this decomposition we find that indeed $\det(\rho(\sigma)) = (-1)^{|\sigma|\binom{\dim(A)}{2}}$. For the last statement notice that

$$\frac{1}{2}\dim(A)(\dim(A)-1) \equiv \begin{cases} 0(2) & \text{if } \dim(A) \equiv 0 \text{ or } 1(4) \\ 1(2) & \text{if } \dim(A) \equiv 2 \text{ or } 3(4) \end{cases}$$

7.3. **Super-grading.** As is well known there are only two characters for $\mathbf{S}_n$: the trivial and the determinant. We will accordingly define the *parity* with values in $\mathbf{Z}/2\mathbf{Z}$

$$\tilde{\sigma} \equiv \begin{cases} 0\ (2) & \text{if we choose the trivial character} \\ |\sigma|\ (2) & \text{if we choose the non-trivial character} \end{cases} \tag{7.3}$$

To unify the notation, we set the parity index $p = 0$ in the first case, which we call even, and $p = 1$ in the second case, which we call odd.

In both cases

$$\tilde{\sigma} = (-1)^{p|\sigma|} \tag{7.4}$$

7.4. **Intersection algebra structures.** For $\sigma_1, \dots, \sigma_m \in \mathbf{S}_n$ we define the following Frobenius algebras:

$$A_\sigma := (A^{l(\sigma)}, \eta^{\otimes l(\sigma)}, 1^{\otimes l(\sigma)}) \tag{7.5}$$

$$A_{\sigma_1,\dots,\sigma_m} := (A^{\otimes |\langle \sigma_1,\dots,\sigma_m\rangle \backslash \bar{n}|}, \eta^{\otimes |\langle \sigma_1,\dots,\sigma_m\rangle \backslash \bar{n}|}, 1^{\otimes |\langle \sigma_1,\dots,\sigma_m\rangle \backslash \bar{n}|}) \tag{7.6}$$

Notice that the multiplication μ gives rise to a series of maps by contractions. More precisely given a collection of subsets of $\bar{n}$ we can contract the tensor components of $A^{\otimes n}$ belonging to the subsets by multiplication. Given a permutation we can look at its cycle decomposition which yields a decomposition of $\bar{n}$ into subsets. We define $\mu(\sigma)$ to be the above contraction. Notice that due to the associativity of the multiplication the order in which the contractions are performed is irrelevant.

These contractions have several sections. The simplest one being the one mapping the product to the first contracted component of each of the disjoint contractions. We denote this map by j or in the case of contractions given by $I(\sigma)$ for some $\sigma \in \mathbf{S}_n$ by $j(\sigma)$.

E.g. $\mu((12)(34))(a \otimes b \otimes c \otimes d) = ab \otimes cd$ and $j((13)(24))(ab \otimes cd) = ab \otimes cd \otimes 1 \otimes 1$.

Thus we define the following maps

$$r_\sigma : A_e \to A_\sigma \quad ; \quad r_\sigma := \mu(\sigma) \tag{7.7}$$

$$i_\sigma : A_\sigma \to A_e \quad ; \quad i_\sigma := j(\sigma) \tag{7.8}$$

Moreover the same logic applies to the spaces $A_{\sigma_1,\dots,\sigma_m}$ and we similarly define $r_{\sigma_1,\dots,\sigma_m}, i_{\sigma_1,\dots,\sigma_m}$ where the indices are symmetric and maps

$$r^{\sigma_1,\dots,\sigma_{m-1}}_{\sigma_1,\dots,\sigma_m} : A_{\sigma_1,\dots,\sigma_{m-1}} \to A_{\sigma_1,\dots,\sigma_m}, i^{\sigma_1,\dots,\sigma_{m-1}}_{\sigma_1,\dots,\sigma_m} : A_{\sigma_1,\dots,\sigma_m} \to A_{\sigma_1,\dots,\sigma_{m-1}} \tag{7.9}$$

where the again the indices are symmetric.

We also notice that $A_\sigma = A_{\sigma^{-1}}$ and $A_{\sigma,\sigma} = A_\sigma$.

7.5. **Proposition.** The maps r_σ, i_σ make A_σ, η_σ into a special $\mathbf{S}_n$ reconstruction data. A choice of parity $\tilde{\sigma}$ fixes the character to be:

$$\chi_\sigma = (-1)^{p|\sigma|} \tag{7.10}$$

Furthermore the collection of maps $r^{\sigma_1,\ldots,\sigma_{m-1}}_{\sigma_1,\ldots,\sigma_m}$ turns the collection of $A_{\sigma_1,\ldots,\sigma_m}$ into special intersection $\mathbf{S}_n$ reconstruction data.

Proof. It is clear that all the A_σ are cyclic A_e modules. The sections i_σ also satisfy the condition (1.2) of being section of A_e-modules and is is clear that $A_\sigma = A_{\sigma^{-1}}$.

Also the η_σ remain unscaled since $(-1)^{p|\sigma|}\chi_\sigma \equiv 1$.

What remains to be shown is that the character is indeed given by $\chi_\sigma = (-1)^{p|\sigma|}$ and that the trace axiom holds.

This is a nice exercise. We are in the graded case and moreover the identity is up to scalars the only element with degree zero — unless ($\dim A = 1$) and we are in the case of $pt/\mathbf{S}_n$ which was considered in 4.3.6. So if $c \in A_{[\sigma,\sigma']} : c \neq \lambda 1_e$ then the trace axiom is satisfied automatically.

Therefore we only need to consider the case $c = 1 \in A_{[\sigma,\sigma']}$ with $[\sigma,\sigma'] = e$. In this case, we see that σ' acts on $A_\sigma \simeq A^{\otimes l(\sigma)}$ as a permutation. Indeed the normalizer of σ is the semi–direct product of permutations of the cycles and cyclic groups whose induced action on A_σ is given by permutation and identity respectively.

