HOMOLOGICAL MIRROR SYMMETRY, DEFORMATION QUANTIZATION AND NON-COMMUTATIVE GEOMETRY

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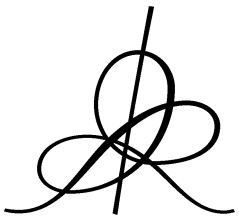

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To Alain Connes on his 55th birthday

1 Introduction

Mathematical foundation of mirror symmetry belongs both to geometry (Strominger-Yau-Zaslow conjecture, see [SYZ]) and algebra (Kontsevich's homological mirror symmetry program, see [Ko1]). Present paper deals with some aspects of the latter. Our methods and conjectures have an algebraic (categorical) nature. On the other hand, based on the ideas of [Ko1], [KoSo1], [KoSo2], we stress the role of *non-commutative* geometry of mirror symmetry, making an attempt to connect it with deformation quantization. The latter can be thought of generalization of the theory of differential operators (or D-modules) to arbitrary symplectic manifolds.

In homological mirror symmetry and in the theory of D-modules one meets similar objects. They are pairs (L, ρ) where L is a Lagrangian manifold, and ρ is a flat bundle on L (local system). In the framework of homological mirror symmetry such pairs are objects of the so-called Fukaya category, which is the principal mathematical structure of the genus zero part of the A-model. In the framework of D-modules they are holonomic D-modules (in C^∞ category). It is natural to compare the categories themselves. This comparison is the main theme of present paper.

One can object any relationship between mirror symmetry and deformation quantization. Let us mention some of possible objections.

a) Theory of D-modules (or more generally modules over the ring of microdifferential operators) "lives" on the cotangent bundle T^*X, while the Fukaya category is defined for any symplectic manifold, hence is more general.

b) Theory of D-modules works well in algebraic or complex analytic framework, while the Fukaya category (and in general Floer theory) exists in C^∞-category only.

c) In mirror symmetry one considers series in exponentially small (with respect to the symplectic structure) parameter, while in deformation quantization the parameter "is of the size" of the symplectic structure.

From our point of view these are rather problems than objections. In regard to the point a) this means that one should work out a version of the theory of microdifferential operators for arbitary symplectic manifolds. This theory is known as deformation quantization of symplectic manifolds. In regard to the point b) we remark that finding of a complex analog of the Floer theory is an interesting problem. At this time we have only speculations in this directions.

The aim of present paper is to summarize our present understanding of the topic and make some conjectures. It is a part of an ongoing project. The details will appear elsewhere. Our main conjecture can be briefly formulated such as follows:

i) there is a category of holonomic modules over the quantized algebra of smooth functions on a symplectic manifold;

ii) it is possible to change morphisms in this category by a kind of integral transformation, so that it becomes equivalent (at least locally) to the Fukaya category of the same symplectic manifold.

Let us mention two possible applications of this idea. First, it can help in constructing of an algebraic model of the Fukaya category (analogy: de Rham complex is an algebraic model of Morse complex). Second, it might help to resolve some difficulties in the definition of the Fukaya category (non-transversality of supports, non-existence of identity morphisms, etc.) In our opinion current situation is not completely satisfactory despite the recent progress (cf. for ex. [FOOO]). Third, one can go beyond Lagrangian submanifolds by considering coisotropic submanifolds or even non-smooth varieties(cf. [KO]). Finally, one hopes to achieve a deeper understanding of the relationship at the level of chiral algebras (see [BD],[MSV], [KaV]). Hence the subject of this paper is just the first approximation to the full picture. We hope to discuss it elsewhere.

The paper is organized as follows. In Section 2 we briefly recall basics on

the deformation theory and A_∞-categories. The purpose of this section is to fix the language. Details of the formalism will be explained in [KoSo2].

In Section 3 we recall main facts about the Fukaya category, which is one of the main structures of homological mirror symmetry. Section 4 is a reminder on deformation quantization of symplectic manifolds. Section 5 is devoted to the comparison of the Fukaya category with the category of holonomic modules over the quantized algebra of functions.

Main idea of Section 3 goes back to the pioneering paper [Ko1]. As explained in loc. cit. homological mirror symmetry is not a statement about individual categories, but rather about families of A_∞-categories over $\mathbf{Z}$-graded formal schemes. Geometrically this structure is modelled by a "family of non-commutative differential-graded manifolds over a commutative differential-graded base". The framework in which the latter phrase has precise meaning will be explained in [KoSo2]. The reader can keep in mind the example of the deformation theory of an associative algebra. In this case one has a family of associative algebras parametrized by the formal moduli space $\mathcal{M}$ of deformations of the given algebra A. The moduli space $\mathcal{M}$ is a formal pointed differential-graded manifold (dg-manifold for short) (see for ex. [Ko2], [KoSo3]). Then by definition $\mathcal{M}$ is a base of a family of associative algebras $A_\gamma, \gamma \in \mathcal{M}$. Each algebra gives rise to a "non-commutative scheme" $Spec(A_\gamma)$, hence one has the desired structure.

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2 Reminder on A_∞-categories and deformation theory

In this section we recall some facts about homological algebra of mirror symmetry. Full details including the necessary language of non-commutative geometry will appear in [KoSo2]. Some of the material can be found in ex-

isting literature, for example in [Ko2]. We are not going to discuss in detail motivations for all of the definitions and notions below. Main purpose of this section is to fix the language.

Let A be a free $\mathbf{Z}$-graded k-module over a unital commutative ring k of characteristic zero (main applications deal with the case when k is a field).

Definition 1 *An A_∞-algebra A over k is given by the following data:*

a) A $\mathbf{Z}$-graded free k-module A.

b) A codifferential d on the cofree coalgebra $T(A[1]) = \oplus_{n\geq 1} A[1]^{\otimes n}$, where $A[1]$ denotes the graded free k-module such that $A[1]^i = A^{i+1}$ (A with shifted grading).

(We recall that a codifferential means a coderivation d of the coalgebra satisfying the condition $d^2 = 0$).

