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THE ALTERNATING GROUP OF DEGREE 6 IN GEOMETRY OF THE LEECH LATTICE AND K3 SURFACES

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Abstract. The alternating group of degree 6 is located at the junction of three series of simple non-commutative groups : simple sporadic groups, alternating groups and simple groups of Lie type. It plays a very special role in the theory of finite groups. We shall study its new roles both in a finite geometry of certain pentagon in the Leech lattice and also in a complex algebraic geometry of $K3$ surfaces.

1. Introduction

The alternating group A_6 plays a very special role in the theory of finite groups [Su]. Indeed, A_6 is a simple group which is located at a sort of junction of three series of simple non-commutative groups : simple sporadic groups, alternating groups and simple groups of Lie type. More explicitly, there are distinguished isomorphisms $[M_{10}, M_{10}] \simeq A_6 \simeq \mathrm{PSL}(2,9)$; see for instance Conway-Sloane [CS, Chapter 10]. Though M_{10}, the Mathieu group of degree 10, is not itself a simple group, it falls into a sequence $M_{10} < M_{11} < M_{12}$ of maximal subgroups of the smallest sporadic simple groups M_{11} and M_{12}.

From a slightly different view, in contrast to the fact that $\mathrm{Aut}(A_n) \simeq S_n$, whence $\mathrm{Out}(A_n) \simeq C_2$ when $n \neq 6$ and $n \geq 3$, the outer automorphism group $\mathrm{Out}(A_6)$ is isomorphic to a bigger group $C_2^{\oplus 2}$. Corresponding to the three involutions, $\mathrm{Aut}(A_6)$ has three index 2 subgroups $A_6 < G < \mathrm{Aut}(A_6)$, which are S_6, $\mathrm{PGL}(2,9)$ and M_{10}. According to Suzuki [Su, Page 300], it is this extraordinary property which seems to make the classification of simple groups deep and difficult. This property of A_6 also plays a crucial role in our note (the proof of Proposition 2.6).

The aim of this note is to study roles played by A_6 both in a finite geometry in the Leech lattice Λ (or in a slightly different language, in the set of Leech roots of $II_{1,25} := \Lambda \oplus U$) and also in a complex algebraic geometry of $K3$ surfaces (Theorems 2.3, 3.1, 5.1 and Proposition 3.5).

We first show that A_6 can be characterized as the pointwise stabilizer group of some uniquely determined pentagon in the Leech lattice, or equivalently, a special configuration of Leech roots of Coxeter-Dynkin type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$ (Theorem 2.3). This is an analogue of results by Curtis for S-lattices [Cu] and by Finkelstein for some maximal subgroups of the Conway group $\cdot 3$ [Fi] and a table in [CS, Page 291]. On the Leech lattice, Leech roots and the Conway groups, we refer the readers to the standard reference book [CS, Chapter 10, 28]; see also Section 2 below for a

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brief summary. We note that our A_6 is in the subgroup $\cdot 3$ of $\cdot 0 (:= O(\Lambda))$ but not in the Mathieu subgroups $M_{22} < M_{24}$ embedded in $\cdot 0$ in a standard way [Cu].

We then apply this characterization in our study of group symmetries on $K3$ surfaces. By definition, a $K3$ surface is a simply-connected compact complex surface admitting a nowhere vanishing global holomorphic 2-form. They form 20-dimensional moduli; for more details about $K3$ surfaces, see for instance [BPV]. According to Mukai [Mu, Main theorem], there are exactly 11 maximal finite groups each of which acts on some $K3$ surface symplectically. They all can be embedded into the Mathieu group M_{23}. Among 11 such groups, simple groups are only $\mathrm{PSL}(2,7)$ and the present $A_6 = \mathrm{PSL}(2,9)$.

Our goal is to show the existence and uniqueness of the triplet $(F, \tilde{A}_6, \rho_F)$ of a $K3$ surface F and its finite group action $\rho_F : \tilde{A}_6 \times F \longrightarrow F$ of $\tilde{A}_6$ on F, up to isomorphisms (Theorems 3.1 and 5.1). Here the group $\tilde{A}_6$ is an extension of A_6 by $\mu_4 \simeq C_4$, which turns out to be the unique maximal possible finite extension of A_6 in the automorphism groups of $K3$ surfaces (Theorem 5.1 and Proposition 4.1). This part is much inspired by the work of Kondo [Ko2, Ko3]. We remark that a work for the other simple group $\mathrm{PSL}(2,7)$ has been carried out in [OZ2].

Our $K3$ surface F turns out to be isomorphic to the minimal resolution of the branched double cover of the elliptic modular surface with level 3 structure. We will write down explicitly the bidegree $(2,3)$ equation for a canonical model of F in $\mathbf{P}^1 \times \mathbf{P}^2$ (Proposition 3.5). On the other hand, we also see that $\mathrm{Pic}(F)^{\tilde{A}_6} = \mathbf{Z}H_F$ and $(H_F^2) = 20$ (Propositions 4.1 and 4.5). So the action of $\tilde{A}_6$ on F is not induced by $\mathrm{PGL}(\mathbf{P}^1) \times \mathrm{PGL}(\mathbf{P}^2)$. The invariant degree $(H_F^2) = 20$ also tells that our example (F, A_6) $(A_6 < \tilde{A}_6)$ is not isomorphic to Mukai's example (X_6, A_6) of polarized $K3$ surface of degree 6 with symplectic group action of A_6 in [Mu, Example 0.4, No. 2]. One can check that in Mukai's example the maximal extension of A_6 in the full automorphism group $\mathrm{Aut}(X_6)$ is the symmetric group S_6.

We also remark that one can construct a smooth non-isotrivial family of $K3$ surfaces $f : \mathcal{X} \longrightarrow \mathbf{P}^1$ such that the fibres $\mathcal{X}_t$ admit A_6-actions in exactly the same manner as in [OZ2, Appendix].

It would be very interesting to see the full automorphism group $\mathrm{Aut}(F)$, which is of infinite order [SI]. In this direction, readers may refer to [Vi], [Ko1], [KK] and [DK] for other $K3$ surfaces.

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2. Uniqueness of the Leech roots of Coxeter-Dynkin type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$

First we briefly recall some necessary notations and facts about Leech lattice from [CS, Chapter 10], [Ko1] and [DK]. Let

$$\Omega := \mathbf{P}^1(\mathbf{F}_{23}) = \{\infty\,,\, 0\,,\, 1\,,\, \cdots\,,\, 22\}\,.$$

We denote by 2^Ω the power set of Ω. 2^Ω has a structure of 24-dimensional vector space over $\mathbf{F}_2$, in which the sum is defined to be the symmetric difference. Let $N := \Omega - \{a^2 | a \in \mathbf{F}_{23}\}$, and $N_\infty := \Omega$, $N_i := \{n - i | n \in N\}$ $(i \in \mathbf{F}_{23})$. The

subspace $\mathcal{C}$ spanned by N_i ($i \in \Omega$) is called the binary Golay code. We call an element of $\mathcal{C}$ a $\mathcal{C}$-set. Let K be a $\mathcal{C}$-set, i.e. $K \in \mathcal{C}$. Then as a subset of Ω, $|K|$ is either 0, 8, 12, 16 or 24. An element $K \in \mathcal{C}$ is called an octad (resp. a dodecad) if $|K| = 8$ (resp. if $|K| = 12$). It is well-known that the set of octads forms the so-called Steiner system $St(5, 8, 24)$ of Ω. There are exactly 759 octads and they are explicitly listed in [To].

Next consider the 24-dimensional Euclidean space $\mathbf{R}^{24}$ with orthonormal basis $\langle \mu_i \rangle_{i\in\Omega}$, i.e. $\mu_i.\mu_j = \delta_{ij}$ for $i, j \in \Omega$. For any subset A of Ω, let μ_A denote the vector $\sum_{i\in A} \mu_i$. Let Λ be the $\mathbf{Z}$-submodule of $\mathbf{R}^{24}$ spanned by the vectors $2\mu_K$ and $\mu_\Omega - 4\mu_\infty$, where K runs through all octads. Define the bilinear form on Λ by $(U, V) := -U.V/8$ for $U, V \in \Lambda$. Then it is well known that $(U, V) \in \mathbf{Z}$ and $\Lambda := (\Lambda, (*, *))$ forms an even unimodular negative definite lattice of rank 24. To be precise, this is the so called *Leech lattice*. It is also well known that Λ contains no element V with $(V^2) = -2$.

The following Theorem (see for instance [CS, Chapter 10, Theorem 25]) gives us a more concrete picture of the Leech lattice:

Theorem 2.1. *The vector $\sum_{i\in\Omega} x_i\mu_i$ ($x_i \in \mathbf{Z}$) is in Λ if and only if the following four conditions are satisfied:*

(i) $x_\infty \equiv x_0 \equiv x_1 \equiv \cdots \equiv x_{21} \equiv x_{22} \,(mod 2)$; denote by $m \equiv x_\infty \,(mod 2)$.
(ii) For each $j \in \{0, 1, 2, 3\}$, the set $\{i \in \Omega \,|\, x_i \equiv j \,(mod 4)\}$ is a $\mathcal{C}$-set.
(iii) $\sum_{i\in\Omega} x_i \equiv 4m \,(mod 8)$.

For instance, from this theorem, one knows that every element V with $(V^2) = -4$ has one of the forms: $((\pm 2)^8, 0^{16})$, where the non-zero coordinates have positive product and are in the place of an octad; $(\mp 3, (\pm 1)^{23})$, where the lower signs are taken on a $\mathcal{C}$-set; and $((\pm 4)^2, 0^{22})$ with no extra condition.

The orthogonal group of the Leech lattice is denoted by $\cdot 0$. It is well known that $\cdot 0$ acts on the set of vectors $(V^2) = -2m$ ($m = 2$ or 3) transitively. We denote the stabilizer group of a vector V with $(V^2) = -2m$ by $\cdot m$. For more details about the Leech lattice, see for instance [CS, Chapter 10].

Let Λ be the Leech lattice and $\Pi = II_{1,25} := \Lambda \oplus U$ the unique even unimodular lattice of index $(1, 25)$. Here U is the hyperbolic plane, i.e. the lattice $\mathbf{Z}^2$ equipped with a bilinear form $((l_1, m_1), (l_2, m_2)) = l_1m_2 + l_2m_1$. This U is the unique even unimodular hyperbolic lattice. Let $w := (0, 0, 1) \in \Pi$ be the Weyl vector. Set

$$\Pi_2 := \{r \in \Pi \,|\, (r^2) = -2,\ (r, w) = 1\}.$$

An element of Π_2 is called a Leech root of Π. The positive cone, denoted by $\mathcal{P}$, is the one of the two connected components of $\{v \in \Pi \otimes \mathbf{R} | (v^2) > 0\}$ whose closure contains w. We set

$$\mathcal{D} = \{v \in \mathcal{P} \,|\, (v, r) \geq 0, \quad \forall r \in \Pi_2\}.$$

It is well known that the correspondence

$$\Lambda \ni X \leftrightarrow x := (X, 1, -\frac{X^2}{2} - 1) \in \Pi_2$$

between Λ and Π_2 is bijective, and under this bijection, we have a natural identification $\cdot\infty = \mathrm{Aut}(\mathcal{D})$ [CS, Chapter 27, Theorem 1]. Here $\cdot\infty = \Lambda : \cdot 0$ is the group of affine isometries (namely including translations) of Λ. We also note that the Weyl vector w is stable under $\mathrm{Aut}(\mathcal{D})$, i.e. $\varphi(w) = w$ for each $\varphi \in \mathrm{Aut}(\mathcal{D})$.