We claim the trace has the value

$$\mathrm{Tr}\varphi_\sigma|_{A_{\sigma'}} = \dim(A_{\sigma,\sigma'}) \tag{7.11}$$

This is seen as follows. Looking at the permutation action on the factors of A_σ, we see that the trace has entries 0 and 1 in any fixed basis of A_σ induced by a fixed choice of basis of A. The value 1 appears if the pure tensor element has exactly the same entry in all tensor components labelled by elements which are in the same cycle of σ (acting on A_σ). But these are precisely the elements that span $A_{\sigma,\sigma'}$ to be more precise there is a canonical isomorphism of these element with $A_{\sigma,\sigma'}$ given by tensors of iterated diagonal maps $\Delta : A \to A \otimes A, \Delta(a) = a \otimes a$.

Thus the trace axiom can be rewritten as:

$$\chi_\sigma \varphi_{\sigma,\sigma'}(-1)^{p|\sigma|} = \chi^{-1}_{\sigma'}\varphi_{\sigma',\sigma}(-1)^{p|\sigma'|} \tag{7.12}$$

In particular if $\sigma' = e$

$$(-1)^{p|\sigma|}\dim(A_\sigma) = \chi_\sigma \mathrm{Tr}(\rho(\sigma)|_{A^{\otimes n}})$$

so that

$$\chi(\sigma) = (-1)^{p|\sigma|}$$

Combining the above we find that:

$$\chi_\sigma \mathrm{STr}(\phi_\sigma|_{A_{\sigma'}}) = (-1)^{p(|\sigma||\sigma'|+|\sigma|+|\sigma'|)}\dim(A_{\sigma,\sigma'}) \tag{7.13}$$

which is an expression completely symmetric in σ, σ' and invariant under a change $\sigma \mapsto \sigma^{-1}$.

For the last statement we only need to notice that consecutive contractions yield commutative diagrams which are co–Cartesian. The structural isomorphisms being clear since they can all be given by the identity morphism — there is no rescaling.

7.6. **Proposition (Discrete Torsion).** Fix the $sign \equiv 1$ and $sign^\sigma \equiv 1$ and set $(-1)^{\tilde{\sigma}^{\sigma'}} = \det_{V^{\sigma'}}(\sigma) = (-1)^{\mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'})}$ where $\det_{V^{\sigma'}}(\sigma)$ is the determinant of the induced action of σ on the fixed point set of σ'. Furthermore fix $\chi_\sigma^{\sigma'}$ by $(-1)^{p(\mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'}))}$. Then $sign$ and the $sign^\sigma$ are compatible and

$$\epsilon(\sigma,\sigma') = (-1)^{p(|\sigma||\sigma'|)}(-1)^{p|\sigma|}(-1)^{p(\mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'}))} \tag{7.14}$$

or in the notation of 6.12

$$\epsilon(\sigma',\sigma) = \begin{cases} (-1)^{p(k|\sigma|+k+1)} & \text{if } \sigma' = \tau_k \\ (-1)^{p((k-1)|\sigma|+(k-1))} & \text{if } \sigma' = c_k \end{cases}$$

Proof. First:

$$\nu(\sigma,\sigma') \equiv \mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'}) + \mathrm{codim}(\mathrm{V}^\sigma)(2)$$

which satisfies 3.14, since

$$\mathrm{codim}_{\mathrm{V}^\sigma}(\mathrm{V}^{\sigma,\sigma'}) + \mathrm{codim}(\mathrm{V}^\sigma) = \mathrm{codim}(\mathrm{V}^{\sigma,\sigma'}) = \mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'}) + \mathrm{codim}(\mathrm{V}^{\sigma'})$$

Now just by definition

$$\epsilon(\sigma,\sigma') = (-1)^{p(|\sigma||\sigma'|)}(-1)^{p|\sigma|}(-1)^{p(\mathrm{codim}_{\mathrm{V}^{\sigma'}}(\mathrm{V}^{\sigma,\sigma'}))}$$

and lastly: $\mathrm{codim}_{\mathrm{V}^\sigma}(\mathrm{V}^{\tau_k,\sigma}) = 1$ and $\mathrm{codim}_{\mathrm{V}^\sigma}(\mathrm{V}^{c_k,\sigma}) = 0$.

7.6.1. **Remark.** This algebraic discrete torsion indeed reproduces the effect that turning it on yields the super–structure on the twisted sectors as postulated in [D]. The computation of the discrete torsion in [D] was however done for $pt/\mathbf{S}_n$ with the choice of cocycle γ given by a Schur multiplier, see 1.10.

7.7. **Proposition.** Any cocycle γ compatible with the special reconstruction data is normalizable and hence unique after the normalization.

Proof. Verbatim the proof of 6.10.

So from now on we can and will deal with normalized cocycles.

7.7.1. **Lemma.** For any minimal decomposition T of σ' into transpositions $\sigma' = \tau_1 \dots \tau_{|\sigma'|}$

$$\check{r}_\sigma(1_\sigma) = \prod_i \gamma_{\tau_i,\tau_i} \tag{7.15}$$

Proof. Notice that $I_\sigma = \bigoplus I_{\tau_i}$ and thus $I_\sigma \prod_{i\in I}\gamma_{\tau_i} = 0$. Furthermore $\deg(\prod_{i\in I}\gamma_{\tau_i}) = d|\sigma| = s^+(\sigma) = 2d_\sigma = \deg(\gamma_{\sigma,\sigma^{-1}})$ and $\dim(I_\delta)^{dl(\sigma)} = \dim(A^{\otimes n}) - 1$ where the superscript denotes the part of homogeneous degree. This follows from the equalities: $\dim((I_\sigma)^{dl(\sigma)}) = \dim(\mathrm{Ker}(r_\sigma^{dl(\sigma)})) = \dim(A^{\otimes n}) - \dim(\mathrm{Im}(r_\sigma^{dl(\sigma)})) = \dim(A^{\otimes n}) - 1$. We split $(A^{\otimes n})^{dl(\sigma)} = (I_\sigma)^{dl(\sigma)} \oplus L$ where L is the line generated by $i_\sigma(\rho_\sigma)$.