Since d is uniquely defined on cogenerators, it gives rise to "higher multiplications" $m_n : A^{\otimes n} \to A, n \geq 1$ of degrees $2 - n$ satisfying a system of quadratic equations which follows from the equation $d^2 = 0$.

Definition 2 *An L_∞-algebra on A is given by the following data:*

a) A $\mathbf{Z}$-graded vector space A.

b) A codifferential on the cofree cocommutative coalgebra $C(A[1]) = \oplus_{n\geq 1} S^n(A[1])$, where $S^n(V)$ denotes the nth symmetric power in the symmetric monoidal category of $\mathbf{Z}$-graded k-modules.

The codifferential d defines a sequence of "higher Lie brackets" $m_n : A^{\otimes n} \to A, n \geq 1$ of degrees $2 - n$ satisfying a system of quadratic equations which follows from the equality $d^2 = 0$.

It is useful to have in mind geometric picture for both algebraic structures defined above. We start with L_∞-algebras.

An L_∞-algebra gives rise to a *formal pointed $\mathbf{Z}$-graded manifold* X, which carries a vector field d_X of degree $+1$ such that d_X vanishes at the marked point, and satisfies the condition $[d_X, d_X] = 0$. This structure is called *formal pointed differential-graded manifold* in [Ko2], [KoSo2], [KoSo3] (it was introduced by A. Schwarz under the name of Q-manifold). Algebra of formal functions of X is isomorphic to the graded dual to the coalgebra $C(A[1])$. Thus we can write $X = Spf((C[1])^*)$, where Spf stands for the formal spectrum.

Remark 1 *a) One should remember that formal* **Z***-graded manifolds have only nilpotent points.*

b) It is useful to interpret maps m_n as Taylor coefficient of the vector field d_X at the marked point.

Let us consider two illustrating examples.

Example 1 *Let A be an associative algebra. Then its truncated Hochschild complex $C^\bullet_+(A,A)[1] = \oplus_{n\geq 1} Hom_k(A^{\otimes n}, A)[1]$ carries a structure of differential-graded Lie algebra, hence defines a formal pointed dg-manifold (see [Ko2]).*

One can interpret the DGLA from the example as a DGLA of coderivations of the tensor coalgebra cogenerated by $A[1]$. Equivalently, it is the DGLA of vector fields on the formal pointed graded manifold X vanishing at the marked point. Then the DGLA structure is the natural one on vector fields.

One also has the notion of *formal differential-graded manifold*, where the condition of vanishing at the marked point is dropped. Algebraically this means that we allow the condition $m_0 \neq 0$.

Example 2 *In the previous example we consider the full Hochschild complex $C^\bullet(A,A)[1] = \oplus_{n\geq 0} Hom_k(A^{\otimes n}, A)[1]$. It gives rise to a formal dg-manifold.*

Importance of L_∞-algebras and formal dg-manifolds in deformation theory is based on the fact that they define deformation functors (see for ex. [Ko1], [KoSo2]). Let us briefly recall the construction. If $g = \oplus_{n\geq 0} g^n$ is an L_∞-algebra (g^n is the nth graded component) then one has the deformation functor from commutative nilpotent algebras (possibly graded) to groupoids. Namely to a commutative nilpotent ring R one assigns the groupoid $Def_g(R)$ consisting for $\gamma \in g^1 \otimes R$ which satisfy the Maurer-Cartan equation

$$m_1(\gamma) + m_2(\gamma,\gamma)/2! + ... + m_n(\gamma,...,\gamma)/n! + ... = 0,$$

where m_n are the higher Lie brackets.

Formal deformations of many algebraic and geometric structures give rise to formal pointed dg-manifolds. Formal dg-manifolds without marked points arise, for example, when one deforms categories (i.e. when objects of a category are deformed). In Example 2 the corresponding dg-manifold controls deformations of the category with one object X such that $Hom(X,X) = A$.

One can develop a similar geometric language for A_∞-algebras. Namely, an A_∞-algebra gives rise to a *non-commutative formal pointed dg-manifold.* It is modelled by the cofree coalgebra $T(A[1])$ which carries a codifferential d_X. One also has the notion of *non-commutative formal dg-manifold* (no marked point is specified). In this case one can have a non-zero map $m_0 : k \to A$. Geometrically this structure corresponds to a vector field d_X of degree $+1$ such that $[d_X, d_X] = 0$, without any condition at the marked point. The corresponding algebraic structure is defined in the following way.

Definition 3 *We say that a codifferential on the coalgebra $k \oplus T(A[1])$ defines a structure of generalized A_∞-algebra on A.*

The notion of a small A_∞-category is a natural generalization of the notion of A_∞-algebra. Traditionally, such a category is defined by a set of objects $Ob(\mathcal{C})$, $\mathbf{Z}$-graded free k-modules of morphisms $Hom(X, Y)$, and structures of A_∞-algebras on the spaces $\oplus_{0 \leq i,j \leq n} Hom(X_i, X_j)$ for any collection of objects $X_0, ..., X_n, n \geq 1$. These structures are given by the higher compositions

$$m_n : \otimes_{0 \leq i \leq n} Hom(X_i, X_{i+1}) \to Hom(X_0, X_n),$$

which are maps of $\mathbf{Z}$-graded free k-modules of degrees $2-n$ satisfying quadratic relations similar to those for A_∞-algebras. The structures of A_∞-algebras are compatible with inclusions of collections of objects.

One can think of an A_∞-category $\mathcal{C}$ as of the large A_∞-algebra $End(\oplus_{X \in Ob(\mathcal{C})} X)$ (compare with the relation between additive categories and associative algebras). The Hochschild complex and Hochschild cohomology of an A_∞-category can be defined in terms of this A_∞-algebra.

For any object X the k-module $End(X) = Hom(X, X)$ is an A_∞-algebra. Its truncated Hochschild complex gives rise to a formal pointed dg-manifold $\mathcal{M}_X$. Then the formal dg-manifold $\mathcal{M} = \sqcup_{X \in Ob(\mathcal{C})} \mathcal{M}_X$ "controls" A_∞-deformations of the category $\mathcal{C}$ with the fixed set of objects.