Let us consider the following 6 vectors in Λ:

$$C = 4\nu_\infty + \nu_\Omega\,, \quad Z = 0\,, \quad X_0 = 4\nu_\infty + 4\nu_0\,,$$

$$R_0 = 2\nu_{K_0}\,, \quad X_1 = 2\nu_{K_1}\,, \quad X_2 = 2\nu_{K_2}\,,$$

where

$$K_0 = \{\infty, 1, 2, 3, 4, 6, 15, 18\}\,,$$

$$K_1 = \{\infty, 0, 1, 2, 3, 5, 14, 17\}\,, \quad K_2 = \{\infty, 0, 1, 2, 4, 13, 16, 22\}\,.$$

Let

$$c = (C, 1, 2)\,, \quad z = (0, 1, -1)\,, \quad x_0 = (X_0, 1, 1)\,,$$

$$r_0 = (R_0, 1, 1)\,, \quad x_1 = (X_1, 1, 1)\,, \quad x_2 = (X_2, 1, 1)$$

be the corresponding Leech roots. We employ here the same notation as in [DK], but we rename X there as C here. Set:

$$\mathcal{R} := \{c, z, x_0, r_0, x_1, x_2\}\ .$$

It is easy to verify that $\mathcal{R}$ forms a Coxeter-Dynkin diagram of type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$:

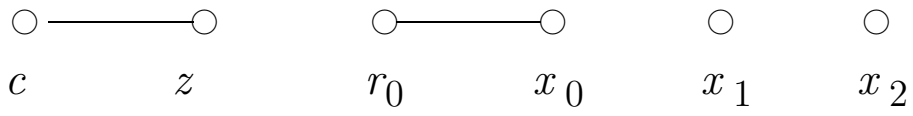

Figure 1

We first remark the following easy but important fact. We recall a few necessary notations. Let $(M, \langle *, * \rangle)$ be an even non-degenerate lattice and $M^* = \mathrm{Hom}\,(M, \mathbf{Z})$ its dual. We embed $M \subseteq M^*$ by the non-degeneracy of $\langle *, * \rangle$, and define $A_M := M^*/M$. Since M is even, we have a natural quadratic form $q_M : A_M \longrightarrow \mathbf{Q}/2\mathbf{Z}$ given by: $v \bmod M \mapsto v^2 \bmod 2\mathbf{Z}$.

Lemma 2.2. *Let $\mathcal{R}' = \{c', z', x_0', r_0', x_1', x_2'\} \subset \Pi_2$ be a set consisting of six Leech roots forming a Coxeter-Dynkin diagram of type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$. Let R' be the sublattice of Π generated by $\mathcal{R}'$. Then R' is isomorphic to $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$ and R' is a primitive sublattice of Π.*

Proof. Note that the discriminant group $A_{R'} \simeq (\mathbf{Z}/3)^2 \oplus (\mathbf{Z}/2)^2$ is generated by $(c' + 2z')/3$, $(r_0' + 2x_0')/3$, $x_1'/2$, $x_2'/2$. A direct calculation shows that $A_{R'}$ contains no isotropic element, whence $R'^* \cap \Pi = R'$. This completes the proof. □

Note also that the five vectors $C, X_0, R_0, X_1, X_2 \in \Lambda$, which are projections of the Leech roots c, x_0, r_0, x_1, x_2, form the following pentagon, where the number on an edge is the intersection number of vectors joined by the edge, e.g. $(R_0,\, X_0) = -1$, $C^2 = -6$ (and the meaning of numbers is different from that in [Cu]):

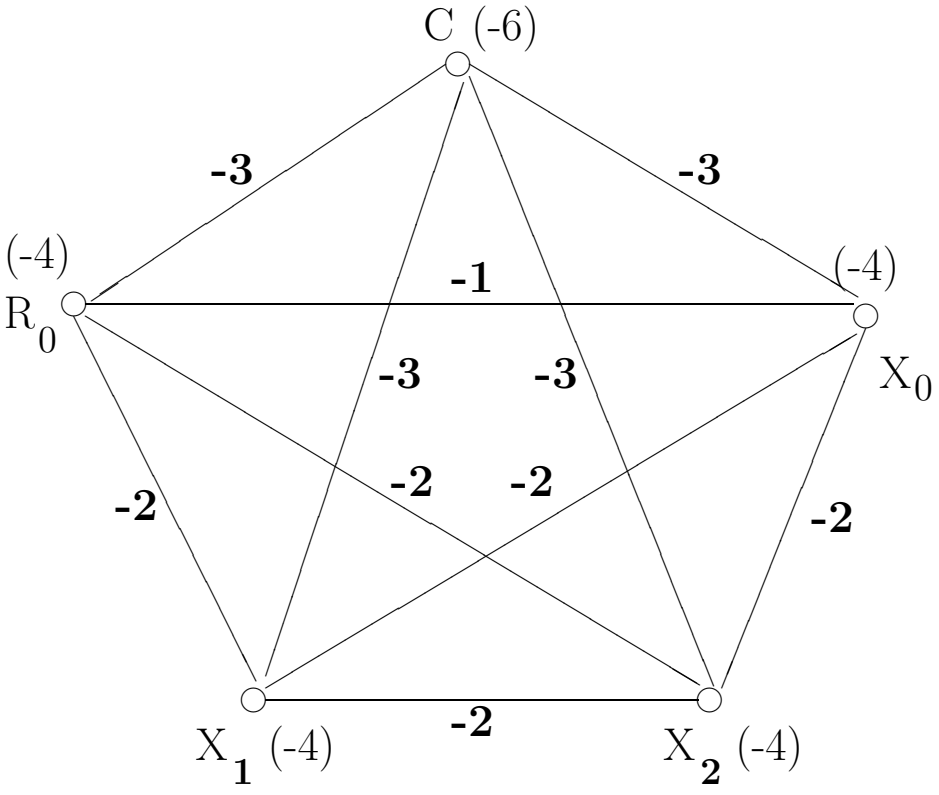

Figure 2

The aim of this section is to characterize the alternating group A_6 as the subgroup of $\mathrm{Aut}(\mathcal{D})$ fixing the six vertices of a Coxeter-Dynkin diagram of type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$.

Theorem 2.3. *(1) Up to Aut($\mathcal{D}$), the graph $\mathcal{R}$ is the unique Coxeter-Dynkin diagram of type $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$, i.e. if $\mathcal{R}' \subset \Pi_2$ is another set of 6 Leech roots forming $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$, then there is $\varphi \in Aut(\mathcal{D})$ such that $\varphi(\mathcal{R}') = \mathcal{R}$.*
(2) The group $Aut(\mathcal{D}, \mathcal{R})$, the pointwise stabilizer of the set $\mathcal{R}$, is isomorphic to the alternating group A_6.

Proof. As before, we denote by X the element of Λ correspondng to $x \in \Pi_2$. Since $Z = 0$, the stabilizer group of z is contained in $\cdot 0$ under the identification $\mathrm{Aut}(\mathcal{D}) = \cdot\infty$. The four vectors $C - X_0$, $C - R_0$, R_0 and X_1 in Λ form the following diagram:

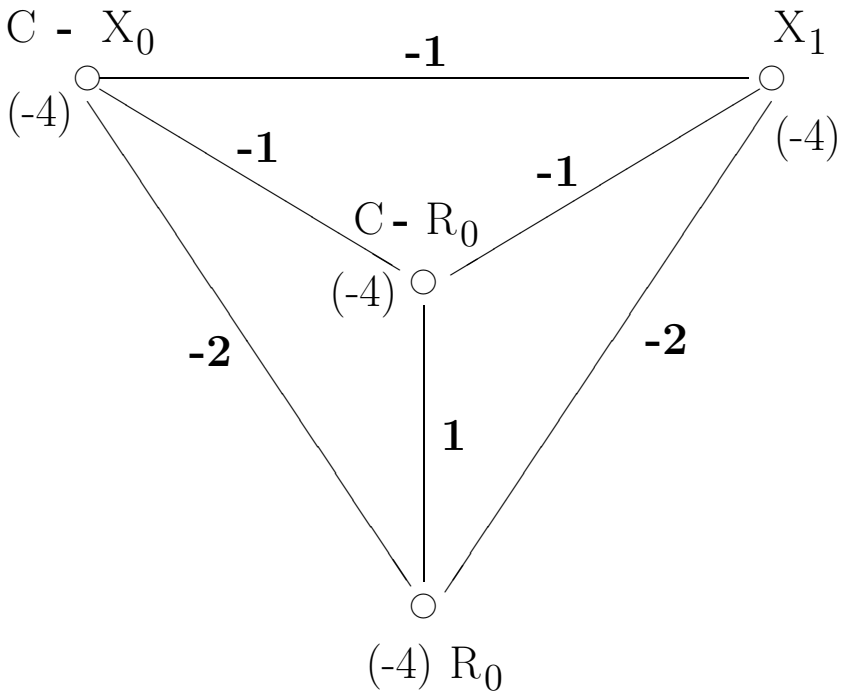

Figure 3

The four vectors generate a 4-dimensional lattice containing exactly nine (-4)-vectors and six (-6)-vectors, yielding a 4-dimensional S-lattice of type $2^9 3^6$ given in [Cu, Page 554]. By [ibid, main theorem] (see also Page 567), such a diagram is unique up to $\cdot 0$.

Let us consider the subgroup of $\cdot 0$ pointwise stabilizing $\{C - X_0, C - R_0, R_0, X_1\}$. This subgroup is obviously the same as the one pointwise stabilizing $\{C, X_0, R_0, X_1\}$. We denote this subgroup by G_0.

Note that $C^2 = -6$. So G_0 is a subgroup of $\cdot 3 < \cdot 0$. By [ibid], $G_0 = 3^4 : A_6$, a semi-direct product of the elementary abelian group $3^4 (= C_3^{\oplus 4})$ and A_6. By [ibid], the lattice M generated by $\{C, X_0, R_0, X_1\}$ is the maximal sublattice on which G_0 acts trivially.

Consider the set

$$\mathcal{S} := \{V \in \Lambda | (V, C) = -3\,, (V, X_0) = (V, R_0) = (V, X_1) = -2\,, V^2 = -4\}.$$

Note that $X_2 \in \mathcal{S}$. By calculating the intersection matrix, we also see that $V \notin \mathbf{Q}\langle C\,,\, X_0\,,\, R_0\,,\, X_1\rangle$ for any $V \in \mathcal{S}$. In particular, $\mathcal{S}^{3^4 : A_6} = \emptyset$ by the maximality of M (this fact will be used in Lemma 2.5).

Lemma 2.4. *We have $|\mathcal{S}| = 81$. In other words, there are exactly 81 Leech roots orthogonal to the five roots c, z, x_0, r_0 and x_1.*

Proof. Since $V^2 = -4$, we have $V = ((\pm 2)^8, 0^{16})$, $(\mp 3, (\pm 1)^{23})$, or $((\pm 4)^2, 0^{22})$.

Case 1: $V = ((\pm 2)^8, 0^{16})$. In this case $V = 2\nu_K$, where K is an octad with $K \ni \infty\,, 0$, $|K \cap K_0| = |K \cap K_1| = 4$. The number of such octads is 30.

Case 2: $V = (\mp 3, (\pm 1)^{23})$. In this case $V = 4\nu_\infty + \nu_\Omega - 2\nu_K$, where $K = \{\infty, j_1, j_2, \ldots, j_7\}$ is an octad with $K \not\ni 0$, $|K \cap K_0| = |K \cap K_1| = 2$. The number of such octads is 48.