We have to show that

$$\eta(\prod_{i\in I}\gamma_\tau, b) = \eta_\sigma(1_\sigma r_\sigma(b))$$

This is certainly true if $\deg(b) \neq dn - d|\sigma| = dl(\sigma)$ since then both sides vanish. This is also the case if $b \in I_\sigma$. It remains to show that $\eta(\prod_{i\in I}\gamma_{\tau_i,\tau_i}, i_\sigma(\rho_\sigma)) = 1$.

We do this by induction on $|\sigma|$ the statement being clear for $|\sigma| = 1$. Let $\tau_{|\sigma|} = (ij)$ and set $\sigma' = \sigma\tau_{|\sigma|}$ then

$$i_{\sigma'}(\rho_{\sigma'}) = \gamma_{\tau_{|\sigma|},\tau_{|\sigma|}} i_\sigma(\rho_\sigma)$$

which follows from the equation $\rho \otimes 1\gamma_{(12),(12)} = \rho \otimes \rho$ and its pull back. So

$$1 = \eta(\prod_{i=1}^{|\sigma|-1} \gamma_{\tau_i,\tau_i}, i_{\sigma'}(\rho_{\sigma'})) = \eta(\prod_{i=1}^{|\sigma|-1} \gamma_{\tau_i,\tau_i}, \gamma_{\tau_{|\sigma|},\tau_{|\sigma|}} i_\sigma \rho_\sigma)$$

Another way to see this is to use the isomorphism $A_\sigma \simeq A_{\tau_1,\ldots,\tau_{|\sigma|}}$ and the iterated restriction maps for the pull-back, noticing, that indeed the $\gamma_{\tau,\tau}$ pull back onto each other in the various space.

Using the same rationale we obtain:

7.8. **Corollary.**

$$\check{r}^{\sigma\sigma'}_{\sigma,\sigma'}(1_{\sigma,\sigma'}) = \pi_{\sigma\sigma'}(\prod_{i\in I_{\sigma,\sigma'}} \gamma_{\tau_i,\tau_i}) \tag{7.16}$$

where $I_{\sigma,\sigma'} = \{i \in I : |\langle\sigma\sigma',\tau\rangle\backslash\bar{n}| < |\langle\sigma\sigma'\rangle\backslash\bar{n}|\}$ or in other words the γ_{τ_i,τ_i} that do not get contracted.

7.9. **Grading and shifts.** The meta–structure for symmetric powers is given by treating A^n as the linear structure, just like the variables in the Jacobian case (see [K]). In particular we fix the following degrees and shifts

$$\deg(1_\sigma) = d|\sigma|$$

$$\begin{aligned} s^+_\sigma &= d|c(\sigma)|, \quad s^-_\sigma = 0 \\ s_\sigma &= \frac{1}{2}(s^+_\sigma + s^-_\sigma) = \frac{d}{2}|c(\sigma)| \end{aligned}$$

Notice that as always there is no ambiguity for s^+, not even in the choice of dimension of A_σ, but the choice for s^- is a real one which is however the only choice which extends the natural grading if A is Jacobian.

This view coincides with realization of $A^{\otimes n}$ the n-th tensor product of the extension of coefficients to A of the Jacobian algebras for $f = z^2$.

7.10. **Notation.** The geometry of $\mathbf{S}_n$–Frobenius algebras is given by the subspace arrangement of fixed point sets $V_\sigma = Fix(\sigma) \subset k^n$ of the various $\sigma \in \mathbf{S}_n$ acting on k^n as well as their intersections $V_{\sigma,\sigma'} = V_\sigma \cap V_{\sigma'}$, etc., which were introduced in §5.

Recall that $|\sigma| = \mathrm{codim}_V(V_\sigma)$. We also define $|\sigma,\sigma'| := \mathrm{codim}(V_\sigma \cap V_{\sigma'})$ and set

$$\begin{aligned} d_{\sigma,\sigma'} &:= \frac{1}{d}\deg(\gamma_{\sigma,\sigma'}) = \frac{1}{2}(|\sigma| + |\sigma'| - |\sigma\sigma'|) \\ n_{\sigma,\sigma'} &:= \frac{1}{d}\deg(\check{r}^{\sigma\sigma'}_{\sigma,\sigma'}(1_{\sigma,\sigma'})) = \mathrm{codim}_{\mathrm{V}_{\sigma\sigma'}}(\mathrm{V}_{\sigma,\sigma'}) \\ &= |\sigma,\sigma'| - |\sigma\sigma'| \\ \tilde{g}_{\sigma,\sigma'} &:= \frac{1}{d}\deg(\tilde{\gamma}_{\sigma,\sigma'}) = d_{\sigma,\sigma'} - n_{\sigma,\sigma'} \\ &= \frac{1}{2}(|\sigma| + |\sigma'| + |\sigma\sigma'| - 2|\sigma,\sigma'|) \end{aligned} \tag{7.17}$$

Now given two elements $\sigma, \sigma' \in \mathbf{S}_n$ their representation on k^n naturally splits k^n into a direct sum, which is given by the smallest common block decomposition of both σ and σ'. More precisely:

Fix the standard basis e_i of k^n. For a subset $B \in \bar{n}$ we set $V_B = \bigoplus_{i \in B} k e_i \subset k^n$. Given σ, σ' we decompose

$$V := k^n = \bigoplus_{B \in \langle \sigma, \sigma' \rangle \backslash \bar{n}} V_B$$

and decompose

$$V_\sigma = \bigoplus_{B \in \langle \sigma, \sigma' \rangle \backslash \bar{n}} V_{\sigma;B}; \quad V_{\sigma,\sigma} = \bigoplus_{B \in \langle \sigma, \sigma' \rangle \backslash \bar{n}} V_{\sigma,\sigma;B} \tag{7.18}$$

where $V_{\sigma;B} := V_\sigma \cap V_B; V_{\sigma,\sigma';B} := V_{\sigma,\sigma'} \cap V_B$ and we used the notation of 5.1.

Notice that $\dim(V_{\sigma,\sigma';B}) = 1$ and we can decompose $\tilde{\gamma}_{g,h} = \bigotimes_B \tilde{\gamma}_{g,h;B}$.