Replacing in the above discussion the truncated Hochschild complex by the full Hochschild complex one obtains a new structure called *generalized A_∞-category.* Generalized A_∞-categories do not have a fixed set of objects. This is due to the fact that now for the A_∞-algebra $End(X)$ one can have $m_0 \neq 0$. As before, one can derive the formal dg-manifold $\mathcal{M}$ (now using the generalized A_∞-algebras $End(X)$). For a commutative nilpotent k-algebra R one can consider k-points $\mathcal{M}(R)$. If $\mathcal{M}_0 \subset \mathcal{M}$ is the subset of zeros of the odd vector field $d_\mathcal{M}$ then one can speak about objects of some A_∞-category.

The objects are parametrized by $\mathcal{M}_0$. We omit here the description which can be given in terms of R-points of $\mathcal{M}_0$ and $\mathcal{M}$.

We will keep the name of generalized A_∞-category for a slightly more general structure. The point is that we allow the higher compositions m_n be defined not for all collections of objects, but only for some of them (transversal collections). The Fukaya category discussed in the next section will be this kind of generalized A_∞-category.

We summarize without details the data defining a generalized A_∞-category in the following way:

a) We are a given a formal dg-manifold $Ob(\mathcal{C}) = \mathcal{M}$.

b) For any $n \geq 1$ we are given a formal dg-submanifold $\mathcal{M}^n_{tr} \subset \mathcal{M}^n$ called the space of transversal n-families. It is assumed that $\mathcal{M}^1_{tr} \subset \mathcal{M}$ (i.e. every object is transversal to itself).

c) For a point $(X, Y) \in \mathcal{M}^2_{tr}$ we are given a $\mathbf{Z}$-graded free k-module $Hom_{\mathcal{C}}(X, Y)$ called a space of morphisms between X and Y. These k-modules are organized in a formal dg-bundle $Hom^{\mathcal{C}} \to \mathcal{M}^2_{tr}$.

d) For $n \geq 1$ and any $(X_0, ..., X_n) \in \mathcal{M}^n_{tr}$ we are given a higher composition map $m_n^{(X_0,...,X_n)}$ which gives rise to a morphism of the obvious pullbacks of $Hom^{\mathcal{C}}$ to $\mathcal{M}^n_{tr}$. The composition maps give rise to a structure of generalized A_∞-algebra $A(X_0, ..., X_n)$, or, equivalently to a non-commutative formal dg-manifold $\mathcal{M}(X_0, ..., X_n)$.

e) Let $\mathcal{M}(\mathcal{C})$ be the inductive limit of $\mathcal{M}(X_0, ..., X_n)$ taken over increasing collections of transversal objects. This is a non-commutative formal dg-ind-scheme (it can be properly defined as an inductive limit in the appropriate category).

f) Finally $\mathcal{M}(\mathcal{C})$ is a non-commutative formal dg-manifold over the commutative scheme $Spec(k)$.

The structure defined in a)-e) is called generalized A_∞-category. It gives rise to the usual k-linear A_∞-category, which we will call associated to $\mathcal{C}$. The latter is defined by a non-commutative formal dg-ind-subscheme $\mathcal{M}(\mathcal{C})_0 \subset \mathcal{M}(\mathcal{C})$ of zeros of the odd vector field $d_{\mathcal{M}(\mathcal{C})}$.

Remark 2 *a) One can define the Hochschild complex of a generalized A_∞-category. It gives rise to a formal dg-manifold. The corresponding deformation functor describes the formal deformation theory of the generalized A_∞-category in the class of generalized A_∞-categories. If the category has one object X we have the full Hochschild complex of the corresponding generalized A_∞-algebra $End(X)$.*

b) We do not discuss the delicate problem of unital A_∞-algebras (more generally, A_∞-categories with identity morphisms). It is an interesting question because in practice identity morphisms may be defined up to a homotopy only. All this can be formulated in the language of non-commutative geometry.

There is a notion of A_∞-functor between two generalized A_∞-categories. A_∞-functors form themselves a generalized A_∞-category. Using A_∞-functors one defines the notion of equivalence of A_∞-categories (see [KoSo2]).

Let us illustrate the notion of equivalence in the case when both categories have only one object. Then we are dealing with generalized A_∞-algebras, say A and B. Assume in addition that $m_0^A = 0$ and $m_0^B = 0$ (i.e. we have ordinary A_∞-algebras). If A is equivalent to B then the complexes (A, m_1^A) and (B, m_1^B) are quasi-isomorphic. Geometrically this means that the tangent spaces at marked points of equivalent non-commutative formal pointed dg-manifolds are quasi-isomorphic. The converse is also true (this is the A_∞-version of the inverse function theorem). A generalized A_∞-algebra with $m_0 \neq 0$ is equivalent to one which has $m_0 \neq 0$, and $m_{n\geq 1} = 0$ (cf.: vector field which is non-trivial at a point is locally equivalent to a constant one). The latter observation explains why generalized A_∞-categories should be studied in families rather than individually. Indeed generalized A_∞-categories with $m_0 \neq 0$ are trivial in the sense that all higher compositions can be killed by an appropriate equivalence functor.

Similarly to the case of formal pointed dg-manifolds there is a theory of minimal models of non-commutative formal pointed dg-manifolds. For such a theory one needs to assume that the ground ring k is a field of characteristic zero (there is more complicated theory for non-pointed dg-manifolds).

If two A_∞-categories are equivalent then the formal pointed dg-manifolds of their deformations are quasi-isomorphic (i.e. tangent complexes at the marked points are quasi-isomorphic).

Finally, there is a theory of generalized A_∞-categories over a formal dg-base. In the above discussion we discussed the case when the base was an ordinary scheme $Spec(k)$. We can also assume that the base is a formal scheme. As we will see in the next section the latter case is important in symplectic geometry.