Case 3: $V = ((\pm 4)^2, 0^{22})$. In this case $V = 4\nu_\infty + 4\nu_j$, where $j \in \{1, 2, 3\}$. The number of such vectors is 3. □

Now we return to the proof of Theorem 2.3. The subgroup G_0 acts on the set $\mathcal{S}$. Moreover, G_0 acts on the following set $\mathcal{S}'$ of unordered pairs of vectors of norm -4:

$$\mathcal{S}' := \{\{W, C - W\} | W \in \Lambda\,,\, (W, C) = -3\,,\, W^2 = -4\}.$$

This action was studied by L. Finkelstein[Fi]. Note that $|\mathcal{S}'| = 276$. By [ibid, Lemma 7.2], there is a subgroup of G_0, isomorphic to A_6, whose action on $\mathcal{S}'$ has orbit decomposition $[1^4, 10, 20^2, 30^2, 36^2, 45^2]$. By the fact that A_6 is simple, it also follows that $G_0 = 3^4 : A_6$ for any $A_6 < G_0$. Note also that $\mathcal{S} \subset \mathcal{S}'$ in a natural manner :

$$\mathcal{S} \ni V \mapsto \{V, C - V\} \in \mathcal{S}'\ .$$

We also recall from [ibid, Lemma 4.8, Lemma 5.11] that our 3^4 in G_0 is a subgroup of the subgroup 3^5 of $\cdot 3$, which is unique up to conjugate, and that the action of 3^5 on $\mathcal{S}'$ has orbit decomposition $[3^{11}, 243]$.

Lemma 2.5. *The action of 3^4 on $\mathcal{S}$ is equivalent to the regular representation, i.e. the left action of 3^4 on 3^4.*

Proof. Since $243 = |3^5|$, the action of 3^5 on the last orbit is equivalent to the regular representation. Therefore, the action of our 3^4 on $\mathcal{S}'$ has at most 33 fixed points. In particular, its action on $\mathcal{S}$ is non-trivial.

Since $|\mathcal{S}| = |3^4|$, it suffices to show that the action is transitive.

Let $\mathcal{S} = \cup_{k=1}^n \mathcal{O}_k$ be the orbit decomposition. Assume to the contrary that $n \geq 2$. Then $|\mathcal{O}_k| = 3^{m_k}$ for some integer $0 \leq m_k \leq 3$ and $\sum_{k=1}^n 3^{m_k} = 81$. We may assume that $m_1 \geq m_2 \geq \cdots \geq m_n$. Then m_1 is either 1, 2, or 3, i.e. $|\mathcal{O}_1|$ is either 3, 9, or 27.

Note that A_6 permutes these n orbits $\mathcal{O}_k$. In what follows, we may often use the *fact* that A_6 has no non-trivial homomorphism to S_m with $m \leq 5$. This *fact* is true because A_6 is simple.

First consider the case where $|\mathcal{O}_1| = 27$. There are at most 3 orbits of cardinality 27. So, the orbit $\mathcal{O}_1$ is stable under A_6. Thus, $\mathcal{O}_1$ must be a union of some orbits of the action of A_6 on $\mathcal{S}'$. However, the shape of $[1^4, 10, 20^2, 30^2, 36^2, 45^2]$ does not allow such a union, a contradiction.

Next consider the case where $|\mathcal{O}_1| = 9$. There are at most 9 orbits of cardinality 9. By the above *fact*, there are two cases: either there are exactly m such orbits, where m is either 6, 8, 9, and A_6 acts on the set $\{\mathcal{O}_k\}_{k=1}^m$ of the m orbits transitively, or at least one $\mathcal{O}_k$ with $|\mathcal{O}_k| = 9$ is A_6-stable. In the first case, if m is 8 or 9, then the stabilizer group of the action would be of order $360/8 = 45$ or $360/9 = 40$. However, A_6 has no subgroup of order 40 or 45 (see for instance the table of maximal subgroups of A_6), a contradiction. Assume that $m = 6$ in the first case. Note that 3 elements of $\mathcal{S}'$ corresponding to X_0, X_1, R_0 are all fixed by A_6. So, there is in $\mathcal{S}$ at most one element being fixed by A_6. Thus $9 \times 6 = 54$ must be a sum of the entries of $[1, 10, 20^2, 30^2, 36^2, 45^2]$. However, from the shape, we see that there is no such sum, a contradiction.

In the second case, this $\mathcal{O}_k$ must be a union of some orbits of the action of A_6 on $\mathcal{S}'$. However, the shape of $[1^4, 10, 20^2, 30^2, 36^2, 45^2]$ does not allow such a union, a contradiction.

Finally, consider the case where $|\mathcal{O}_1| = 3$. As it is observed above, there is in $\mathcal{S}$ at most one element being fixed by A_6. Note also that there are only three ways to decompose 81 into a sum of the entries of $[1, 10, 20^2, 30^2, 36^2, 45^2]$. Namely, $81 = 1 + 10 + 20 + 20 + 30$, $81 = 1 + 20 + 30 + 30$ and $81 = 36 + 45$. In the first two cases, there is an element of $\mathcal{S}$ which is fixed by $3^4 : A_6$, a contradiction to the observation immediately preceding Lemma 2.2. In the last case, all the orbits $\mathcal{O}_k$ $(1 \leq k \leq 27)$ are of order 3, because $33 < 36$ and $33 < 45$, and the orbit decomposition type of A_6 on $\{\mathcal{O}_k\}_{k=1}^{27}$ is $[12, 15]$. Then, A_6 would have a subgroup H of order $360/12 = 30$. By the table of maximal subgroups of A_6, this H would then be an index 2 subgroup of A_5, whence normal, a contradiction to the fact that A_5 is simple. □

Thus, the configuration of pentagon formed by $\{C, X_0, R_0, X_1, X_2\}$ in Λ is unique up to $\cdot 0$. This implies the first assertion of Theorem 2.1. The second assertion is now also clear, because the action of 3^4 on $\mathcal{S}$ is equivalent to the regular representation. □

The next proposition is also important in our proof of Theorems 3.1 and 5.1.

Proposition 2.6. *Let $Sym(\mathcal{R})$ be the group of symmetries of the Coxeter-Dynkin diagram of $\mathcal{R}$. Let $R\,(\simeq A_2^{\oplus 2} \oplus A_1^{\oplus 2})$ be the sublattice of Π generated by $\mathcal{R}$, the six Leech roots c, z, x_0, r_0, x_1, x_2. Then*

(1) The natural homomorphism $Sym(\mathcal{R}) \longrightarrow O(A_R, q_R)$ is an isomorphism, and

$$Sym(\mathcal{R}) \simeq O(A_R, q_R) \simeq D_8 \times \mathbf{Z}/2,$$

where D_8 is the dihedral group of order 8.

(2) Each element of $Sym(\mathcal{R})$ is induced by an element in $Aut(\mathcal{D})$.

Proof. The assertion (1) is obvious. Let us prove the assertion (2). One can find three Leech roots u_1, u_2, u_3 such that $x_2, u_1, c, z, u_2, r_0, x_0, u_3, x_1$ form a Coxeter-Dynkin diagram of type A_9 (Figure 4). For example, $u_1 = (4\nu_0 + \nu_\Omega - 2\nu_K, 1, 1)$, where $K = \{0, 5, 12, 13, 16, 20, 21, 22\}$, $u_2 = (4\nu_0 + \nu_\Omega, 1, 2)$, $u_3 = (\nu_\Omega - 4\nu_5, 1, 1)$. There are also three Leech roots u_2, v_1, v_2 such that $c, z, u_2, r_0, x_0, v_1, v_2, x_1, x_2$ form one of type D_9 (Figure 5). For example, $v_1 = (\nu_\Omega - 4\nu_7, 1, 1)$, $v_2 = (2\nu_K, 1, 1)$, where $K = \{\infty, 0, 6, 7, 10, 12, 15, 18\}$.

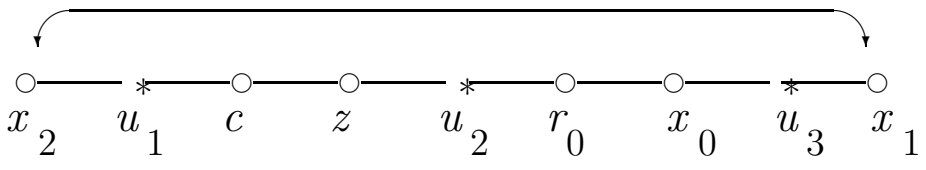

Figure 4

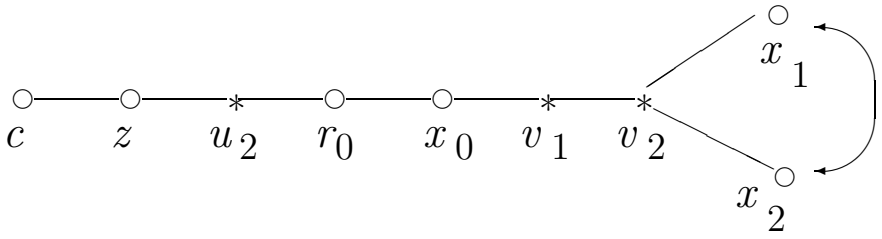

Figure 5

Thus, by Borcherds [Bo, Lemma 9.6, Theorem 9.5], there are two isometries in $\mathrm{Aut}(\mathcal{D})$ whose restrictions on R give rise to two involutions in $Sym(\mathcal{R})$, one switching the pair of A_2's as well as the pair of A_1's, and the other switching the pair of A_1's only. We need one more involution. From Theorem 2.1 and Lemma 2.5, we know that the pointwise stabilizer group $\mathrm{Aut}(\mathcal{D},\mathcal{R})$ of the pentagon formed by $\{C, X_0, R_0, X_1, X_2\}$ is a subgroup of $\cdot 3$ isomorphic to A_6, and that the action of $\mathrm{Aut}(\mathcal{D},\mathcal{R})$ on the set $\mathcal{S}'$ has only 4 fixed points $\{X_0, C-X_0\}$, $\{R_0, C-R_0\}$, $\{X_1, C-X_1\}$, $\{X_2, C-X_2\}$ (the maximality of the lattice M preceding Lemma 2.4). By [Fi, Lemma 7.2], the normalizer $N_{\cdot 3}(\mathrm{Aut}(\mathcal{D},\mathcal{R}))$ in $\cdot 3$ is isomorphic to $\mathbf{Z}/2\times\mathrm{Aut}(A_6)$, and hence has quotient $N_{\cdot 3}(\mathrm{Aut}(\mathcal{D},\mathcal{R}))/\mathrm{Aut}(\mathcal{D},\mathcal{R}) \simeq \mathbf{Z}/2\times(\mathbf{Z}/2)^2$. So, it is easy to see that the first factor $\mathbf{Z}/2$ sends the pentagon to its dual pentagon $\{C, C-X_0, C-R_0, C-X_1, C-X_2\}$, and the second $(\mathbf{Z}/2)^2$ gives the full symmetry group of the pentagon. From this it is obvious to see that there is an element of $\cdot 3$ interchanging R_0 and X_0, and leaving C, X_1, and X_2 fixed. Since $\cdot 3 < \mathrm{Aut}(\mathcal{D})$, this gives an isometry in $\mathrm{Aut}(\mathcal{D})$ whose restriction on $\mathcal{R}$ is the involution switching r_0 and x_0 while leaving the remaining four roots fixed. Clearly this involution, together with the previous two, generate $Sym(\mathcal{R})$. This completes the proof. □

By the above lemma, one can choose an isometry $\phi_4 \in \mathrm{Aut}(\mathcal{D})$ inducing the order 4 element of $Sym(\mathcal{R})$:

$$c \mapsto x_0 \mapsto z \mapsto r_0 \mapsto c\,, \quad x_1 \leftrightarrow x_2.$$

The choice of ϕ_4 is unique up to $\mathrm{Aut}(\mathcal{D},\mathcal{R}) \simeq A_6$. Also ϕ_4^4 is in A_6.