Using the notation:

$$|\sigma|_B := \mathrm{codim}_{\mathrm{V_B}}(\mathrm{V}_{\sigma;\mathrm{B}}), \quad |\sigma, \sigma'|_{\mathrm{B}} := \mathrm{codim}_{\mathrm{V_B}}(\mathrm{V}_{\sigma,\sigma;\mathrm{B}})$$

set

$$\begin{aligned} d_{\sigma,\sigma';B} &:= \frac{1}{2}(|\sigma|_B + |\sigma'|_B - |\sigma, \sigma'|_B) \\ n_{\sigma,\sigma';B} &:= |\sigma, \sigma'|_B - |\sigma\sigma'|_B = \mathrm{codim}_{\mathrm{V}_{\sigma\sigma';\mathrm{B}}}(\mathrm{V}_{\sigma,\sigma';\mathrm{B}}) \\ \tilde{g}_{\sigma,\sigma';B} &:= d_{\sigma,\sigma';B} - n_{\sigma,\sigma';B} = \frac{1}{d}\deg(\tilde{\gamma}_{\sigma,\sigma';B}) \\ &= \frac{1}{2}(|\sigma|_B + |\sigma'|_B + |\sigma\sigma'|_B - 2|\sigma, \sigma'|_B) \end{aligned} \tag{7.19}$$

Notice that all the above functions take values in $\mathbf{N}$.

7.10.1. **Triple intersections.** For any number of elements σ_i we can analogously define the above quantities. We will do this for the triple intersections, since we need these to show associativity and although tedious we do this in order to fix the notation.

We regard the triple intersections $V_{\sigma,\sigma',\sigma''} = V_\sigma \cap V_{\sigma'} \cap V_{\sigma''}$.

Recall that $|\sigma| = \mathrm{codim}_V(V_\sigma)$. We also define $|\sigma, \sigma', \sigma''| := \mathrm{codim}(V_{\sigma,\sigma',\sigma''})$ and set

$$\begin{aligned} d_{\sigma,\sigma',\sigma''} &:= \frac{1}{d}\deg(\gamma_{\sigma,\sigma'}\gamma_{\sigma\sigma',\sigma''}) \\ &= \frac{1}{2}(|\sigma| + |\sigma'| - |\sigma\sigma'| + |\sigma\sigma'| + |\sigma''| - |\sigma\sigma'\sigma''|) \\ &= \frac{1}{2}(|\sigma| + |\sigma'| + |\sigma''| - |\sigma\sigma'\sigma''|) \\ n_{\sigma,\sigma',\sigma''} &:= \frac{1}{d}\deg(\check{r}^{\sigma\sigma'\sigma''}_{\sigma,\sigma',\sigma''}(1_{\sigma,\sigma'\sigma''})) = \mathrm{codim}_{\mathrm{V}_{\sigma\sigma'\sigma''}}(\mathrm{V}_{\sigma,\sigma',\sigma''}) \\ &= |\sigma, \sigma', \sigma''| - |\sigma\sigma'\sigma''| \\ \tilde{g}_{\sigma,\sigma',\sigma'} &:= \frac{1}{d}\deg(\tilde{\gamma}_{\sigma,\sigma',\sigma''}) = d_{\sigma,\sigma',\sigma''} - n_{\sigma,\sigma',\sigma''} \\ &= \frac{1}{2}(|\sigma| + |\sigma'| + |\sigma''| + |\sigma\sigma'\sigma''| - 2|\sigma, \sigma', \sigma''|) \end{aligned} \tag{7.20}$$

where $\tilde{\gamma}_{\sigma,\sigma',\sigma''}$ was defined in (3.6).

As above given three elements $\sigma, \sigma', \sigma'' \in \mathbf{S}_n$ their representation on k^n naturally splits k^n into a direct sum, which is given by the smallest common block decomposition of σ, σ' and σ'. More precisely:

Again, fix the standard basis e_i of k^n. For a subset $B \in \bar{n}$ we set $V_B = \bigoplus_{i \in B} ke_i \subset k^n$. Given σ, σ' we decompose

$$V := k^n = \bigoplus_{B \in \langle \sigma, \sigma', \sigma'' \rangle \backslash \bar{n}} V_B$$

and decompose

$$V_\sigma = \bigoplus_{B \in \langle \sigma, \sigma', \sigma'' \rangle \backslash \bar{n}} V_{\sigma;B};\ V_{\sigma,\sigma'} = \bigoplus_{B \in \langle \sigma, \sigma', \sigma'' \rangle \backslash \bar{n}} V_{\sigma,\sigma';B}$$

$$V_{\sigma,\sigma',\sigma''} = \bigoplus_{B \in \langle \sigma, \sigma', \sigma'' \rangle \backslash \bar{n}} V_{\sigma,\sigma',\sigma'';B} \tag{7.21}$$

where $V_{\sigma;B} := V_\sigma \cap V_B; V_{\sigma,\sigma';B} := V_{\sigma,\sigma'} \cap V_B V_{\sigma,\sigma',\sigma'';B} := V_{\sigma,\sigma',\sigma''} \cap V_B$

Notice that $\dim(V_{\sigma,\sigma',\sigma'';B}) = 1$.

We will also use the notation:

$$|\sigma|_B := \mathrm{codim}_{\mathrm{V_B}}(\mathrm{V}_{\sigma;\mathrm{B}}), |\sigma, \sigma'|_{\mathrm{B}} := \mathrm{codim}_{\mathrm{V_B}}(\mathrm{V}_{\sigma,\sigma;\mathrm{B}})$$

and

$$|\sigma, \sigma', \sigma''|_B := \mathrm{codim}_{\mathrm{V_B}}(\mathrm{V}_{\sigma,\sigma,\sigma'';\mathrm{B}})$$

7.11. **The cocycle in terms of $\gamma_{\tau,\tau}$s.** Let $\gamma_{\sigma,\sigma'}$ be given by the following:

For transversal σ, σ' we set $\gamma_{\sigma,\sigma'} = 1$.