3 Fukaya category

3.1 Fukaya category and non-commutative geometry

The Fukaya category $F(X)$ of a smooth symplectic manifold X is a generalized A_∞-category over a base. It can be constructed as an A_∞-deformation of the following trivial $\mathbf{C}$-linear category $F_0(X)$. Objects of $F_0(X)$ are pairs (L,ρ), where L is a Lagrangian submanifold of X and ρ is a local system on L. Transversal collections of objects correspond to transversal collections of Lagrangian submanifolds (in fact we need more sofisticated transversality condition, see [KoSo1]). We set $Hom_{F_0(X)}((L_0,\rho_0),(L_1,\rho_1)) = \oplus_{x\in L_0\cap L_1} Hom(\rho_{0x},\rho_{1x})$ and $Hom_{F_0(X)}((L,\rho),(L,\rho)) = \Omega^\bullet(L,End(\rho))$. All compositions $m_n, n \geq 1$ are trivial for collections of different objects. Otherwise $m_{n\geq 3} = 0$, and m_2 is the natural product on the differential-graded algebra $\Omega^\cdot(L,End(\rho))$.

To a pair (L,ρ) one can associate a generalized A_∞-algebra $A(L,\rho)$. Let assume for simplicity that ρ is a trivial rank one local system. Then the corresponding generalized A_∞-algebra $A(L)$ is generated by geometric cycles in L. Higher multiplications $m_n(C_1,...,C_n)$ between generic cycles C_i are given by a kind of quantum cohomology construction. Namely, one counts with some weight pseudo-holomorphic maps $f : (D^2,\partial D^2) \to (X,L)$ with marked points $x_1,...,x_n \in \partial D^2$. Here $D^2 \subset \mathbf{C}$ is the standard disc. It is required that the point x_i is mapped to C_i. The weight is $exp(-\frac{1}{\epsilon}\int_{D^2} f^*(\omega))$, where ω is the symplectic form on X, and ϵ is a parameter. The idea of this construction of $A(L)$ was suggested by Kontsevich. Difficult analytic details have been worked out in [FOOO]. In what follows we will assume the conditions on the data imposed in [FOOO]. The resulting generalized A_∞-algebra $A(L)$ is defined over the valuation ring $\mathbf{C}_\epsilon^{\geq 0} = \{f = \sum_{i\geq 1} a_i e^{-\lambda_i/\epsilon}\}$, where $a_i \in \mathbf{C}, a_1 \neq 0$, and $\lambda_i \geq 0$ is a monotonically increasing sequence of real numbers such that $lim_{i\to+\infty}\lambda_i = +\infty$. The valuation map is given by $v(f) = \lambda_1$. The corresponding valuation field $\mathbf{C}_\epsilon$ consists of series f as above, with $\lambda_i \in \mathbf{R}$. It is useful to notice that $\mathbf{C}_\epsilon^{\geq 0}$ contains the maximal ideal $\mathbf{C}_\epsilon^{>0}$ consisting of series with all $\lambda_i > 0$. One observes that the composition $m_0 = 0$ modulo $\mathbf{C}_\epsilon^{>0}$, but in general $m_0 \neq 0$.

It can be proved that the disjoint union of non-commutative dg-manifolds associated with generalized A_∞-algebras $A(L,\rho)$ gives rise to a generalized A_∞-category $F_\epsilon(X)$ over the formal spectrum of the ring $\mathbf{C}_\epsilon^{\geq 0}$. Its reduction modulo the ideal $\mathbf{C}_\epsilon^{>0}$ is equivalent to $F_0(X)$. If the symplectic form ω sat-

isfies certain rationality conditions then one can introduce a new parameter $q = exp(-\frac{1}{\epsilon})$, thus replacing the ring $\mathbf{C}_{\epsilon}^{\geq 0}$ by the ring of formal series $\mathbf{C}[[q]]$. Then the maximal ideal is just $q\mathbf{C}[[q]]$, and the field $\mathbf{C}_{\epsilon}$ coincides with the field of Laurent series $\mathbf{C}((q))$.

The Fukaya category $F(X)$ is defined as an A_{∞}-category obtained from $F_{\epsilon}(X)$ by restriction to zeros of the odd vector field. In particular, for an object A of $F(X)$ the composition m_0 vanishes: $m_0^A = 0$. The condition $m_0^A = 0$ defines a "subvariety" of the non-commutative moduli space $\mathcal{M}^{NC}_{Ob(F_{\epsilon}(X))}$. Objects of the Fukaya category exist only along this "subvariety". This geometric picture explains why it is too naive to work with the Fukaya category for fixed ϵ, even if one can prove convergence of the series defining m_n (the latter is still an open problem).

Let us assume for simplicity the above-mentioned rationality conditions of ω. Then the Fukaya category is defined over the formal spectrum $Spf(\mathbf{C}[[q]])$. In fact one can extend the definition so that the base will be $Spf(\mathbf{C}[[q]] \otimes (\otimes_{i \neq 2}\mathbf{C}[t_{i,\mu}]))$, where q has degree zero, and $t_{i,\mu}$ are parameters of degrees $2-i$ corresponding to some basis in the graded vector space of cohomology $H^i(X, \mathbf{C})$. We introduce new parameters z, t of degree zero by setting $z = qe^t$. Then, inverting z we obtain a family of A_{∞}-categories over the field $\mathbf{C}((z))$ parametrized by $Spf(\mathbf{C}[[t]] \otimes (\otimes_{i \neq 2}\mathbf{C}[t_{i,\mu}]))$. This family should be thought of as formal deformation of a certain A_{∞}-category over $\mathbf{C}((z))$. The tangent space to the moduli space of the formal deformations of this category is isomorphic to the cohomology $H^{\bullet}(X, \mathbf{C}((z)))$. The latter cohomology group is isomorphic to $\oplus_{i \geq 0} Ext^i(Id, Id)$, where Id is the identity functor, and the extensions are taken in the properly defined A_{∞}-category of endofunctors. This description is useful for the purposes of quantum cohomology (Yoneda product on functors gives rise to the quantum product on the cohomology group).

3.2 Conventional approach to the Fukaya category

Below we briefly recall the "naive" definition of the Fukaya category, when the composition m_0 is ignored. It is useful in some questions, for example in mirror symmetry for abelian varieties (see [KoSo1]). As we will discuss below, this "naive" Fukaya category is related to deformation quantization.

