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Laplace Transforms for Integrals of Markov Processes

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Abstract. Laplace transforms for integrals of stochastic processes have been known in analytically closed form for just a handful of Markov processes: namely, the Ornstein-Uhlenbeck, the Cox-Ingerssol-Ross (CIR) process and the exponential of Brownian motion. In virtue of their analytical tractability, these processes are extensively used in modelling applications. In this paper, we construct broad extensions of these process classes. We show how the known models fit into a classification scheme for diffusion processes for which Laplace transforms for integrals of the diffusion processes and transitional probability densities can be evaluated as integrals of hypergeometric functions against the spectral measure for certain self-adjoint operators. We also extend this scheme to a class of finite-state Markov processes related to hypergeometric polynomials in the discrete series of the Askey classification tree.

1. Introduction

Let $(X_t)_{t\geq 0}$ be a time-homogenous, real-valued Markov process on the filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ and consider the Laplace transform $L_{T-t}(X_t, \vartheta)$ defined as follows:

$$L_{T-t}(X_t, \vartheta) = E^{\mathrm{P}}\left[e^{-\vartheta \int_t^T \phi(X_s) ds}\ q(X_T) \Big| \mathcal{F}_t \right] \tag{1.1}$$

where $t \leq T$, $\vartheta \in \mathbb{C}$ and $\phi, q : \mathbb{R} \to \mathbb{R}$ two Borel functions. In this paper, we address the question of whether it is possible to compute the Laplace transform $L_{T-t}(X_t, \vartheta)$ in analytically closed form. Our work builds upon several streams of research often motivated by applications to various fields of Physics and Finance, and unifies them to obtain a broad classification scheme for Laplace transforms expressible in analytic closed form.

For $q \equiv 1$, a class of examples for which analytic closed form solutions are available is represented by the so called *affine models* which are characterized by a representation of the form

$$L_{T-t}(X_t, \vartheta) = e^{m(T-t,\vartheta)X_t + n(T-t,\vartheta)}. \tag{1.2}$$

The archetypical affine models are based on diffusion processes and are described by stochastic differential equations of the form

$$dX_t = (a - bX_t)dt + \sigma X_t^{\beta} dW_t \tag{1.3}$$

where a, b, σ are constants and $\beta = 0$ or $\frac{1}{2}$. The case $\beta = 0$ corresponds to the Gaussian Ornstein-Uhlenbeck process [33] and the case $\beta = \frac{1}{2}$ corresponds to the Cox-Ingersoll-Ross (CIR) process [7]. The case of the CIR process was generalized to bridges by Pitman

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and Yor in [27]. It has been shown in [4], [5] and in [11] that any affine process which is a time-homogenous, nonnegative diffusion is necessarily of the CIR type. However, there are also affine processes with jumps. General non-negative affine processes correspond to the so called conservative CBI-processes (continuous state branching processes with immigration) and have been well studied, among others, by Kawazu and Watanabe in [19] and Filipović in [13].

An extension of the affine class, known as the quadratic class, postulates the Laplace transform being of the form

$$L_{T-t}(X_t, \vartheta) = e^{l(T-t,\vartheta)X_t^2 + m(T-t,\vartheta)X_t + n(T-t,\vartheta)}. \tag{1.4}$$

The first examples of quadratic models appeared in the double square root model of Longstaff in [23] and in the nonlinear equilibrium model by Beaglehole and Tenney in [3]. Rogers [30] also uses examples where the pricing kernel is a quadratic function of the Markov process. Most recently, Filipović [14] proved that if one represents the forward rate as a polynomial function of the diffusion process, the maximal consistent order of the polynomial is two. Consistency in this context means that the interest rate model will produce forward rate curves belonging to the parameterized family. Finally, Leippold and Wu [21] formulated a general asset and derivative pricing framework for the quadratic class.

A separate class of models for which the Laplace transform can be expressed in analytically closed form is represented by the exponential Brownian motion of equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1.5}$$

where $\mu, \sigma > 0$ are positive constants. This case was first considered by Yor in [35] who arrived to an expression involving a triple integral. An earlier related result for bond prices given in terms of an integral over modified Bessel functions was formulated by Dothan in [10]. As an alternative, Geman and Yor [17] derive a closed-form expression for the Laplace transform in terms of confluent hypergeometric functions (see Donati-Martin et al. [9] and Yor [36] for further references). For applications to finance, one needs to compute the inverse Laplace transform, for which numerical methods have been developed by Geman and Eydeland [16], Fu et al. [15], Craddock et al. [8], Shaw [31]. Dufresne in [12] and Linetsky in [22] develop analytical methods and alternative expansions.