If σ and σ' are *not* transversal using Theorem 5.9 we set

$$r_{\sigma,\sigma'}(\gamma_{\sigma,\sigma'}) = r_{\sigma\sigma'}(\prod_{i \in I} \gamma_{\tau_i,\tau_i}) = \prod_{i \in I'} \pi_{\sigma\sigma'}(\gamma_{\tau_i,\tau_i}) \prod_{j \in I''} r_{\sigma\sigma'}(\gamma_{\tau_j,\tau_j})$$

$$=: \gamma^t_{\sigma,\sigma'} \gamma^\perp_{\sigma,\sigma'} \tag{7.22}$$

where

$$I' = \{i \in I : \pi_{\sigma,\sigma'}(\gamma_{\tau_i,\tau_i}) = \pi_{\sigma\sigma'}(\gamma_{\tau_i,\tau_i})\}$$

$$I'' = \{i \in I : \pi_{\sigma,\sigma'}(\gamma_{\tau_i,\tau_i}) \neq \pi_{\sigma\sigma'}(\gamma_{\tau_i,\tau_i})\} \tag{7.23}$$

and $\gamma^t_{\sigma,\sigma'} \in i^{gh}_{g,h}(A_{g,h})$

7.12. **Proposition.** The equations of 7.11 are well defined and yield a group cocycle compatible with the reconstruction data. Furthermore

$$\gamma^\perp_{\sigma,\sigma'} = r^{\sigma\sigma'}_{\sigma,\sigma'}(1_{\sigma,\sigma'}) \tag{7.24}$$

$$\gamma^t_{\sigma,\sigma'} = i^{\sigma\sigma'}_{\sigma,\sigma'}(\bigotimes_{B \in \langle \sigma, \sigma' \rangle \backslash \bar{n}} e^{g(\sigma,\sigma',B)}) \tag{7.25}$$

$$\gamma_{\sigma,\sigma'} = r^{\sigma\sigma'}_{\sigma,\sigma'}(\bigotimes_{\in \langle \sigma, \sigma' \rangle \backslash \bar{n}} e^{g(\sigma,\sigma',B)}) = \check{r}^{\sigma,\sigma'}_{\sigma,\sigma'}(\tilde{\gamma}_{g,h}) \tag{7.26}$$

Proof. We need to check that indeed equation (7.22) is well defined. From Lemma 7.7.1 and the Corollary 7.8 we know that (7.24) is true and that the the product over I'' is well defined.

For (7.25) we notice that if a γ_{τ_i,τ_i} gets contracted, then

$$\pi_{\sigma\sigma'}(\gamma_{\tau_i,\tau_i}) = 1 \otimes \cdots \otimes 1 \otimes e \otimes 1 \otimes \cdots \otimes 1 \tag{7.27}$$

where $e = \mu\check{\mu}(1)$ is the Euler class which sits in the image of the k-th factor which is the same as the image of the l-th factor under the map $\pi_{\sigma,\sigma'}$ if $\tau_i = (kl)$.

The well definedness then follows by decomposition into $V_{\sigma\sigma',B}$ from the statement for one–dimensional $V_{\sigma,\sigma'}$ where it is clear from grading.

Finally (7.26) follows from (7.24) and (7.25) via Proposition 3.5.

For the associativity we use the general theory of intersection algebras 3.8. Here we notice that indeed the number of $\gamma_{\tau_i,\tau_i} : i \in I''$ contracted in each component B by $r^{\sigma,\sigma'}_{\sigma\sigma',\sigma''}$ is given by

$$\begin{aligned} &n_{\sigma,\sigma';B} - |\sigma,\sigma',\sigma''|_B + |\sigma\sigma',\sigma''|_B \\ &\qquad = |\sigma,\sigma'|_B - |\sigma\sigma'|_B - |\sigma,\sigma',\sigma''|_B + |\sigma\sigma',\sigma''|_B := q(\sigma,\sigma',\sigma'';B) \end{aligned} \tag{7.28}$$

so that by commutativity of (3.5)

$$r^{\sigma\sigma'}_{\sigma\sigma',\sigma''}(\gamma_{\sigma,\sigma'}) = i^{\sigma\sigma',\sigma''}_{\sigma,\sigma',\sigma''}(\bigotimes_B e^{q(\sigma,\sigma',\sigma'';B)})\check{r}^{\sigma\sigma',\sigma''}_{\sigma,\sigma',\sigma''}(1_{\sigma,\sigma',\sigma''})$$

and thus the $\gamma^\perp$ match also by commutativity. What remains to be calculated is the power of e in each of the components B. This power is given by

$$\begin{aligned} & \frac{1}{2}(|\sigma\sigma'|_B + |\sigma''|_B + |\sigma\sigma'\sigma''|_B - 2|\sigma\sigma',\sigma''|_B) \\ + \quad & \frac{1}{2}(|\sigma|_B + |\sigma'|_B + |\sigma\sigma'|_B - 2|\sigma,\sigma'|_B) \\ + \quad & (|\sigma,\sigma'|_B - |\sigma\sigma'|_B - |\sigma,\sigma',\sigma''|_B + |\sigma\sigma',\sigma''|_B) \\ = \quad & \frac{1}{2}(|\sigma|_B + |\sigma'|_B + |\sigma''|_B + |\sigma\sigma'\sigma''|_B - 2|\sigma,\sigma',\sigma''|_B) \\ = \quad & \tilde{g}_{\sigma,\sigma',\sigma'';B} \end{aligned}$$

q.e.d.

Putting together the Propositions 7.7 and 7.12 of this section we obtain:

7.13. **Theorem.** There exists a unique normalized cocycle compatible with the above special reconstruction data. There is only one compatible cocycle in the all even case.

In the super–case there are two choices of parity for the twisted sectors: all even or the parity of $p|\sigma| \equiv |\sigma|(2)$. Fixing the parity fixes the non–abelian cocycle.

In other words, there is a unique multiplicative $\mathbf{S}_n$ Frobenius algebra structure on the tensor powers of A and there are two G–actions labelled by parity.