Objects of $F^{naive}(X)$ are pairs (L, ρ), where $L \subset X$ is a Lagrangian submanifold and ρ is a local system on L. Morphisms between (L_0, ρ_0) and (L_1, ρ_1) are defined only if L_0 and L_1 intersect transversally. In this case

$Hom((L_0,\rho_0),(L_1,\rho_1)) = \oplus_{x\in L_0\cap L_1} Hom(\rho_{0x},\rho_{1x})\otimes \mathbf{C}_\epsilon$. The space of morphisms is $\mathbf{Z}$-graded by means of Maslov index. Thus we are dealing with graded Lagrangian manifolds (cf. [Se1]). There are higher compositions $m_n, n\geq 1$, which are linear maps of degrees $2-n$:

$$m_n : \otimes_{0\leq i\leq n} Hom((L_i,\rho_i),(L_{i+1},\rho_{i+1})) \to Hom((L_0,\rho_0),(L_n,\rho_n)).$$

They are defined by means of the Floer-type construction associated with the "transversal" collection of Lagrangian submanifolds $L_i, 0\leq i\leq n$. It is usually said that the maps m_n give rise to an A_∞-structure on $F^{naive}(X)$. We refer the reader to [KoSo1] about the details of this definition and to [FOOO] about definitions of the related moduli spaces.

There are several problems with the naive definition of the Fukaya category. One of the most essential is the presence of pseudo-holomorphic discs with the boundary mapped to a Lagrangian submanifold. This amounts to non-trivial maps $m_0 : \mathbf{C}_\epsilon \to Hom(X,X)$. As a result, the axioms of A_∞-category are not satisfied, and one has to work with generalized A_∞-categories. It was explained in the preceeding sections (for all details see [KoSo2]) how a consistent theory of such can be developed in the framework of non-commutative formal geometry. One can still work with $F^{naive}(X)$ with understanding that it is only a part of the "true" Fukaya category $F(X)$.

4 Reminder on deformation quantization

We recall that a symplectic 2n-dimensional manifold (X,ω) gives rise to an abelian category $\mathcal{C}(X)$ of modules over a non-commutative algebra $A(X)$ (deformation quantization of the algebra $C^\infty(X)$ of smooth functions on X). Such a deformation quantization is non-unique. We will use the one which has the characteristic class $[\omega]/t$ (see [De]). The algebra $A(X)$ is a topological algebra over the ring of formal series $\mathbf{C}[[t]]$. As $\mathbf{C}[[t]]$-module it is isomorphic to the algebra of formal series $C^\infty(X)[[t]]$. The algebra $A(X)$ consists of global sections of a sheaf of non-commutative algebras A_X, such that locally A_X is isomorphic to the sheaf of t-pseudo-differential operators on $\mathbf{R}^n$ (the latter are locally series $P=\sum_{|I|\geq 0} a_I(x)(t\partial_x)^I$). The Poisson structure induced on $C^\infty_X \simeq A_X/tA_X$ coincides with the one given by the symplectic form.

One has a category of $A(X)$-modules M such that M is t-adically complete, flat as $\mathbf{C}[[t]]$-module, and M/tM is the space of sections of a sheaf of modules over the sheaf of smooth functions C^∞_X. The category of A_X-modules will be denoted by $\mathcal{C}(X)$. Morphisms are defined as $\mathbf{C}[[t]]$-linear homomorphisms of topological modules. We will keep the same notation for the related category defined over the field $\mathbf{C}((t))$. It is obtained from $\mathcal{C}(X)$ respectively $\mathcal{C}_X$ by adding t^{-1}, so that modules V and tV become equivalent.

Let $hol(X)$ be a full subcategory of $\mathcal{C}(X)$ which consists of modules M such that the support $Supp(M/tM)$ is a Lagrangian submanifold.

We will call objects of $hol(X)$ *holonomic*. The Lagrangian support $Supp(M)$ of a holonomic module will be sometimes called its *characteristic variety* of M and denoted by $Ch(M)$. The category $hol(X)$ contains objects $V_{(L,\rho)}$ which correspond to pairs (L,ρ) where $L \subset X$ is a Lagrangian submanifold and ρ is a local system on L. In what follows only such objects will be considered.

Remark 3 *In [KaMa] the authors constructed (for every Lagrangian submanifold L satisfying some topological conditions) an A_X-module V_L such that $Ch(V_L) = L$. One can easily generalize their construction including local systems on L. We will call the corresponding objects Karasev-Maslov modules.*

We will need symplectic manifolds $X^n, n \geq 2$. The corresponding symplectic forms are given by $(\omega, -\omega, -\omega, ..., -\omega)$.

The identity functor Id_{A_X-mod} is represented by the $A_{X\times X}$-module K_Δ supported on the diagonal $\Delta \subset X \times X$. It can be identified with the sheaf A_X. Deformations of $A_X - mod$ as an A_∞-category correspond to the deformations of the identity functor Id_{A_X-mod}. The latter are described by the deformations of K_Δ as an object of $A_{X\times X} - mod$. The tangent space at K_Δ to the moduli space of its deformations is isomorphic to $\oplus_{i\geq 0} Ext^i_{A_{X\times X}-mod}(K_\Delta, K_\Delta)$. After changing scalars to $\mathbf{C}((t))$ the latter sum can be identified with $H^\bullet(X, \mathbf{C}((t)))$. We can restrict the deformation functor to the subcategory $hol(X)$. It is not difficult to see that the support of a holonomic module remains Lagrangian. These observations lead to the following result.

Proposition 1 *a) The Hochschild cohomology of the category $A_X - mod$ is isomorphic to $H^\bullet(X, \mathbf{C}((t)))$. (Here we consider $A_X - mod$ as an A_∞-category with $m_{n\neq 2} = 0$ and m_2 given by the usual composition of morphisms).*

b) The tangent space to the deformations of an object $(L,\rho) \in hol(X)$ is isomorphic to $H^\bullet(L, End(\rho)) \otimes \mathbf{C}((t))$.