In this article, we obtain far reaching extensions of the representation formula for the Laplace transform for the integrals of stochastic processes over geometric Brownian motions. The key idea is to seek expansions of similar form as those in [12] and [22] but expressed in terms of more general hypergeometric functions, see [1], as pioneered by Wong in [34]. Let us recall that general hypergeometric functions are denoted as follows:

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) \tag{1.6}$$

for $p \leq q+1, \gamma_j \in \mathbb{C} \backslash -\mathbb{Z}_+$, and are represented by the following Taylor expansion around $z = 0$:

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\gamma_1)_n \ldots (\gamma_q)_n} \frac{z^n}{n!}. \tag{1.7}$$

The Kummer functions in the work by Geman and Yor [17] are in the family ${}_1F_1$ of the so called confluent hypergeometric functions. Gaussian hypergeometric functions are in the family ${}_2F_1$ and admit the functions of type ${}_1F_1$ as limits. Both Gaussian and

confluent hypergeometric functions solve differential equations of the Fuchsian class, see [18]. Specifically

$$z(1-z)\ {}_2F_1''(\alpha,\beta;\gamma;z)+(\gamma-(1+\alpha+\beta)z)\ {}_2F_1'(\alpha,\beta;\gamma;z)-\alpha\beta\ {}_2F_1(\alpha,\beta;\gamma;z)=0 \quad (1.8)$$

and

$$z\ {}_1F_1''(\alpha;\gamma;z)+(\gamma-z)\ {}_1F_1'(\alpha;\gamma;z)-\alpha\ {}_1F_1(\alpha;\gamma;z)=0. \quad (1.9)$$

In general, higher order hypergeometric functions are not associated to a differential equation. However, in some particularly important cases, they provide solutions of finite difference equations. The Askey classification scheme, see [20] and [32], gives a complete list of all orthogonal polynomials solving either a differential or a finite difference equation and in addition satisfy a recurrence relation. All of these polynomials descend as particular or limiting cases from the so-called Racah polynomials, which are particular cases of the hypergeometric functions ${}_4F_3$.

We first consider the case of diffusion processes and next the case of finite state Markov processes. In the diffusion case, we construct a classification scheme based on reduction to eigenvalue problems admitting solutions within the class of Gaussian and confluent hypergeometric functions ${}_2F_1$ and ${}_1F_1$. In the second case, the problem is more difficult for several reasons, as there is no discrete equivalent of a theory of Fuchs type equations and, in addition, the groups of conformal transformations and diffeomorphisms do not extend to lattices. What we do in the discrete case is to take the moves from the Askey classification scheme for orthogonal polynomials and show how to extend the previous spectral decomposition to the case of Meixner, dual Hahn and Racah polynomials, which are special cases of ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ hypergeometric functions.

For a diffusion process, on a domain $D_x \subset \mathbb{R}$, of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (1.10)$$

we define the transitional probability density $p_{T-t}(x,y)$ as the density of the Markov semigroup of the process X_t:

$$E^{\mathrm{P}}\left[\, f(X_T)\big|\mathcal{F}_t\,\right] = \int_D f(y)\ p_{T-t}(x,y)dy. \quad (1.11)$$

We are interested in building a classification scheme for the drift and volatility functions $\mu(x),\sigma(x)$ such that the calculation of both functions p and L can be reduced to computing an integral over hypergeometric functions. The transitional probability density and Laplace transform can be computed in terms of the spectral resolution for the infinitesimal generator of the process X_t

$$\mathcal{L} = \frac{\sigma(x)^2}{2}\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x} \quad (1.12)$$

and the Feynman-Kac operator

$$\tilde{\mathcal{L}} = \mathcal{L} - \vartheta\phi(x) = \frac{\sigma(x)^2}{2}\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x} - \vartheta\phi(x). \quad (1.13)$$

In fact, we have that

$$p_{T-t}(x,y) = e^{(T-t)\mathcal{L}}(x,y) \quad (1.14)$$

and

$$L_{T-t}(x,\vartheta) = \int_D q(y)\ e^{(T-t)\tilde{\mathcal{L}}}(x,y)dy. \quad (1.15)$$

As we show in detail in Section 2, these operators are conjugated by a non-singular transformation to self-adjoint operators which admit a spectral resolution. The calculation of the transitional probability density in (1.14) and the Laplace transform in (1.15) is thus reduced to the resolution of the differential eigenvalue problems

$$\mathcal{L}\ f(x) = \lambda\ f(x) \quad , \quad \tilde{\mathcal{L}}\ \bar{f}(x) = \bar{\lambda}\ \bar{f}(x). \tag{1.16}$$

To properly define the classification scheme, we specify by what means the reduction can be accomplished.