We take a basis of the discriminant group $A_R \simeq (\mathbf{Z}/6)^2$:

$$e_1 = (c+2z)/3 + (r_0+2x_0)/3 + x_1/2\,, \quad e_2 = (c+2z)/3 - (r_0+2x_0)/3 + x_2/2,$$

whose intersection matrix is

$$\begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}.$$

Then ϕ_4 acts on A_R as follows:

$$\phi_4 : e_1 \mapsto e_2\,, \quad e_2 \mapsto -e_1.$$

Definition 2.7. (Definition of $\tilde{A}_6$) The subgroup of $\mathrm{Aut}(\mathcal{D})$ generated by ϕ_4 and $\mathrm{Aut}(\mathcal{D},\mathcal{R})$ is denoted by $\tilde{A}_6$. This is a uniquely determined group.

We denote by $A_6.\mu_4$ *any* group G which falls into the exact sequence:

$$1 \longrightarrow A_6 \longrightarrow G \longrightarrow \mu_4 \longrightarrow 1$$

There are many isomorphism classes of groups of the form $A_6.\mu_4$. The direct product group $A_6 \times \mu_4$ is clearly one of $A_6.\mu_4$. Our group $\tilde{A}_6$ is also an extension of

A_6 by μ_4, and is one of $A_6.\mu_4$. Indeed, $\operatorname{Ker} f = A_6$ and $\operatorname{Im} f \simeq \mathbf{Z}/4$ for the natural homomorphism $f : \tilde{A}_6 \longrightarrow Sym(\mathcal{R})$.

We close this section by introducing a very important vector $h \in R_\Pi^\perp$, the orthogonal complement of R in Π, and its properties. This vector will play a crucial role in Sections 3 and 5:

Lemma 2.8. *Let w' be the orthogonal projection of the Weyl vector w onto $R_\Pi^\perp \otimes \mathbf{Q}$, or more explicitly, w' is the vector such that*

$$w = w' + w_R, \quad \text{where} \quad w_R = -(c + z + r_0 + x_0 + (x_1 + x_2)/2).$$

We set $h := 2w'$. Then

(1) $h^2 = 20$, $h \in R_\Pi^\perp$ and h is primitive in $R_\Pi^\perp$.
(2) $\varphi(h) = h$ for any element φ of $Aut(\mathcal{D})$ such that $\varphi(\mathcal{R}) = \mathcal{R}$.
(3) There is no element $x \in R_\Pi^\perp$ such that $x^2 = -2$ and $(x, h) = 0$.

Proof. The assertions (1) and (2) are obvious. Let us show (3). Recall that the Leech lattice Λ has no element of norm -2 and that z and w generate the second direct summand of the (original) orthogonal decomposition $\Pi = \Lambda \oplus U$. Since

$$h^\perp_{R_\Pi^\perp} \simeq \langle R\,,\, w\rangle_\Pi^\perp \subset \Lambda,$$

every $x \in R_\Pi^\perp$ with $(x, h) = 0$ must have norm $x^2 < -2$. □

3. Existence of a $K3$ surface with an $\tilde{A}_6$-action

The goal of this section is to construct a triplet $(F, \tilde{A}_6, \rho_F)$ consisting of a $K3$ surface F and a faithful $\tilde{A}_6$-action $\rho_F : \tilde{A}_6 \times F \longrightarrow F$ (Theorem (3.1)), and then to give an explicit description of F (Proposition (3.5)). In the next two sections, it turns out that such a triplet $(F, \tilde{A}_6, \rho_F)$ is actually unique up to isomorphisms.

Before stating our main result of this section, we recall some facts about such triplets.

Throughout this note, by a $K3$ surface, we mean a simply-connected compact complex surface X admitting a nowhere vanishing global holomorphic 2-form ω_X. The second cohomology group $H^2(X, \mathbf{Z})$ together with a cup product becomes an even unimodular lattice of index $(3, 19)$ and is isomorphic to the so-called $K3$ lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where E_8 is the *negative* definite even unimodular lattice of rank 8. We denote by $S(X)$ the Néron-Severi lattice of X. This is a primitive sublattice of $H^2(X, \mathbf{Z})$ generated by the classes of line bundles. We denote by $T(X)$ the transcendental lattice of X, i.e. the minimal primitive sublattice whose $\mathbf{C}$-linear extension contains the class ω_X, or equivalently $T(X) = S(X)^\perp$ in $H^2(X, \mathbf{Z})$. If X is projective, then $S(X) \cap T(X) = \{0\}$ and $S(X) \oplus T(X)$ is a finite-index sublattice of $H^2(X, \mathbf{Z})$.

Let (X, G, ρ) be a triplet consisting of a $K3$ surface, a finite group G and a faithful action $\rho : G \times X \longrightarrow X$. Then G has a 1-dimensional representation on $H^0(X, \Omega_X^2) = \mathbf{C}\omega_X$ given by $g^*\omega_X = \alpha(g)\omega_X$, and we have an exact sequence, called the basic sequence:

$$1 \longrightarrow G_N := \operatorname{Ker} \alpha \longrightarrow G \xrightarrow{\alpha} \mu_I \longrightarrow 1\ .$$

We call G_N the symplectic part and $\mu_I := \langle \zeta_I \rangle$ (resp. I), where $\zeta_I = \exp(2\pi\sqrt{-1}/I)$, the transcendental part (resp. the transcendental value) of the action ρ. We note that if $A_6.\mu_4$ acts faithfully on a $K3$ surface then $G_N \simeq A_6$ and the transcendental

part is isomorphic to μ_4. This follows from the fact that A_6 is simple and also maximal among all finite groups acting on a $K3$ surface symplectically [Mu]. We also note that X is projective if $I \geq 2$ [Ni1].

We say that 2 triplets (X, G, ρ) and (X', G', ρ') are isomorphic if there are a group isomorphism $f : G' \simeq G$ and an isomorphism $\varphi : X' \simeq X$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times X & \stackrel{\rho}{\longrightarrow} & X \\ {\scriptstyle f\times\varphi}\uparrow & & \uparrow{\scriptstyle \varphi} \\ G' \times X' & \stackrel{\rho'}{\longrightarrow} & X' \end{array}$$

The aim of this section is to show the following:

Theorem 3.1. *There is a triplet $(F, \tilde{A}_6, \rho_F)$ consisting of a $K3$ surface F and a faithful group action $\rho_F : \tilde{A}_6 \times F \longrightarrow F$ of $\tilde{A}_6$ on F. Here $\tilde{A}_6$ is the group defined in (2.7).*

Proof. Let F be a $K3$ surface such that the transcendental lattice $T(F) = \mathbf{Z}\langle t_1, t_2\rangle$ has the intersection matrix

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} ,$$

and $\omega_F := t_1 + \sqrt{-1}t_2$ is a holomorphic 2-form of F. Such a $K3$ surface exists and is unique [SI]. We claim that the surface F admits an action of $\tilde{A}_6$ as a group of automorphisms.

In order to find an action of $\tilde{A}_6 = \langle \mathrm{Aut}(\mathcal{D}, \mathcal{R}), \phi_4\rangle$ on F, we first relate the Néron-Severi lattice $S(F)$ to the lattices $\Pi = \Lambda \oplus U$ and R. Here and hereafter, we shall freely use the lattices and their elements introduced in Section 2.

The Picard lattice $S(F)$ is isometric to $U \oplus E_8 \oplus E_8 \oplus \langle -6\rangle \oplus \langle -6\rangle$. Indeed, one has $(A_{S(F)}, q_{S(F)}) \simeq (A_{T(F)}, -q_{T(F)})$, and the genus of $S(F)$ is the single element set $\{S(F)\}$ [Ni2, Theorem 1.14.2]. We set $L := H^2(F, \mathbf{Z})$. Since $(A_{S(F)}, q_{S(F)}) \simeq (A_R, -q_R)$, one has also an isomorphism

$$\Phi : S(F) \simeq R_\Pi^\perp \subset \Pi ,$$

and the diagram (depending on Φ):

$$T(F) \subset L \supset S(F) \simeq_\Phi R_\Pi^\perp \subset \Pi \supset R .$$

Since L and Π are both unimodular, these primitive inclusions and Φ naturally induce the isomorphisms, depending on Φ, of the discriminant groups:

$$(A_{T(F)}, q_{T(F)}) \simeq (A_{S(F)}, -q_{S(F)}) \simeq_\Phi (A_{R_\Pi^\perp}, -q_{R_\Pi^\perp}) \simeq (A_R, q_R) .$$

We also recall that $t_1/6$ and $t_2/6$ (resp. e_1 and e_2 defined in the previous section) are generators of $A_{T(F)}$ (resp. A_R) with intersection form

$$\begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix} .$$

Next, we shall transfer the group action of $\tilde{A}_6$ on Π to an effective Hodge isometric action on L. For this, it is more convenient to choose a special isomorphism Φ given by the next Lemma:

Lemma 3.2. *There is an isomorphism $\Phi : S(F) \simeq R_\Pi^\perp$ such that:*
(1) $H := \Phi^{-1}(h)$ is an ample class of F, and

(2) under the isomorphism $(A_{T(F)}, q_{T(F)}) \simeq (A_R, q_R)$ *above, we have* $t_1/6 \leftrightarrow e_1$ *and* $t_2/6 \leftrightarrow e_2$.

Here $h \in R_\Pi^\perp$ *is the vector defined in Lemma (2.8).*

Proof. Let us choose any isomorphism $\Phi_0 : S(F) \simeq R_\Pi^\perp$. By Lemma (2.8), there is no vector $x \in S(F)$ such that $(x, \Phi_0^{-1}(h)) = 0$ and $(x^2) = -2$. This means that a product σ of (-2)-reflections of $S(F)$ and $-1_{S(F)}$ transforms $\Phi_0^{-1}(h)$ to an ample divisor class H on F. So, the new isomorphism $\Phi_1 := \Phi_0 \circ \sigma^{-1} : S(F) \simeq R_\Pi^\perp$ enjoys the property (1). Let e_1' and e_2' be the generators of A_R corresponding to $t_1/6$ and $t_2/6$ under Φ_1. Then, by Proposition (2.6), there is $\eta \in \mathrm{Aut}(\mathcal{D})$ such that $\eta(\mathcal{R}) = \mathcal{R}$ and $\eta(e_1') = e_1$ and $\eta(e_2') = e_2$ (on A_R). Note also that $\eta(h) = h$ by Lemma (2.8)(2). Now, the new isomorphism $\Phi := \eta \circ \Phi_1 : S(F) \simeq R_\Pi^\perp$ satisfies the properties (1) and (2). □

Using this Φ*, we shall identify* $S(F) = R_\Pi^\perp$ *and* $H = h$ *from now on*:

$$T(F) \subset L \supset S(F) = R_\Pi^\perp \subset \Pi \supset R\ .$$

Let us construct an action of $\tilde{A}_6 = \langle \mathrm{Aut}(\mathcal{D}, \mathcal{R}), \phi_4 \rangle$ on L. First, observe that the group $\tilde{A}_6$ acts on $R^\perp = S(F)$, faithfully, i.e. $\tilde{A}_6$ can be viewed as a subgroup of $O(S(F))$. Indeed, if $\varphi | S(F) = id$ for $\varphi \in \tilde{A}_6$, then $\varphi | A_R = id$ as well. Since $Sym(\mathcal{R}) \simeq O(A_R, q_R)$, we have then $\varphi | R = id$ and consequently $\varphi = id$ on Π.