7.14. **Definition.** We call the *symmetric power of a Frobenius algebra* the $\mathbf{S}_n$–twisted Frobenius algebra obtained from $T^nA, (r_\sigma)$ by using the unique normalized cocycle with all even sectors and the *super–symmetric power of a Frobenius algebra* the $\mathbf{S}_n$–twisted Frobenius algebra obtained from $T^nA, (r_\sigma)$ by using the unique normalized cocycle with the parity given by $A_\sigma \equiv |\sigma|\ (2)$.

7.15. **Definition.** We define *the second quantization of a Frobenius algebra A* to be the sum of all symmetric powers of A and *the second super–symmetric quantization of a Frobenius algebra A* to be the sum of all super–symmetric powers of A. We consider this sum either as formal or as a direct sum, where we need to keep in mind that the degrees of the summands are not equal.

7.16. **Comparison with the Lehn and Soerger construction.** In [LS] Lehn and Soerger constructed a non–commutative multiplicative structure in the special setting of symmetric powers. By the uniqueness result of the last section we know —since their cocycles are also normalized— that their construction has to agree with ours. In this section we make this explicit. Our general considerations of intersection algebras explain the appearance of their cocycles as the product over the Euler class to the graph defect times contribution stemming from the dual of the contractions.

7.16.1. **Definition.** (The graph defect) For $B \in \langle\sigma,\sigma'\rangle\backslash\bar{n}$ define the graph defect as [LS]

$$g(\sigma,\sigma';B) := \frac{1}{2}(|B| + 2 - |\langle\sigma\rangle\backslash B| - |\langle\sigma'\rangle\backslash B| - |\langle\sigma,\sigma'\rangle\backslash B|) \tag{7.29}$$

The equality of the two multiplications follows from:

7.16.2. **Proposition.**

$$g(\sigma,\sigma';B) = \tilde{g}_{\sigma,\sigma';B} = \frac{1}{2}(|\sigma|_B + |\sigma'|_B + |\sigma\sigma'|_B - 2|\sigma',\sigma'|_B) = d_{\sigma,\sigma';B} - n_{\sigma,\sigma';B}$$

Proof. By the above:

$$\begin{aligned} g(\sigma,\sigma';B) &= \frac{1}{2}(\dim(V_B) + 2\dim_{V_B}(V_{\sigma,\sigma';B}) - \dim(V_{\sigma,B}) - \\ &\quad - \dim(V_{\sigma';B}) - \dim(V_{\sigma\sigma';B})) \\ &= \frac{1}{2}(\dim(V_B) - \dim(V_{\sigma,B}) + \dim(V_B) - \dim(V_{\sigma';B}) \\ &\quad + \dim(V_B) - \dim(V_{\sigma\sigma';B}) - (2\dim(V_B) - \dim_{V_B}(V_{\sigma,\sigma';B}))) \\ &= \frac{1}{2}(|\sigma|_B + |\sigma'|_B + |\sigma\sigma'|_B - 2|\sigma,\sigma'|_B) \end{aligned}$$

7.16.3. **Remark.** The above equation makes it obvious that $g \in \mathbf{N}$, since $d_{\sigma,\sigma';B}$, $n_{\sigma,\sigma';B} \in \mathbf{N}$ and both $|\sigma| + |\sigma'| \geq |\sigma,\sigma'|$ and $|\sigma\sigma'| \geq |\sigma,\sigma'|$. The first inequality follows from $V_{\sigma,\sigma'} = V_\sigma \cap V_{\sigma'}$ and the second one from $V_{\sigma,\sigma'} \subset V_{\sigma\sigma'}$.

Appendix A

A.1. **Theorem.** For any non–abelian $\mathbf{S}_n$ cocycle φ with values in k^* one of the following holds: $\forall\sigma,\tau, |\tau| = 1$:

$$\varphi_{\sigma,\tau} = 1$$

We call this case even and set the parity $p = 0$. Or $\forall\sigma,\tau|\tau| = 1$

$$\varphi_{\sigma,\tau} = (-1)^{|\sigma|}$$

We call this case odd and set the parity $p = 1$. In unified notation:

$$\varphi_{\sigma,\tau} = (-1)^{p|\sigma|} \tag{A-1}$$

with $p \in \{0,1\}$.

Proof. First notice for $|\tau| = 1$, $\tau^2 = e$ and thus $\varphi_{\tau,\sigma} = \pm 1$ for any $\sigma \in \mathbf{S}_n, [\sigma,\tau] = e$.

Fix $\tau_0 = (12)$ and $\tau_0' = (34)$ and define the parity by

$$\varphi_{\tau_0,\tau_0} =: (-1)^p \tag{A-2}$$

Then since $\tau \in \mathbf{S}_n \, \exists \sigma \in \mathbf{S}_n : \sigma\tau\sigma^{-1} = \tau_0$ using (1.21)

$$\varphi_{\tau,\tau} = \varphi_{\tau_0,\tau_0} = (-1)^p \tag{A-3}$$

If $[\tau,\tau'] = e$ then either $\tau = \tau'$ or if $n \geq 4 : \exists \sigma \in \mathbf{S}_n : \sigma\tau\sigma^{-1} = (12)$, $\sigma\tau'\sigma^{-1} = (34)$ again using (1.21)

$$\forall \tau,\tau'; \tau \neq \tau'; [\tau,\tau'] = e : \ \varphi_{\tau,\tau'} = \varphi_{(12),(34)} = \varphi_{\tau_0,\tau_0'} \tag{A-4}$$

Furthermore

$$\varphi_{(12),(34)} = \varphi_{(24)(12)(14),(34)} = \varphi_{(14),(34)}\varphi_{(12),(13)}\varphi_{(24),(23)} \tag{A-5}$$

Thus the $\varphi_{\tau,\tilde{\tau}}$ can all be expressed in terms of $\varphi_{\tau,\tau'}$ with $[\tau,\tau'] = \tau''$ where $\tau = (ij), \tau' = (jk), \tau'' = (ik)$.