We also mention the following proposition (see [So1]). It should be compared with the definition of $Hom's$ in the Fukaya category.

Proposition 2 *Let $V_{(L,\rho)}$ denotes the object of hol_X correspodning to the pair (L,ρ). If L_0 is transversal to L_1 then*

a) $Ext^i(V_{(L_0,\rho_0)}, V_{(L_1,\rho_1)})$ is trivial if $i \neq n$, where $n = 1/2 \dim X$;

b) $Ext^n(V_{(L_0,\rho_0)}, V_{(L_1,\rho_1)}) \simeq \oplus_{x \in L_0 \cap L_1} Hom(\rho_{0x}, \rho_{1x}) \otimes \mathbf{C}((t))$;

c) the algebra $Ext^\bullet(V_{(L,\rho)}, V_{(L,\rho)})$ is isomorphic to the cohomology $H^\bullet(L, End(\rho)) \otimes \mathbf{C}((t))$ (cf. part b) of the previous Proposition).

Let $D^b_\infty(hol(X))$ be the A_∞-category associated with the category $hol(X)$. It is in fact a dg-category. In order to construct it one choses injective resolutions of A_X-modules I_M and I_N of two holonomic modules M and N. Then one defines $Hom_{D^b_\infty(hol(X))}(M,N) = Hom^\bullet(I_M, I_N)$. In this way one obtains an A_∞-model for the derived category of the category of holonomic modules.

The discussion above shows certain similarity between $D^b_\infty(hol(X))$ and $F^{naive}(X)$. At the same time their deformation theories induce different products on the cohomology of X. In case of the Fukaya category it is the quantum product, while in case of $hol(X)$ it is the usual cup product. The reader will also notice that Maslov index is not visible in the case of $hol(X)$.

On the other hand we will explain in the next section that:

a) The algebras $End_{D^b_\infty(hol(X))}((L,\rho))$ and $End_{F^{naive}(X)}((L,\rho))$ are A_∞-equivalent.

b) If L and L' are Hamiltonian isotopic, then we can "twist" the space $Hom_{D^b_\infty(hol(X))}((L,\rho),(L',\rho'))$ in such a way that it becomes quasi-isomorphic to the corresponding complex of morphisms in $F^{naive}(X)$.

5 Comparison of the categories

Let (L,ρ) be a pair as before, i.e. L is a Lagrangian submanifold of X and ρ is a local system on L. Let us denote by $E_{(L,\rho)}$ the correspodning object of $F(X)$, and by $V_{(L,\rho)}$ the corresponding object of $hol(X)$. We assume that $m_0 = 0$ in the A_∞-algebra $A(L,\rho)$. This means that in fact we are dealing with the category $F^{naive}(X)$. We also assume the conditions imposed on L in [FOOO]. This allows us to make necessary choices without further explanations. In particular L is relatively spin in the sense of [FOOO], so the moduli spaces of pseudo-holomorphic discs are orientable. Taking

the Hochschild complex of $A(L,\rho)$ we can construct the formal pointed dg-manifold $\mathcal{M}_{E_{(L,\rho)}}$ of deformations of $E_{(L,\rho)}$. Similarly we can start with the Lie algebra $Hom_{D^b_\infty(hol(X))}(V_{(L,\rho)}, V_{(L,\rho)})$ and construct the formal pointed dg-manifold $\mathcal{M}_{V_{(L,\rho)}}$ of deformations of $V_{(L,\rho)}$.

Let us imagine that both $F^{naive}(X)$ and $D^b_\infty(hol(X))$ are "sheaves of A_∞-categories" on the "moduli space of objects". Let us also imagine that there is a well-defined moduli space $Lagr_X$ of Lagrangian submanifolds of X. Then we should have a natural projection $\pi : \mathcal{M}_X \to Lagr_X$. If L is a Lagrangian submanifold, and $[L] \in Lagr_X$ the corresponding point of the moduli space then the fiber $\pi^{-1}([L])$ consists of local systems supported on L (or on any Lagrangian submanifold representing the same equivalence class in the moduli space). We would like to compare $F^{naive}(X)$ and $hol(X)$ in a "small neighborhood of $[(L,\rho)]$". From the categorical point of view we have two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ with the same "space" of objects, and such that for any object X the A_∞-algebras $End_{\mathcal{A}}(X)$ and $End_{\mathcal{B}}(X)$ are equivalent. We would like to find a functor $\Phi : \mathcal{A} \to \mathcal{A}$ such that changing morphisms in $\mathcal{A}$ to $Hom^{new}_{\mathcal{A}}(X,Y) = Hom_{\mathcal{A}}(X, \Phi(Y))$ one gets a new A_∞-category equivalent to $\mathcal{B}$.

5.1 Main conjecture

We will denote by $hol(L)$ the full subcategory of $hol(X)$ consisting of holonomic modules with the given support L. For simplicity we will assume the rationality condition imposed on the symplectic form ω. Hence the Fukaya category is defined over $\mathbf{C}((q))$ (this is not a serious restriction because one can consider deformation quantization over any pro-nilpotent algebra, in particular $\mathbf{C}_\varepsilon$).

Our main idea can be explained such as follows. Having in mind the intuitive picture of the previous subsection we consider both $F^{naive}(X)$ and $D^b_\infty(hol(X))$ in a small neighborhood of a given $[L] \in Lagr_X$. For a pair L_1, L_2 sufficiently close to L we would like to find a functor $\Phi_{L_1,L_2} : D^b_\infty(hol(L_2)) \to D^b_\infty(hol(L_1))$, such that $Hom(\rho_1, \Phi_{L_1,L_2}(\rho_2))$ (morphism as objects of $D^b_\infty(hol(L_1))$) is quasi-isomorphic to $Hom_{F^{naive}(X)}((L_1,\rho_1),(L_2,\rho_2))$. Such a functor should be represented by a bimodule. Let us describe all this more precisely.