Let us define the isometry $\psi_4 \in O(T(F))$ by $t_1 \mapsto t_2$, $t_2 \mapsto -t_1$. Then, by using Nikulin [Ni2, Corollary 1.5.2], one can find an isometry $\tilde{\phi}_4 \in O(L)$ such that $\tilde{\phi}_4 | S(F) = \phi_4 | S(F)$ and $\tilde{\phi}_4 | T(F) = \psi_4$. Let $\phi \in \mathrm{Aut}(\mathcal{D}, \mathcal{R})$. Then, $\phi | A_{S(F)} = id$ and we have an isometry $\tilde{\phi} \in O(L)$ such that $\tilde{\phi} | S(F) = \phi | S(F)$ and $\tilde{\phi} | T(F) = id$.

Let $\tilde{A}_6{}'$ be the subgroup of $O(L)$ generated by $\tilde{\phi}_4$ and these $\tilde{\phi}$:

$$\tilde{A}_6{}' := \langle \tilde{\phi}\, (\forall \phi \in \mathrm{Aut}(\mathcal{D}, \mathcal{R}))\ ,\ \tilde{\phi}_4 \rangle < O(L)\ .$$

Lemma 3.3. *(1) The natural homomorphism* $\iota : \tilde{A}_6{}' \longrightarrow \tilde{A}_6 (< O(S(F))$*, induced by the restriction of the action on* $S(F)$*, is an isomorphism.*

(2) Each element of $\tilde{A}_6{}'$ *is an effective Hodge isometry of* L.

Proof. As we have already observed, $\phi | S(F)$ ($\phi \in \mathrm{Aut}(\mathcal{D}, \mathcal{R})$) and $\phi_4 | S(F)$ generate $\tilde{A}_6 (< O(S(F)))$. Thus, ι is surjective. Let $\varphi \in \tilde{A}_6{}'$ such that $\varphi | S(F) = id$. Then, $\varphi | A_{S(F)} = id$ and $\varphi | A_{T(F)} = id$ as well. Observe that the natural homomorphism $D_8 \simeq O(T(F)) \longrightarrow O(A_{T(F)}, q_{T(F)}) \simeq D_8 \times C_2$ is injective. Then, $\varphi | T(F) = id$ and hence $\varphi | L = id$. This means that ι is also injective.

Let us show the assertion (2). It suffices to check it for the generators. It is clear that $\tilde{\phi}$ ($\phi \in \mathrm{Aut}(\mathcal{D}, \mathcal{R})$) preserves the Hodge decomposition of L. By $\omega_F = t_1 + \sqrt{-1} t_2$, we have $\psi_4(\omega_F) = -\zeta_4 \omega_F$. Thus $\tilde{\phi}_4$ also preserves the Hodge decomposition of L. In addition, our group $\tilde{A}_6{}' \simeq \tilde{A}_6$, being a subgroup of $\mathrm{Aut}(\mathcal{D})$, fixes the Weyl vector w and the set $\mathcal{R}$ by the definition, whence it fixes $H = h$ by Lemma (2.8). Since H is ample, the action of $\tilde{A}_6{}'$ on L is also effective. □

Thus the group $\tilde{A}_6{}' \simeq \tilde{A}_6$ realizes as a group of automorphisms of F by the Torelli Theorem for $K3$ surfaces [PSS], [BR]. This completes the proof of Theorem (3.1). □

Remark 3.4. It will turn out that $S(F)^{\tilde{A}_6} = S(F)^{A_6} = \mathbf{Z}H$ by Proposition (4.5) and Lemma (2.8).

We shall close this section by giving an explicit equation of a canonical model of F in $\mathbf{P}^1 \times \mathbf{P}^2$.

Proposition 3.5. *Let F be a $K3$ surface constructed in Theorem (3.1), i.e. the unique $K3$ surface whose transcendental lattice has the intersection matrix*

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

(1) The $K3$ surface F is isomorphic to the minimal resolution of the double cover $\overline{F}$ of the (rational) elliptic modular surface E with level 3 structure. The double cover is branched along two of a total of 4 singular fibres of the same type I_3 and $\overline{F}$ has 6 ordinary double points.

(2) $\overline{F}$ is isomorphic to a surface in $\mathbf{P}^1 \times \mathbf{P}^2$ given by the following equation, where $([S:T],[X:Y:Z])$ are coordinates of $\mathbf{P}^1 \times \mathbf{P}^2$:

$$S^2(X^3+Y^3+Z^3) - 3(S^2+T^2)XYZ = 0.$$

Proof. Let $E := \{\lambda(X^3+Y^3+Z^3) - 3\mu XYZ = 0\} \subset \mathbf{P}^1 \times \mathbf{P}^2$ be the (rational and smooth) elliptic modular surface with level 3 structure. It is easy to see that the elliptic fibration $E \longrightarrow \mathbf{P}^1$, induced from the projection $\mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^1$, has exactly four singular fibres of the same type I_3 lying over the points $[0:1]$, $[1:\zeta_3]$, $[1:\zeta_3^2]$ and $[1:1]$. Let $\mathbf{P}^1 \to \mathbf{P}^1$, $[S:T] \mapsto [S^2 : S^2+T^2]$, be the double cover branched at $[0:1]$ and $[1:1]$. Then the pull back $\overline{F}$ of E in the fibre product $\mathbf{P}^1 \times \mathbf{P}^2 \cong \mathbf{P}^1 \times_{\mathbf{P}^1} \mathbf{P}^2$ is given by the equation:

$$S^2(X^3+Y^3+Z^3) - 3(S^2+T^2)XYZ = 0.$$

Now $\overline{F} \to E$ is branched along the fibres E_1 and E_∞. Here we let E_t (resp. E_∞) be the fibre lying over $[1:t]$ (resp. $[0:1]$). Note that $\overline{F}$ has six singular points of Dynkin type A_1 lying over the six intersection points in E_1 and E_∞ and is smooth everywhere else. Let $F \to \overline{F}$ be the minimal resolution of these 6 singular points. We shall show that this F is isomorphic to its namesake constructed in Theorem (3.1).

The adjunction formula shows that $\overline{F}$ has trivial canonical line bundle. By a cohomology exact sequence and the Kawamata-Viehweg vanishing theorem, we see that the irregularity $q(\overline{F}) = 0$. So F is a $K3$ surface. The elliptic fibration on E lifts to an elliptic fibration $F \to \mathbf{P}^1$ with a section. Now E_1 and E_∞ lift to singular fibres of type I_6, while each of the singular fibres E_{ζ^i} $(i = 1, 2)$ splits into two singular fibres of the same type I_3. Thus, by [Sh], $\rho(X) \geq 2 + 4 \cdot 2 + 2 \cdot 5 = 20$, whence $\rho(X) = 20$. According to [SZ, Table 2, No. 5], the transcendental lattice $T(F)$ has the intersection matrix (a_{ij}) of rank 2 with $a_{11} = a_{22} = 6$ and $a_{12} = a_{21} = 0$. So the surface F is exactly the same $K3$ surface as in Theorem (3.1) by [SI]. This proves the proposition. □

Remark 3.6. (1) The surface F is not a Kummer surface, as its transcendental lattice is not the double of an even lattice [Mor].

(2) The surface F is the universal double cover of an Enriques surface, i.e. has a fixed point free involution. This can be seen indirectly by the criterion in [Ke].

4. Uniqueness of the $K3$ surface admitting an $A_6.\mu_4$-action

In this section, we shall show the uniqueness of the $K3$ surface admitting an $A_6.\mu_4$-action and the maximality of the extension of A_6 by μ_4. Our main results of this section are Propositions (4.1) and (4.5).

In what follows, we set $L := H^2(X, \mathbf{Z})$ for a $K3$ surface X. We define

$$L^{G_N} := \{x \in L \,|\, g^*x = x \text{ for all } g \in G_N\}$$

$$L_{G_N} := (L^{G_N})^{\perp}_L = \{x \in L \,|\, (x, y) = 0 \text{ for all } y \in L^{G_N}\}.$$

Proposition 4.1. *Let G be a finite group acting faithfully on a $K3$ surface X. Assume that $A_6 < G$ and $I \geq 2$, where I is the transcendental value. Then we have*

(1) $G_N = A_6$.
(2) $\mathrm{rank}\, L^{G_N} = 3$. In particular, $S(X)^{G_N} = \mathbf{Z}H$, where H is an ample class, and $\mathrm{rank}\, T(X) = 2$.
(3) $I = 2$, or 4.

Proof. As we remarked in Section 3, the statement (1) follows from the fact that $[A_6, A_6] = A_6$ and the maximality of A_6 as a symplectic $K3$ group [Mu]. Since X is projective by $I \geq 2$, the second statement of (2) follows from the first one. Let us prove the first statement of (2).

Recall that the order structure of A_6 is as follows:

order[conjugacy class]	1 [1A]	2 [2A]	3 [3A]	3 [3B]	4 [4A]	5 [5A]	5 [5B]
cardinality	1	45	40	40	90	72	72

Moreover, by [Ni1], the number of the fixed points of the symplectic action is as follows.

$ord(g)$	1	2	3	4	5	6	7	8
$\lvert X^g \rvert$	X	8	6	4	4	2	3	2

Set $\tilde{H}(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z})$. Now, by applying the topological Lefschetz fixed point formula for $G_N = A_6$, we calculate that

$$\mathrm{rank}\, \tilde{H}(X, \mathbf{Z})^{A_6} = \frac{1}{|A_6|} \sum_{g \in A_6} \mathrm{tr}(g^* | \tilde{H}(X, \mathbf{Z}))$$

$$= \frac{1}{360}(24 + 8 \cdot 45 + 6 \cdot 80 + 4 \cdot 90 + 4 \cdot 144) = 5 \ .$$

This implies the result.

Let us show the assertion (3). By (2), we have $\mathrm{rank}\, T(X) = 2$. Thus, $I = 2, 4, 3$ or 6 because the Euler function $\varphi(I)$ divides rank $T(X)$. It suffices to rule out the case $I = 3$.

Let us first determine the irreducible decomposition of $S(X) \otimes \mathbf{C}$ as A_6-modules. In the description, we use Atlas notation for irreducible characters of A_6 in the Table below. We also use the same letters for the representations.