We will show by induction that we can scale such that

$$\forall \tau,\tau', [\tau,\tau'] \neq e : \varphi_{\tau,\tau'} = (-1)^p \tag{A-6}$$

Notice that (A-4), (A-5) together with (A-6) and (A-3) imply

$$\forall \tau,\tau'; |\tau| = |\tau'| = 1 : \varphi_{\tau,\tau'} = (-1)^p \tag{A-7}$$

Induction for (A-7).

Assume that (A-7) holds for $\tau,\tau' \in \mathbf{S}_n \subset \mathbf{S}_{n+1}$.

Now scale with

$$\begin{aligned} \lambda_{(ij)} &:= (-1)^p \varphi_{(n-1\,n+1),(n\,n+1)} \text{ for } i,j \leq n \\ \lambda_{(i\,n+1)} &:= (-1)^p \varphi_{(in),(n\,n+1)} \text{ for } i < n \\ \lambda_{(n\,n+1),(n\,n+1)} &:= 1 \end{aligned} \tag{A-8}$$

Notice this implies that

$$\begin{aligned} \tilde{\varphi}_{(in),(n\,n+1)} &= \frac{\lambda_{(n\,n+1)}}{\lambda_{(i\,n+1)}} \varphi_{(in),(n\,n+1)} \\ &= (-1)^p \frac{1}{\varphi_{(in),(n\,n+1)}} \varphi_{(in),(n\,n+1)} = (-1)^p \end{aligned} \tag{A-9}$$

$$\tilde{\varphi}_{(in),(i\,n+1)} = \tilde{\varphi}^{-1}_{(in),(n\,n+1)} = (-1)^p \tag{A-10}$$

$$\tilde{\varphi}_{(ij),(kl)} = \varphi_{(ij),(kl)} = (-1)^p \text{ if } i,j,k,l \leq n \tag{A-11}$$

where the last statement follows by induction.

We need to show

$$\tilde{\varphi}_{\tau,\tau'} = (-1)^p \tag{A-12}$$

For $n = 2,3$ (A-3) and (A-9) respectively show the claim. For $n = 4$ we also need to consider (A-4), but by using induction and (A-9) with $n = 3, n+1 = 4$ we see that

$$\begin{aligned}\varphi_{(12),(34)} &= \varphi_{(24)(12)(14),(34)} = \varphi_{(14),(34)}\varphi_{(12),(13)}\varphi_{(24),(23)}\\ &= \varphi_{(14),(34)}\varphi_{(12),(13)}\varphi^{-1}_{(24),(34)} = (-1)^p(-1)^p(-1)^p\\ &= (-1)^p \qquad \text{(A-13)}\end{aligned}$$

So we assume $n \geq 4$ and from (A-4), (A-13):

$$\forall \tau, \tau'; |\tau| = |\tau'| = 1; [\tau, \tau'] = e: \quad \varphi_{\tau,\tau'} = (-1)^p \qquad \text{(A-14)}$$

Thus by induction, (A-3) and (A-14), we need to check the cases

i) $\tau = (ij), \tau' = (j\, n+1); i, j \in \{1, \dots, n-1\}; i \neq j$
ii) $\tau = (i\, n+1), \tau' = (j\, n+1); i \in \{1, \dots, n\}, j \in \{1, \dots, n-1\}; i \neq j$
iii) $\tau = (in+1), \tau' = (ij); i, j \in \{1, \dots, n\}$

Notice that $\tilde{\varphi}_{(i\,n+1),(ij)} = \tilde{\varphi}^{-1}_{(i\,n+1),(j\,n+1)}$ and thus i) implies iii) if $j \neq n$. Else iii) follows by (A-9) and thus it suffices to show i) and ii).

For i)

$$\begin{aligned}\tilde{\varphi}_{(ij),(j\,n+1)} &= \frac{\lambda_{(j\,n+1)}}{\lambda_{(i\,n+1)}}\varphi_{(ij),(j\,n+1)} = \frac{\varphi_{(jn)(nn+1)}}{\varphi_{(in)(nn+1)}}\varphi_{(ij),(j\,n+1)}\\ &= \varphi_{(jn),(n\,n+1)}\varphi_{(in),(i\,n+1)}\varphi_{(ij),(j\,n+1)} = \varphi_{(in)(ij)(jn),(n\,n+1)}\\ &= \varphi_{(ij),(n\,n+1)} = (-1)^p\end{aligned}$$

by (A-14).

for ii)

$$\begin{aligned}\tilde{\varphi}_{(i\,n+1),(j\,n+1)} &= \frac{\lambda_{(j\,n+1)}}{\lambda_{(ij)}}\varphi_{(i\,n+1),(j\,n+1)} = \frac{\varphi_{(jn)(n\,n+1)}}{\varphi_{(n-1\,n+1)(n\,n+1)}}\varphi_{(i\,n+1),(j\,n+1)}\\ &= \varphi_{(jn),(n\,n+1)}\varphi_{(n-1\,n+1),(n-1\,n)}\varphi_{(i\,n+1),(j\,n+1)}\\ &= \varphi_{(i\,n+1)(jn)(n-1\,n+1),(n-1\,n)} = \varphi_{(i\,n+1\,n-1)(jn),(n-1\,n)}\\ &= \varphi_{(n+1\,n-1)(n-1\,i)(jn),(n-1\,n)}\\ &= \varphi_{(jn),(n-1n)}\varphi_{(n-1\,i),(n-1\,j)}\varphi_{(n-1\,n+1),(ij)} = (-1)^p\end{aligned}$$

Finally if $\sigma = \prod_{i=1}^{|\sigma|}\tau_i$

$$\varphi_{\sigma,\tau} = \prod_{i=1}^{|\sigma|}\varphi_{\tau_i,\tilde{\tau}_i} = (-1)^{p|\sigma|}$$

with $\tilde{\tau}_i = (\prod_{j=i+1}^{|\sigma|}\tau_i)\tau(\prod_{j=i+1}^{|\sigma|}\tau_i)^{-1}$, $|\tilde{\tau}_i| = |\tau| = 1$ which proves the Theorem.