Let $M_i = V_{(L_i,\rho_i)} \in hol(L_i), i = 1,2$. We expect there exists a Lagrangian submanifold $\Lambda_{12} = \Lambda(L_1,L_2) \subset X \times X$ and $K(L_1,L_2) = K_{\Lambda_{12}} \in hol(X \times X)$ such that:

1) If L_1 and L_2 have a non-empty intersection then $\Lambda_{12} \circ L_1 = L_2$. Here

$\Lambda \circ L = \pi_2(\pi_1^{-1}(L) \cap \Lambda)$, where $\pi_i : X \times X \to X, i = 1,2$ are the natural projections. In particular we assume that the restrictions of $\pi_i, i = 1,2$ to Λ are coverings.

2) $Hom_{D^b_\infty(hol(X))}(M_1, K(L_1, L_2) \circ M_2) \simeq Hom_{F^{naive}(X)}((L_1, \rho_1), (L_2, \rho_2))$, where $\simeq$ means quasi-isomorphism of complexes, and we consider both categories over the field $\mathbf{C}((q))$ (i.e. $q = t$ in the case of $hol(X)$). The composition $\circ$ for modules is given by the formula $K \circ M = \pi_{2*}(K \otimes \pi_1^*(M))$.

We denote by $Hom^{new}(M_1, M_2)$ the left hand side of 2).

3) For a generic sequence of Lagrangian submanifolds $L_1, L_2, ..., L_n, n \geq 2$ and holonomic modules $M_1, ..., M_n$ such that $M_i = V_{(L_i, \rho_i)}$ for all i, we expect to have an isomorphism of A_{X^n}-modules

$$K(L_1, L_2) \circ K(L_2, L_3) \circ ... \circ K(L_{n-1}, L_n) \to K(L_1, L_n).$$

Such an isomorphism defines a linear map

$$m_n^{new} : \otimes_{1 \leq i \leq n-1} Hom^{new}(M_i, M_{i+1}) \to Hom^{new}(M_1, M_n).$$

We expect the above data satisfy the following

Conjecture 1 *(i) Higher compositions $m_n^{new}, n \geq 1$ give rise to a structure of an A_∞-category on $D^b_\infty(hol(X))$.*

(ii) This category is A_∞-equivalent to $F^{naive}(X, \omega)$ (with $q = t$).

5.2 The conjecture in the case of cotangent bundle

Let us fix a Lagrangian submanifold $L \subset X$, and consider only those L' which are "very close" to L. More precisely we assume that they are not only close to L but also Hamiltonian isotopic to L. We want to "restrict" $F^{naive}(X)$ to this "neighborhood of L". This means that we consider an A_∞-subcategory with the objects taken from the above-mentioned subset, and morphisms same as in $F^{naive}(X)$. We do the same thing with $hol(X)$ and $D^b_\infty(hol(X))$. We would like to compare these categories in the case when $X = T^*Y$ is the cotangent bundle with the standard symplectic structure (notice that a neighborhood of a Lagrangian submanifold L can be identified by a symplectomorphism with a neighborhood of the zero section in T^*L). We are going to consider Lagrangian submanifolds of the type $L_i = \{(x, df_i(x)) | x \in Y\}$. Let ρ_i be local systems on L_i.

For a pair of such Lagrangian submanifolds we have a symplectomorphism $\phi : X \to X$ such that $(x, \xi) \mapsto (x, \xi + df_2(x) - df_1(x))$. Clearly it maps isomorphically L_1 into L_2 .

Let us define $\Lambda = \Lambda_{12}$ as $graph(\phi) \subset X \times X$. The corresponding bimodule K_Λ is the quotient of $A_X \boxtimes A_X^{op}$ by the left ideal generated by the relation $a \otimes 1 = 1 \otimes e^{\frac{1}{t} ad(f_2 - f_1)}(a)$, $a \in A_X$. Here $ad(a)(b) = ab - ba$ (clearly A_X contains C_Y^∞ as a subalgebra, so the ideal is well-defined).

Notice that

$exp\frac{1}{t}(ad(f_2 - f_1))exp\frac{1}{t}(ad(f_3 - f_2))...exp\frac{1}{t}(ad(f_n - f_{n-1})) = exp\frac{1}{t}(ad(f_n - f_1))$. Hence we have an isomorphism $K(L_1, L_2) \circ K(L_2, L_3) \circ ... \circ K(L_{n-1}, L_n) \to K(L_1, L_n)$.

In order to check the conjecture we may assume that $f_1 = 0$. Then we observe that

$$Hom_{D^b_\infty(hol(X))}(\rho_1, K(L_1, L_2) \circ \rho_2) = \Omega^\bullet(Y, \rho_1^* \otimes \rho_2)$$

where the RHS is the complex of de Rham forms with values in the local system. Let $\nabla_i, i = 1, 2$ denotes the flat connection on $\rho_i, i = 1, 2$. Then the differential is given by $\nabla_1^* \otimes 1 + 1 \otimes \nabla_2 + df_2 \wedge (\cdot)$. The latter complex is equivalent to the standard de Rham complex (without df_2) if one twists the sections by $exp(\frac{1}{t} f_2)$, i.e. $s \mapsto s\, exp(\frac{1}{t} f_2)$. Then according to [KoSo1], Section 4 the resulting complex is quasi-isomorphic to $Hom_{F^{naive}(X)}((L_1, \rho_1), (L_2, \rho_2))$. We remark that in the notation of [KoSo1] one has $q = exp(-\frac{1}{\varepsilon})$.