	1A	2A	3A	3B	4A	5A	5B
χ_1	1	1	1	1	1	1	1
χ_2	5	1	2	-1	-1	0	0
χ_3	5	1	-1	2	-1	0	0
χ_4	8	0	-1	-1	0	$(-1-\sqrt{5})/2$	$(-1+\sqrt{5})/2$
χ_5	8	0	-1	-1	0	$(-1+\sqrt{5})/2$	$(-1-\sqrt{5})/2$
χ_6	9	1	0	0	1	-1	-1
χ_7	10	-2	1	1	0	0	0

Claim 4.2. As A_6-modules, one has

$$S(X) \otimes \mathbf{C} = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_6 \ .$$

Proof. Since $S(X)^{A_6} = \mathbf{Z}H$, the irreducible decomposition must be of the following form:

$$S(X) \otimes \mathbf{C} = \chi_1 \oplus a_2\chi_2 \oplus a_3\chi_3 \oplus a_4\chi_4 \oplus a_5\chi_5 \oplus a_6\chi_6 \oplus a_7\chi_7 \ ,$$

where a_i are non-negative integers. Let us determine a_i's. As in (2), using the topological Lefschetz fixed point formula and the fact that $\operatorname{rank} T(X) = 2$, we have

$$\chi_{\mathrm{top}}(X^g) = 4 + \operatorname{tr}(g^*|S(X))$$

for $g \in A_6$. Running g through the 7-conjugacy classes and calculating both sides based on Nikulin's table and the character table above, we obtain the following system of equations:

$$20 = 1 + 5(a_2 + a_3) + 8(a_4 + a_5) + 9a_6 + 10a_7 \ ,$$
$$4 = 1 + (a_2 + a_3) + a_6 - 2a_7 \ ,$$
$$2 = 1 + (2a_2 - a_3) - (a_4 + a_5) + a_7 \ , \ 2 = 1 + (-a_2 + 2a_3) - (a_4 + a_5) + a_7 \ ,$$
$$0 = 1 - (a_2 + a_3) + a_6 \ ,$$
$$0 = 1 - (\frac{1+\sqrt{5}}{2}a_4 + \frac{1-\sqrt{5}}{2}a_5) - a_6 \ , \ 0 = 1 - (\frac{1-\sqrt{5}}{2}a_4 + \frac{1+\sqrt{5}}{2}a_5) - a_6 \ .$$

Now, we get the result by solving this system of Diophantine equations. □

From now, assuming to the contrary that $I = 3$ and $G_N = A_6$, we shall derive a contradiction.

Claim 4.3. $G = G_N \times \mu_3 (= A_6 \times \mu_3)$.

Proof. We shall use the following general proposition by [IOZ] (see also [Og, Proposition (5.1)]):

Proposition 4.4. *Assume that $I = 3$. Let g be an element of G such that $\alpha(g) = \zeta_3$, i.e. $g^*\omega_X = \zeta_3\omega_X$. Set $\mathrm{ord}(g) = 3k$. Then $(k, 3) = 1$. In particular, $G = G_N : \mu_3$, a semi-direct product.*

From this proposition and our assumption, we have $G = A_6 : \mu_3$, a semi-direct product. Let h be an element of G such that $\alpha(h) = \zeta_3$ and $\mathrm{ord}(h) = 3$. Since $\mathrm{Out}(A_6) = C_2^{\oplus 2}$, it follows that there is an element $a \in A_6$ such that $h^{-1}xh = a^{-1}xa$ for all $x \in A_6$. Then $ha^{-1} \in Z(G)$, $\alpha(ha^{-1}) = \zeta_3$, and $\mathrm{ord}(ha^{-1}) = 3l$ with $(l, 3) = 1$. Here $Z(G)$ is the center of G. So, replacing ha^{-1} by $(ha^{-1})^{\pm l}$, one

obtains an element g such that $\alpha(g) = \zeta_3$, $\mathrm{ord}(g) = 3$ and $gx = xg$ for all $x \in G$. This implies the result. □

Let g be a generator of μ_3 in $G = A_6 \times \mu_3$. Then g^* makes the irreducible decomposition in Claim (4.2) stable, i.e. $g^*(\chi_i) = \chi_i$. By the Schur lemma, $g|\chi_i$ is a scalar multiplication. Moreover, by $g^*H = H$, one has $g^*|\chi_1 = 1$. Set $g^*|\chi_2 = \zeta_3^a$, $g^*|\chi_3 = \zeta_3^b$ and $g^*|\chi_6 = \zeta_3^c$, where $a, b, c \in \{0, 1, 2\}$. Note that the action of g^* on $S(X) \otimes \mathbf{C}$ is defined over $\mathbf{Z}$. Thus, the multiplicities of ζ_3 and $\zeta_3^2 (= \zeta_3^{-1})$ must be equal. So by $\dim \chi_2 = \dim \chi_3 = 5$ and $\dim \chi_6 = 9$, one has $c = 0$ and $a + b \equiv 0 \bmod 3$, i.e. $(a, b, c) = (0, 0, 0)$, $(1, 2, 0)$ or $(2, 1, 0)$.

Let us first consider the case $(a, b, c) = (0, 0, 0)$. In this case, we have $g^*|S(X) = id$. Moreover, we have $\mathrm{rank}\, T(X) = 2 = \varphi(3)$. Here φ is the Euler function. Thus, by [OZ1, main Theorem], the intersection matrix of $T(X)$ is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .$$

Therefore by [SI], one has

$$X \simeq \text{ the minimal resolution of } (E_{\zeta_3} \times E_{\zeta_3})/\langle \mathrm{diag}\,(\zeta_3, \zeta_3^2)\rangle \ .$$

So by the main result of Vinberg [Vi], we have

$$\mathrm{Aut}(X) \simeq C_3 \times (C_2^{*12} : ((S_3 \times S_3) : C_2)) \ ,$$

where C_2^{*12} denotes the free product of 12 C_2's. Since $A_6 (< \mathrm{Aut}(X))$ is simple and $C_3 < \mathrm{Aut}\, X$ is normal, we have $A_6 \cap C_3 = \{1\}$. Hence A_6 becomes a subgroup of the quotient group $(C_2^{*12} : ((S_3 \times S_3) : C_2))$. Since C_2^{*12} has no elements of finite order, except involutions and identity, we have again $A_6 \cap C_2^{*12} = \{1\}$ for the same reason, and A_6 becomes a subgroup of the quotient group $(S_3 \times S_3) : C_2$. However, $|(S_3 \times S_3) : C_2| = 72$, while $|A_6| = 360$, a contradiction.

Let us consider the case $(a, b, c) = (1, 2, 0)$ (resp. $(a, b, c) = (2, 1, 0)$). Let $\tau \in A_6$ be an element of the conjugacy class $3A$ (resp. $3B$).

By Claim (4.2) and by the case assumption together with the character table, we have

$$\mathrm{tr}\,(\tau g)^*|S(X) = 1 + 2\zeta_3 - \zeta_3^2 \,,$$

which is a contradiction, as the left hand side is an integer while the right hand side is not. This completes the proof of (3) and also Proposition (4.1). □

The next proposition is the main result of this section.

Proposition 4.5. *Assume that X is a $K3$ surface admitting a faithful group action of $A_6.\mu_4$. Set $G = A_6.\mu_4$. Then*

(1) $X \simeq F$, where F is the surface constructed in Theorem (3.1).
(2) $S(X)^G = S(X)^{G_N} = \mathbf{Z}H$ where H is an ample primitive class with $(H^2) = 20$.

Let us first show the following:

Proposition 4.6. $A_{L^{G_N}} := (L^{G_N})^*/L^{G_N} \simeq \mathbf{Z}/3 \oplus \mathbf{Z}/60$.

Proof. We denote by $N(Rt)$ the (non-Leech) Niemeier lattice whose root lattice is isomorphic to Rt. There are 23 such lattices up to isomorphisms [CS, Chapter 18]. By [Ko2, Lemmas 5 and 6], there are a (non-Leech) Niemeier lattice $N(Rt)$, a primitive embedding $A_1 \oplus L_{G_N} \subset N(Rt)$ and a faithful action of G_N on $N(Rt)$ such that G_N acts on A_1 trivially. Moreover, one can choose this action so that $L_{G_N} =$

$N(Rt)_{G_N}$, and a Weyl chamber (one of whose codimension one faces corresponds to A_1) stable. Here $N(Rt)_{G_N}$ is the orthogonal complement of the sublattice $N(Rt)^{G_N}$ in $N(Rt)$. So G_N can be regarded as a subgroup of the symmetry group of the Coxeter-Dynkin diagram of Rt, i.e. one has

$$G_N < S(N(Rt)) := O(N(Rt))/W(N(Rt)),$$

where $W(N(Rt))$ is the Weyl group. Furthermore, G_N fixes a vertex of the diagram.

Note that $\operatorname{rank} N(Rt)^{G_N} = \operatorname{rank} L^{G_N} + 2 = 5$ and $A_{L^{G_N}} \simeq A_{L_{G_N}}(-1) = A_{N(Rt)_{G_N}}(-1) \simeq A_{N(Rt)^{G_N}}$. Now Proposition (4.6) follows from Lemma (4.7) below. □

Lemma 4.7. *(1) The root lattice Rt is either $A_1^{\oplus 24}$ or $A_2^{\oplus 12}$.*

(2) Assume that the first case in (1) occurs. Then the orbit decomposition of the action of G_N on the 24 simple roots is either $[1,1,1,6,15]$ or $[1,1,6,6,10]$. The intersection matrix of $N(A_1^{\oplus 24})^{G_N}$ (under some integral basis) is then either

$$\begin{pmatrix} -2 & 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ -1 & -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & -8 \end{pmatrix} \text{ or } \begin{pmatrix} -2 & 0 & -1 & -1 & -1 \\ 0 & -2 & -1 & -1 & -1 \\ -1 & -1 & -4 & -1 & -1 \\ -1 & -1 & -1 & -4 & -1 \\ -1 & -1 & -1 & -1 & -6 \end{pmatrix}.$$

In particular, $A_{N(Rt)^{G_N}} \simeq \mathbf{Z}/3 \oplus \mathbf{Z}/60$.

(3) Assume that the second case in (1) occurs. Then the orbit decomposition of the action of G_N on the 24 simple roots is $[1,1,1,1,20]$. The intersection matrix of $N(A_2^{\oplus 12})^{G_N}$ (under some integral basis) is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix}.$$

In particular, $A_{N(Rt)^{G_N}} \simeq \mathbf{Z}/3 \oplus \mathbf{Z}/60$ as well.

Proof. By the description of $S(N(Rt))$ ([CS, Chapter 18], [Ko2, the proof of Main Theorem]), and by the fact that $A_6(< S(N(Rt)))$ fixes a vertex of the Coxeter-Dynkin diagram of Rt, we see that Rt is either $A_1^{\oplus 24}$ or $A_2^{\oplus 12}$. Let

$$\mathcal{N}_2 := \{r_1\,,\, r_2\,,\, \cdots\,, r_{24}\}$$

be the set of the simple roots corresponding to the codimension one faces of the stable Weyl chamber. We may assume that r_1 is fixed by A_6. We shall consider two types of $N(Rt)$ one by one.

First consider the case $N := N(A_1^{\oplus 24})$.

Claim 4.8. The orbit decomposition type of A_6 on $\mathcal{N}_2$ is either (i) $[1,1,1,6,15]$ or (ii) $[1,1,6,6,10]$. (Note in particular that $A_6 < M_{23}$ under the action on $\mathcal{N}_2$.)