Appendix B

B.1. **A detailed proof of Theorem 5.9.** We will assume by induction on r that

$$\gamma_{\sigma',\tau'} = 1 \text{ for } |\tau'| = 1, |\sigma'| \leq r-1 \text{ and } \lambda_\sigma = 1 \text{ for } |\sigma| \leq r \qquad \text{(B-1)}$$

Fix σ with $|\sigma| = r+1$. We need to show that indeed for two decompositions

$$\sigma = \sigma'\tau' = \sigma''\tau'' \qquad \text{(B-2)}$$

indeed

$$\gamma_{\sigma',\tau'} = \gamma_{\sigma'',\tau''} \qquad \text{(B-3)}$$

We set $\sigma''' = \sigma\tau'\tau''$ and $\tau''' = \tau'\tau''\tau'$. It follows

$$\sigma' = \sigma'''\tau'', \sigma'' = \sigma'''\tau''', \tau''' \neq \tau''$$

If $|\sigma'''| = r-1$, we find

$$\begin{aligned}\gamma_{\sigma',\tau'} &= \gamma_{\sigma'''\tau'',\tau'}\gamma_{\sigma''',\tau''} = \gamma_{\sigma''',\tau''\tau'}\gamma_{\tau'',\tau'} \\ &= \gamma_{\sigma''',\tau'''\tau''}\gamma_{\tau''',\tau''} = \gamma_{\sigma'''\tau''',\tau''}\gamma_{\sigma''',\tau'''} = \gamma_{\sigma'',\tau''}\end{aligned}$$

If $|\sigma'''| = r+1$ then if $\tau' = (ij), \quad \tau'' = (kl)$, i,j,k,l must all lie in the same cycle.

Without loss of generality and to avoid too many indices, we assume that this cycle c is just given by $c = (12\cdots h)$ for some $h \leq r+2$. First assume that $\{i,j\} \cap \{k,l\} = \emptyset$. We can then assume $i < j$, $k < l$ and $i < k$. Then there are three possibilities: $i < j < k < l$, $i < k < l < j$ and $i < k < j < l$ where the first two have $|\sigma'''| = r-1$.

So fix $i < k < j < l$. We see that we can decompose

$$\sigma' = \tilde{\sigma}(ilh)(kj), \quad \sigma'' = \tilde{\sigma}(ikh)(jl)$$

with

$$\tilde{\sigma} = \sigma(hljki) \text{ and } |\tilde{\sigma}| = r-3$$

Now

$$\begin{aligned}\gamma_{\sigma',\tau'} &= \gamma_{\tilde{\sigma}(ilh)(kj),(ij)} = \gamma_{\tilde{\sigma}(ilh)(kj),(ij)}\gamma_{\tilde{\sigma}(ilh),(kj)} = \gamma_{\tilde{\sigma}(ilh),(kj)(ij)}\gamma_{(kj),(ij)} \\ &= \gamma_{\tilde{\sigma}(ilh),(ik)(kj)}\gamma_{(ik),(kj)} = \gamma_{\tilde{\sigma}(ilh)(ik),(kj)}\gamma_{\tilde{\sigma}(ilh),(ik)} = \gamma_{\tilde{\sigma}(iklh),(kj)} \\ &= \gamma_{\tilde{\sigma}(ikh)(kl),(kj)}\gamma_{\tilde{\sigma}(ikh),(kl)} = \gamma_{\tilde{\sigma}(ikh),(kl)(kj)}\gamma_{(kl),(kj)} \\ &= \gamma_{\tilde{\sigma}(ikh),(jl)(kl)}\gamma_{(jl),(kl)} = \gamma_{\tilde{\sigma}(ikh)(jl),(kl)}\gamma_{\tilde{\sigma}(ikh),(jl)} = \gamma_{\tilde{\sigma}(ikh)(jl),(kl)} \\ &= \gamma_{\sigma'',\tau''}\end{aligned}$$

since $|\tilde{\sigma}(ilh)| = |\tilde{\sigma}(ikh)| = r-1$.

If $|\{i,j\} \cap \{k,l\}| = 1$ then we can assume that $j = k$ and $i < l$ which leaves us with the cases: $i < j < l$,$j < i < l$ and $i < l < j$; where in the first two cases $|\sigma'''| = r-1$.

Now assume $i < j < k$. We can decompose

$$\sigma' = \tilde{\sigma}(ilh), \quad \sigma'' = \tilde{\sigma}(ijh)$$

with

$$\tilde{\sigma} = \sigma(hlji) \text{ and } |\tilde{\sigma}| = h-4$$

And

$$\begin{aligned}\gamma_{\sigma',\tau'} &= \gamma_{\tilde{\sigma}(ilh),(ij)} = \gamma_{\tilde{\sigma}(il)(lh),(ij)}\gamma_{\tilde{\sigma}(il),(lh)} = \gamma_{\tilde{\sigma}(il),(lh)(ij)}\gamma_{(lh),(ij)} \\ &= \gamma_{\tilde{\sigma}(il),(ij)(lh)}\gamma_{(ij),(lh)} = \gamma_{\tilde{\sigma}(il)(ij),(lh)}\gamma_{\tilde{\sigma}(il),(ij)} = \gamma_{\tilde{\sigma}(ijl),(lh)} \\ &= \gamma_{\tilde{\sigma}(ij)(jl),(lh)}\gamma_{\tilde{\sigma}(ij),(jl)} = \gamma_{\tilde{\sigma}(ij),(jl)(lh)}\gamma_{(jl),(lh)} \\ &= \gamma_{\tilde{\sigma}(ij),(jh)(jl)}\gamma_{(jh),(jl)} = \gamma_{\tilde{\sigma}(ij)(jh),(jl)}\gamma_{\tilde{\sigma}(ij),(jh)} = \gamma_{\tilde{\sigma}(ijh),(jl)} \\ &= \gamma_{\sigma'',\tau''}\end{aligned}$$

since $|\tilde{\sigma}(il)| = |\tilde{\sigma}(ij)| = r-1$.

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