Remark 4 *The case of general symplectic manifold does not follow automatically from the results of this section. Indeed, in order to define morphisms in the Fukaya category for Lagrangian submanifolds in a small neighborhood of a given L one has to consider pseudo-holomorphic discs which do not belong entirely to the neighborhood (we are restricted by the boundary conditions only). Nevertheless, if our main conjecture is true, one can find a family of kernels $K(L_1, L_2)$ which takes care about such discs.*

5.3 Complex structure on the moduli space of holonomic modules

We observe that the tangent space to the moduli space of deformations of a module $M \in hol(X), supp(M) = L$ is isomorphic to $Hom_{D^b_\infty(hol(X))}(M, M)$ (derived deformations of $Hom_{hol(X)}(Id_X, Id_X)$). There is a natural embedding $Hom_{D^b_\infty(hol(L))}(M, M) \to Hom_{D^b_\infty(hol(X))}(M, M)$, corresponding to the

deformations with the fixed support L. On the other hand there is a natural projection $Hom_{D^b_\infty(hol(X))}(M,M) \to \Omega^\bullet(L)$, where $\Omega^\bullet(L)$ denotes the de Rham complex of L. Indeed, a deformation of the module M induce the deformation of the support of M. The latter are controlled by differential forms on the support. We can perform computations in the derived categories. Then we have an exact sequence of the tangent spaces to the formal moduli spaces of deformations:

$$Ext^\bullet_{hol(L)}(M,M) \to Ext^\bullet_{hol(X)}(M,M) \to H^\bullet_{DR}(L).$$

Suppose that M is a simple module. Then

$$RHom_{hol(L)}(M,M) \simeq R\Gamma(L, R\underline{Hom}(M,M)) \simeq R\Gamma(L, \mathbf{C}_L) \simeq \Omega^\bullet(L).$$

Taking first cohomology (this corresponds to "classical" tangent space) we obtain in this case an exact sequence

$$H^1_{DR}(L) \to Ext^1_{hol(X)}(M,M) \to H^1_{DR}(L).$$

This means that the tangent space to the "classical" deformations of M inside of $hol(X)$ is twice as big as the tangent space to the "classical" deformations of L inside of $Lagr_X$. If X is a Calabi-Yau manifold then one hopes to obtain a complex structure on the moduli space of formal deformations of a simple holonomic module M. We expect it is isomorphic to a subvariety in the dual Calabi-Yau manifold. In order to describe the complex structure explicitely we need to identify the tangent space $T_L(Lagr_X)$ with the tangent space $T_M(hol(L))$. It is sufficient to describe the lifting of paths from $Lagr_X$ to $hol(L)$. Given a path $L(t) \subset Lagr_X$ such that $L(0) = L$, we define a path $M(t) \subset hol(L), M(0) = M$ in the following way: $M(t) = M \otimes \rho(t)$. Here $\rho(t)$ is the restriction to $L(t)$ of the unitary bundle over X with a connection ∇ such that $curv(\nabla) = \omega_X$ (it is often called the pre-quantum line bundle). Restriction of ∇ to $L(t)$ is a flat connection.

6 Conclusion

We have suggested the way to compare the Fukaya category with the category of holonomic modules over the quantized algebra of smooth functions. The idea is for a pair of Lagrangian submanifolds L_1, L_2 to find a kernel $K(L_1, L_2)$ which transforms local systems (or more general A_X-modules) supported

on L_2 to local systems supported on L_1. We have conjectured that it is possible to make these choices in such a way that the counting of instantons (i.e. higher compositions in the Fukaya category) can be replaced by pure algebraic operation of taking homomorphisms between local systems having the same Lagrangian support. We have checked the conjecture in the simplest case. It would be interesting to check it in other cases as well as "globalize" this picture, finding kernels $K(L_1, L_2)$ for Lagrangian submanifolds which are not close to each other.

References

[AP] D. Arinkin, A. Polishchuk, Fukaya category and Fourier transform, math.AG/9811023.

[Be] V. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs, 33. Amer. Math. Soc., 1990.

[BD] A. Beilinson, V. Drinfeld, Chiral algebras, preprint.

[BK] S. Barannikov, M. Kontsevich, Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields, alg-geom/9710032.

[De] P. Deligne, Defortmations de l'algebre des fonctions d'une variete symplectique. Selecta Math. 1 (1995), 667-698.

[FOOO] K. Fukaya, Y.G.Oh, H. Ohta, K.Ono, Lagrangian intersection Floer theory-anomaly and obstruction. Preprint Kyoto University (2000).

[Fu1] K. Fukaya, Floer homology of Lagrangian foliation and non-commutative mirror symmetry. Preprint 98-08, Kyoto University, 1998.

[Gi] V. Ginzburg, Characteristic varieties and vanishing cycles, Inv. Math. 84 (1986), 327-402.

[GLO] V. Golyshev, V. Lunts, D. Orlov, Mirror symmetry for abelian varieties, math.AG/9812003.

[GM] M. Grinberg, R. Macpherson, Euler characteristics and Lagrangian intersections, in Symplectic Geometry and Topology (ed. Ya. Eliashberg), 1999, 265-294.

[K] A. Kapustin, D-branes in a topologically nontrivial B-field, hep-th/9909089

[KaMa] M. Karasev, V. Maslov, Pseudo-differential operators and canonical operator on general symplectic manifolds. Izvestia AN SSSR, 47:5 (1983), 999-1029 (in Russian).

[KaV], M. Kapranov, E. Vasserot, Vertex algebras and the formal loop space, math.AG/0107143.

[KO] A. Kapustin, D. Orlov, Remarks on A-branes, mirror symmetry and the Fukaya category, hep-th/0109098

[Ko1] M. Kontsevich, Homological algebra of Mirror symmetry. Proceedings of the ICM in Zurich, vol. 1, p. 120-139.

[Ko2] M. Kontsevich, Deformation quantization of Poisson manifolds, I, q-alg/9709040.

[Ko3] M. Kontsevich, Operads and Motives in Deformation Quantization, math.QA/9904055.

[KR] M. Kontsevich, A. Rosenberg, Non-commutative smooth spaces, math.AG/9812158.

[KoSo1] M. Kontsevich, Y. Soibelman, Homological mirror symmetry and torus fibrations, math.SG/0011041

[KoSo2] M. Kontsevich, Y. Soibelman, Deformation Theory (book, to appear)

[KoSo3] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture", math.QA/0001151.

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, math.AG/9803041.

[Se1] P. Seidel, Vanishing cycles and mutations, math.SG/0007115.

[So1] Y. Soibelman, Quantum tori, mirror symmetry and deformation theory, math.QA/0011162 and Lett. Math.Phys. v.56 (2001).

[SYZ] A. Strominger, S-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B479 (1996), p.243.

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