Proof. Since $\operatorname{rank} N^{A_6} = 5$, $\mathcal{N}_2$ is divided into exactly 5 orbits, and one of the orbits is a one-point set, say $[1,a,b,c,d]$ with $1+a+b+c+d=24$. Since A_6 is a simple group, the action on each orbit is faithful unless it is a one-point orbit. Thus, $a|360$, and $a \geq 6$ unless $a=1$. The same holds for b, c, d. Now assuming $a \leq b \leq c \leq d$, we see that the orbit decompositions is one of the two in (4.8) and three below:

(iii) $[1,1,6,8,8]$, (iv) $[1,1,1,9,12]$, (v) $[1,1,1,1,20]$. In Case (iii) (resp. (iv)), the transitivity of A_6-action on the length-8 (resp. length-12) orbit implies that A_6 has a (stabilizer) subgroup of order $|A_6|/8 = 45$ (resp. $|A_6|/12 = 30$), which is impossible by utilizing the list of maximal subgroups of A_6 in the Atlas. If Case (v) occurs, then $A_6 < M_{24}$ fixes 4 letters and hence $A_6 < M_{20}$ so that $360 = |A_6|$ divides $|M_{20}| = 960$, absurd. This proves the claim. □

By this claim, after re-numbering the elements of $\mathcal{N}_2$, we have

$$Rt^{A_6} = \mathbf{Z}\langle s_1, s_2, s_3, s_4, s_5\rangle \ ,$$

where in Case (i) of (4.8)

$$s_1 = r_1 \ , \ s_2 = r_2 \ , \ s_3 = r_3 \ , \ s_4 = r_4 + \cdots + r_9 \ , \ s_5 = r_{10} + \cdots + r_{24} \ ,$$

and in Case (ii) of (4.8)

$$s_1 = r_1 \ , \ s_2 = r_2 \ , \ s_3 = r_3 + \cdots + r_8 \ , \ s_4 = r_9 + \cdots + r_{14} \ , \ s_5 = r_{15} + \cdots + r_{24} \ .$$

Claim 4.9. According to the cases (i) and (ii), one has:

$$N^{A_6} = \mathbf{Z}\langle s_1, s_2, s_3, \frac{s_1+s_2+s_4}{2}, \frac{s_3+s_5}{2}\rangle \ ,$$

$$N^{A_6} = \mathbf{Z}\langle s_1, s_2, \frac{s_1+s_2+s_3}{2}, \frac{s_1+s_2+s_4}{2}, \frac{s_1+s_2+s_5}{2}\rangle \ .$$

Proof. Recall that the subset $N/Rt(\simeq \mathbf{F}_2^{\oplus 12})$ of $Rt^*/Rt \simeq \mathbf{F}_2^{\oplus 24}$ is the so-called binary Golay code. The element of length 8 (resp. 12) is called an octad (resp. a dodecad). We often identify $\sum_{k\in K} r_k/2$ with the subset $\{r_k | k \in K\}$ of $\mathcal{N}_2$. The set of octads forms the Steiner system $St(5,8,24)$ of $\mathcal{N}_2$. Note that our numbering of elements is different from the one in Section 2. However, since our proof does not involve calculations based on Todd's list, we continue to keep our numbering of elements from 1 to 24 (not from ∞, 0, to 22).

Recall that $Rt^{A_6} \subset N^{A_6} \subset (Rt^*)^{A_6}$, and that the lattice N^{A_6} is generated by Rt^{A_6}, $\sum_{k=1}^{24} r_k/2$ and $\sum_{k\in K} r_k/2$, where K runs through all A_6-invariant octads, dodecads, or the complements of octads. In what follows, we consider the second case, i.e. the case where the orbit decomposition is

$$\mathcal{N}_2 = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \ ,$$

where

$$|\mathcal{O}_1| = |\mathcal{O}_2| = 1 \ , \ |\mathcal{O}_3| = |\mathcal{O}_4| = 6 \ , \ |\mathcal{O}_5| = 10 \ .$$

The first case is easier, so we omit its proof.

By the shape of the orbit decomposition, the candidates of A_6-invariant octads and dodecads are

$$(s_1+s_2+s_3)/2 \, , \, (s_1+s_2+s_4)/2 \, , \, (s_1+s_2+s_5)/2 \, , \, (s_3+s_4)/2 \, .$$

Let us show that the first sum, or equivalently, the set $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$, is indeed an octad. Choose an order 5 element $g \in A_6 (< M_{23})$. Then, by [EDM, Appendix B, Table 5.I], the cycle decomposition type of g on $\mathcal{N}_2$ is $(1^4)(5^4)$. Thus, the cycle type of the action of g on $\mathcal{O}_3$ is $(1)(5)$. So, after re-numbering elements in $\mathcal{O}_3$, we may assume that $g(r_i) = r_{i+1}$ $(3 \le i \le 6)$, $g(r_7) = r_3$ and $g(r_8) = r_8$. By the Steiner property, there is an octad A containing the 5-element set $\{r_3, r_4, r_5, r_6, r_7\}$. Since

$$g(\{r_3, r_4, r_5, r_6, r_7\}) = \{r_3, r_4, r_5, r_6, r_7\}$$

we have by the Steiner property that $A = g(A)$. Let s, t, u be the remaining three elements of A. Since g acts on the fifth orbit $\mathcal{O}_5$ as (5^2), none of them is in the fifth orbit. If two or three of s, t, u are in the fourth orbit $\mathcal{O}_4$, using the fact that the cycle type of g on the fourth orbit is of $(1)(5)$, we have $g^m(A) \neq A$ for a suitable m, a contradiction. Assume that exactly one of s, t, u, say s, is in the fourth orbit $\mathcal{O}_4$. Note that $\mathcal{O}_4$ and $\mathcal{O}_3$ are both order 6 set. Then the cycle type of an involution $\tau \in A_6$, is necessarily of type $(1^2)(2^2)$ both on $\mathcal{O}_4$ and on $\mathcal{O}_3$. So, there is an involution $\tau \in A_6$ such that $\tau(s) \notin A$, but at least one of $\tau(t)$, $\tau(u)$, and at least four of $\tau(r_3), \cdots, \tau(r_7)$ are in A. Thus, $|\tau(A) \cap A| \geq 5$, whence $\tau(A) = A$, by the Steiner property, a contradiction to $s \in A$ but $\tau(s) \notin A$. So, none of s, t, u is in $\mathcal{O}_4 \cup \mathcal{O}_5$. This means that A is the union of the first three orbits and is then A_6-invariant. In the exactly same manner, the union of the first two and the fourth orbits is also an A_6-invariant octad. Then, by taking a symmetric difference and complement, we find that the other two candidates are actually A_6-invariant dodecads. This implies the result. □

The matrices in Lemma (4.7)(2) are nothing but the intersection matrices with respect to these basis in (4.9). Calculating elementary divisors, we also get the last statement of Lemma (4.7)(2).

Next consider the case $N := N(A_2^{\oplus 12})$.

Claim 4.10. The orbit decomposition type of A_6 on $\mathcal{N}_2$ is $[1, 1, 1, 1, 20]$.

Proof. As before, $\mathcal{N}_2$ is divided into exactly 5 orbits, and one of the orbits is a one point set. Since the graph of $\mathcal{N}_2$ consists of 12-connected components,

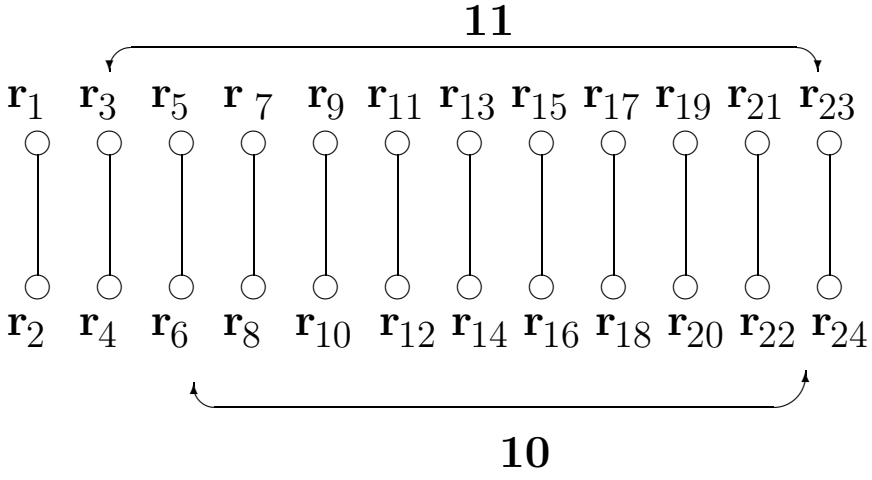

Figure 6

the orbit decomposition is $[1, 1, a, b, c]$ with $a + b + c = 22$ $(a \leq b \leq c)$ and A_6 acts on the set of 11-connected components. Since A_6 is simple, if A_6 acts on m

components with $m \leq 5$, then it acts trivially on each of the m components, whence $m = 1$ and its 2 simple roots are both fixed. Thus, $a = b = 1$ and $c = 20$. □

By this claim, after re-numbering the elements of $\mathcal{N}_2$, we have

$$Rt^{A_6} = \mathbf{Z}\langle s_1, s_2, s_3, s_4, s_5\rangle$$

where

$$s_1 = r_1\,,\ s_2 = r_2\,,\ s_3 = r_3\,,\ s_4 = r_4\,,\ s_5 = r_5 + \cdots + r_{24}\,,$$

and r_i are labeled so that $\{r_{2k-1}, r_{2k}\}$ forms a connected component.

Claim 4.11. One has:

$$N^{A_6} = \mathbf{Z}\langle s_1, s_2, s_3, s_4, s_5\rangle\ .$$

Proof. By [CS, Chapter 18], N^{A_6} is generated by Rt^{A_6} and A_6-invariant elements of the form $v_S := \sum_{k\in S} \pm(r_{2k-1} + 2r_{2k})/3$, where S is an element of the so called ternary Golay code. Such v_S must also satisfy $(v_S^2) \in \mathbf{Z}$. However, by the shape of the orbit decomposition, there is no such A_6-invariant sum. □

Again, the matrix in Lemma (4.7)(3) is nothing but the intersection matrix with respect to this basis. Calculating elementary divisors, we also get the last statement of Lemma (4.7)(3). This proves Lemma (4.7) and also Proposition (4.6). □

In order to complete the proof of Proposition (4.5), we need two more lemmas.

Lemma 4.12. *There is an integral basis $\langle v_1, v_2\rangle$ of $T(X)$ such that $G/G_N = \langle g\rangle$ acts as $g(v_1) = v_2$ and $g(v_2) = -v_1$. With respect to this basis $\langle v_1, v_2\rangle$, the intersection matrix of $T(X)$ has the following form:*

$$\begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix} \text{ for some } m \in \mathbf{Z}\ .$$

Proof. Since $G/G_N \simeq \mu_4$, the result follows from [Ko3, Page 1249] (see also [MO], [OZ2, Lemma 2.8]). □

Lemma 4.13. *Set $l := [L^{G_N} : \mathbf{Z}H \oplus T(X)]$, where $G_N \simeq A_6$ and H is the same as in Proposition (4.1)(1). Then $l = 1$ or 2. Moreover, if $l = 2$, then*

$$L^{A_6} = \mathbf{Z}\langle \frac{H + v_1 + v_2}{2}, v_1, v_2\rangle\ .$$

Here $\langle v_1, v_2\rangle$ is an integral basis of $T(X)$ as in Lemma (4.12).

Proof. The proof is identical to [Ko3, Page 1248], [OZ2, Page 177] or [Og, Lemma 6.10]. □

Now we are ready to complete the proof of Proposition (4.5).

Proof. Let $(H^2) = 2n$ for some positive integer n. Let m be the positive integer in Lemma (4.12). By virtue of [SI], it suffices to show that $(n, m) = (10, 3)$. Let l be the same as in Lemma (4.13).

Assume first that $l = 1$. Then $L^{A_6} = \mathbf{Z}H \oplus T(X)$. However, by Proposition (4.6), we have then $3 \cdot 60 = 2n \cdot 4m^2$, and $nm \notin \mathbf{Z}$, a contradiction.

Therefore $l = 2$ and by Proposition (4.6), we have $4 \cdot 3 \cdot 60 = 2n \cdot 4m^2$, that is, $nm^2 = 90$. Thus, $(n, m) = (90, 1)$ or $(10, 3)$. However if $(n, m) = (90, 1)$, then the

intersection matrix of L^{A_6} with respect to the integral basis $\langle v_1, v_2, (v_1+v_2+H)/2\rangle$ would be

$$\begin{pmatrix} 2m & 0 & m \\ 0 & 2m & m \\ m & m & m+\frac{n}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 46 \end{pmatrix} ,$$

whence $A_{L^{G_N}} \simeq \mathbf{Z}/180$, a contradiction to Proposition (4.6). Thus $(n, m) = (10, 3)$. □

5. Uniqueness of the $A_6.\mu_4$-action

In this section, we shall show the uniqueness of the $A_6.\mu_4$-action.

Theorem 5.1. *Let X be a $K3$ surface admitting a faithful $A_6.\mu_4$-action, say,*

$$\tau : A_6.\mu_4 \times X \longrightarrow X .$$

Then the triplet $(X, A_6.\mu_4, \tau)$ is isomorphic to the triplet $(F, \tilde{A}_6, \rho_F)$ constructed in Theorem (3.1)*.*

Proof. Our argument here is much inspired by [Ko3]. Put $G := G_\tau = A_6.\mu_4$. Note that $G_N = A_6$. By Proposition (4.5), we may assume that $X = F$. We put $\tau_F = \tau$. We denote by H_τ a primitive G_τ-invariant ample class (now) on F. By (4.1), we have $S(F)^{G_\tau} = \mathbf{Z}H_\tau$. As before, we set $L := H^2(F, \mathbf{Z})$ and put $T := T(F)$ and $S := S(F)$.

We denote by H_ρ a primitive $\tilde{A}_6$-invariant ample class on F (see Theorem 3.1 and Remark 3.4).

Let $\mathcal{R} := \{c, z, x_0, r_0, x_1, x_2\}$ be the set defined in Section 2. Recall that the pointwise stabilizer group $\mathrm{Aut}(\mathcal{D}, \mathcal{R})$ is isomorphic to A_6, that the set $\mathcal{R}$ generates the primitive sublattice R, being isomorphic to $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$, and that we have the following diagram constructed in the proof of Theorem (3.1):

$$T \subset L \supset S = R_\Pi^\perp \subset \Pi = \Lambda \oplus U \supset R ,$$

under which $H_\rho = h$ and the generators $t_1/6$, $t_2/6$ of $A_{T(F)}$ correspond to the generators e_1, e_2 of A_R respectively. See Section 2 for the definition of e_1 and e_2.

Consider the group action $\tau^*_{F,S} : A_6 \longrightarrow \mathrm{GL}(S)$ induced by the geometric action of τ_F on F. The action of $\tau^*_{F,S}$ is faithful, because any action of A_6 on F is symplectic (see (4.1)). Since A_6 is simple and $O(A_S, q_S) \simeq D_8 \times \mathbf{Z}/2$ by Proposition (2.6), the natural homomorphism $\tau^*_{F,S}(A_6) \longrightarrow O(A_S, q_S)$ is trivial. Thus, the action $\tau^*_{F,S}$ on S can be extended to the action $\tau_{F,\Pi}$ on Π in such a way that $\tau_{F,\Pi}|S = \tau^*_{F,S}$ and that $\tau_{F,\Pi}|R = id$.

Let w_R be the element of R^* defined in Lemma (2.8):

$$w_R = -(c + z + r_0 + x_0 + \frac{x_1 + x_2}{2}) .$$

Lemma 5.2. *Consider the element $w_\tau := H_\tau/2 + w_R$ of $\Pi \otimes \mathbf{Q}$. Then, w_τ is a primitive element of Π and $(w_\tau^2) = 0$.*

Proof. We can choose t_1 and t_2 as v_1 and v_2 in Lemma (4.12). Then under the natural isomorphism $A_T \simeq A_R$ induced by the diagram above, we have $t_1/6 \leftrightarrow e_1$ and $t_2/6 \leftrightarrow e_2$, and hence

$$A_T \ni (t_1 + t_2)/2 \leftrightarrow w_R \in A_R .$$

On the other hand, we have also $(H_\tau + t_1 + t_2)/2 \in L$ by Lemma (4.13), in which we now know that $l = 2$. Thus $(t_1 + t_2)/2$, $H_\tau/2$ and w_R give the same element of $A_{T(F)} = A_{S(F)} = A_R$ again under the natural identification induced by the diagram above. Since Π is unimodular, this implies $w_\tau \in \Pi$. Since $(w_\tau, z) = 1$, it follows that w_τ is also primitive. We can check $(w_\tau^2) = 0$ by a direct calculation. □

The two elements w_τ and z of Π generate a sublattice U_τ, which is isomorphic to U, of Π. Set $N := U_\tau^\perp \subset \Pi$. This N is a negative definite even unimodular lattice of rank 24 and satisfies $\Pi = N \oplus U_\tau$.

Lemma 5.3. *N is isomorphic to the Leech lattice.*

Proof. Put $K := \tau_{F,\Pi}(A_6)$. Note that $K \simeq A_6$. Since w_τ and z are fixed by K, this group K acts on its orthogonal complement N. We have $\Pi^K = U_\tau \oplus N^K$ and $\Pi_K = N_K = S_K$ in Π. This is because the action of K is trivial on both R and U_τ. Using the unimodularity of Π and Propositions (4.5) and (4.6) and the fact that $|\det S_K| = |\det L_K| = |\det L^K|$, we calculate that

$$|\det \Pi^K| = |\det \Pi_K| = |\det S_K|$$

$$= 180 = |\det (\mathbf{Z}H_\tau \oplus R)|/4 \ .$$

This shows that Π^K is generated by H_τ, R and w_τ, whence, by w_τ and R. On the other hand, the construction in Theorem (3.1)and Proposition 4.6 tell us that $\Pi^{\rho_{F,\Pi}(A_6)}$ is generated by $w = w_\rho$ and R. Thus, comparing intersection forms, one finds that N^K is isomorphic to $\Lambda^{\rho_{F,\Pi}(A_6)}$. In particular, N^K contains no -2 vector.

Assume that N is not isomorphic to the Leech lattice. Then N is isomorphic to some non-Leech Niemeier lattice $N = N(Rt)$. Then $A_6 \simeq K < S(N) := O(N)/W(N)$, where $W(N)$ is the Weyl group of $N = N(Rt)$, and K acts on the set of 24 simple roots forming Rt [Ko2]. Moreover, K has exactly 5-orbits on the set of 24 simple roots, because $\operatorname{rank} N^K = 5$. Let us set orbit decomposition type by $[a, b, c, d, e]$ with $a \le b \le c \le d \le e$. Then $5a \le a+b+c+d+e = 24$, whence $a \le 4$. However, since A_6 is simple, there is no nontrivial homomorphism $A_6 \simeq K \longrightarrow S_a$ with $a \le 4$. Thus $a = 1$, i.e. K fixes at least one root, a contradiction. Now we are done. □

Therefore, $N \simeq \Lambda$ and there is $h_1 \in O(\Pi)$ such that $h_1(w_\tau) = w$. By Theorem (2.3), there is $h_2 \in \operatorname{Aut}(\mathcal{D})$ such that $h_2h_1(\mathcal{R}) = \mathcal{R}$ (and of course $h_2h_1(w_\tau) = w$). Recall that the natural homomorphism $\operatorname{Aut}(\mathcal{D}, \mathcal{R}) \longrightarrow O(A_R, q_R)$ is surjective by Proposition (2.6). So, there is $h_3 \in \operatorname{Aut}(\mathcal{D}, \mathcal{R})$ such that $h_3h_2h_1(w_\tau) = w$, $h_3h_2h_1(\mathcal{R}) = \mathcal{R}$ and $h_3h_2h_1|\mathcal{R} = id$. Set $f = h_3h_2h_1$ and $f_S := f|S$. Note that $f_S = id$ on A_S. So, we can extend f_S to an isometry φ on $L := H^2(F, \mathbf{Z})$ such that $\varphi_S := \varphi|S = f_S$ and $\varphi_T := \varphi|T = id$. Here we put $T := T(F)$. In particular, φ preserves the Hodge decomposition. Recall that $w_\tau = H_\tau/2 + w_R$ by definition (see Lemma (5.2)) and that $(w =)w_\rho = H_\rho/2 + w_R$ by the construction in Lemma (2.8) (see also Remark 3.4). Then, $\varphi_S(H_\tau) = H_\rho$, and hence φ is also effective. Thus, there is $\psi \in \operatorname{Aut}(F)$ such that $\psi^* = \varphi$ by the global Torelli Theorem.

By the construction of f, we have $f \circ \tau_{F,\Pi} \circ f^{-1}(A_6) \subset \operatorname{Aut}(\mathcal{D}, \mathcal{R}) = A_6$, whence $f \circ \tau_{F,\Pi} \circ f^{-1}(A_6) = \operatorname{Aut}(\mathcal{D}, \mathcal{R})$. On the other hand, by the construction of $(F, \tilde{A}_6, \rho_F)$ in Theorem (3.1), we have also $\rho_{F,\Pi}(A_6) = \operatorname{Aut}(\mathcal{D}, \mathcal{R})$. Thus

$$(S, A_6, \psi_S^* \circ \tau_{F,S}^* \circ (\psi_S^*)^{-1}) = (S, A_6, \rho_{F,S}^*) \ ,$$

up to $\mathrm{Aut}(A_6)$, that is, there is $\Psi \in \mathrm{Aut}(A_6)$, such that

$$\psi^* \circ \tau^*_{F,S}(a) \circ (\psi^*)^{-1}(v) = \rho^*_{F,S}(\Psi(a))(v)$$

for all $a \in A_6$ and $v \in S$. Since both G_N and $(\tilde{A}_6)_N$ act on T trivially, we have also

$$(T, A_6, \psi^*_T \circ \tau^*_{F,T} \circ (\psi^*_T)^{-1}) = (T, A_6, \tau^*_{F,T}) = (T, A_6, \rho^*_{F,T})$$

via the *same* $\Psi \in \mathrm{Aut}(A_6)$. Hence, we obtain

$$(L, A_6, \psi^* \circ \tau^*_{F,L} \circ (\psi^*)^{-1}) = (L, A_6, \rho^*_{F,L})$$

again via Ψ. We note that $\psi^* \circ \tau^*_{F,L} \circ (\psi^*)^{-1}$ and $\rho^*_{F,L}$ are both defined over L. So, they coincide on L if they coincide on the finite index sublattice $S \oplus T$ of L. Now, it follows from the injectivity part of the global Torelli theorem for $K3$ surfaces that $(F, A_6, \psi^{-1} \circ \tau_F \circ \psi) \simeq (F, A_6, \rho_F)$, and we can identify $(F, G_N) = (F, (\tilde{A}_6)_N)$. Then $S(F)^G = S(F)^{G_N} = S(F)^{(\tilde{A}_6)_N} = S(F)^{\tilde{A}_6} = \mathbf{Z}H$ by Proposition (4.1)(1) applied for F. Here we denote by $H = H_\rho$ the $\tilde{A}_6$-invariant primitive polarization on F. Thus $G\,, \tilde{A}_6 < \mathrm{Aut}(F, H)$, i.e. G and $\tilde{A}_6$ are subgroups of the automorphism group of the polarized $K3$ surface (F, H). Since $\mathrm{Aut}(F, H)$ is a finite group containing A_6, it follows from Proposition (4.1)(2) that $|\mathrm{Aut}(F, H)| \leq 4|A_6|$. Since $G, \tilde{A}_6 \leq \mathrm{Aut}(F, H)$ and $|G| = |\tilde{A}_6| = 4|A_6|$, we have then $G = \mathrm{Aut}(F, H) = \tilde{A}_6$. This completes the proof. □

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