Definition 1. *The problem of finding the transitional probability density and the Laplace transform for the process in (1.10) is said to be* reducible to a spectral integral over hypergeometric functions *if the two eigenvalue problems in (1.16) can be recast in the form of a differential equation for hypergeometric functions such as either (1.9) or (1.8), by means of a combination of the following three operations T_i:*

1. *T_Z: change of variable $x \mapsto z = Z(x)$ where $Z(x)$ is a diffeomorphism $Z : D_x \to D_z$ such that*
$$\mathcal{L}_x \mapsto \mathcal{L}_z \quad , \quad \tilde{\mathcal{L}}_x \mapsto \tilde{\mathcal{L}}_z, \tag{1.17}$$
2. *T_h: gauge transformation associated to a strictly positive function h such that*
$$\mathcal{L} \mapsto h^{-1}\mathcal{L}\ h \quad , \quad \tilde{\mathcal{L}} \mapsto h^{-1}\tilde{\mathcal{L}}\ h, \tag{1.18}$$
3. *T_{γ^2}: left-multiplication by a strictly positive function γ^2 such that*
$$\mathcal{L} \mapsto \gamma^2\mathcal{L} \quad , \quad \tilde{\mathcal{L}} \mapsto \gamma^2\tilde{\mathcal{L}}. \tag{1.19}$$

The third kind of transformations was first recognized in its generality by Natanzon in the article [25] on integrable Schrödinger equations, see also Milson's paper [24]. The following theorem gives a concise statement of our main classification result:

Theorem 1 (First Classification Theorem). *The most general reducible diffusion process (up to diffeomorphism) according to Definition 1 can be constructed as follows:*

1. *four second order polynomials in x: $A(x)$, $Q(x,\vartheta)$, $R(x)$, $S(x)$, such that $A(x)$ belongs to the set $\{1, x, x(1-x), x^2+1\}$ and $R(x) \geq 0$;*
2. *conditions for the stochastic process X_t on the boundary of the domain D_x, specifying the relative probability of reflection versus absorption upon hitting the boundary;*
3. *a solution of the following equation in D_x for some $\xi \in \mathbb{R}$:*
$$\frac{A(x)^2}{R(x)}h''(x) + \frac{S(x)}{R(x)}h(x) = \xi h(x) \tag{1.20}$$
The function $h(x)$ is a linear combination of hypergeometric functions of the confluent type ${}_1F_1$ if $A(x) \in \{1, x\}$ and of the Gaussian type ${}_2F_1$ if $A(x) \in \{x(1-x), x^2+1\}$.

The process associated to this choice is given by a generic solution to the following stochastic differential equation on the domain D_x:

$$dX_t = 2\frac{h'(X_t)}{h(X_t)}\ \frac{A(X_t)^2}{R(X_t)}\ dt + \frac{\sqrt{2}A(X_t)}{\sqrt{R(X_t)}}\ dW_t, \tag{1.21}$$

with the boundary conditions above. The Laplace transform is specified by

$$\phi(x) = \frac{Q(x,\vartheta)}{\vartheta R(x)}. \tag{1.22}$$

The proof of this theorem is in Section 2. These constructs are based on spectral analysis techniques for which we refer to the book by Reed and Simon [29]. We also make the spectral analysis more explicit and list the expressions for the transitional probability density and Laplace transforms in terms of the kernel of semigroups generated by integrable quantum Schrödinger operators. In Section 3, we specialize further and rediscover the known cases of processes built upon the geometric Brownian motion and on the Ornstein-Uhlenbeck and CIR processes, along with some interesting extensions.

In Sections 4 and 5, we restrict the framework to the special case of hypergeometric polynomials. In the discrete case, studied in Section 5, the process X_t takes on only a discrete set of values, as opposed to following a diffusion process. In this class of models, we base our analysis on the Askey-Wilson theory of orthogonal polynomials, see [2] and [32]. We briefly review the basic notions, following [20].

Definition 2. *An* orthogonal system of polynomials *is given by a sequence of polynomials $Q_n(x)$ of order n for $n \in \mathbb{N}$ on the interval $D \subseteq \mathbb{R}$ which satisfies an orthogonality condition of the form*

$$\int_D Q_n(x)Q_m(x)\rho(dx) = d_n^2\delta_{nm} \quad , \quad n, m \in \mathbb{N}, \tag{1.23}$$

where the d_n are constants and $\rho(dx)$ is a given measure. One distinguishes between continuous polynomials *whereby $\rho(dx)$ is absolutely continuous with respect to the Lebesgue measure, i.e.*

$$\rho(dx) = w(x)dx \tag{1.24}$$

for some weight function $w(x)$, and discrete polynomials *for which*

$$\rho(dx) = \sum_{i=0}^{N} w(x_i)\delta(x-i) \quad , \quad N \in \mathbb{N}. \tag{1.25}$$

All orthogonal polynomials satisfy a *three-term recurrence relation* of the form

$$xQ_n(x) = A_nQ_{n+1}(x) - B_nQ_n(x) + C_nQ_{n-1}(x) \tag{1.26}$$

where $n \geq 1$, $A_n > 0$, $C_n \geq 0$ and $B_n \in \mathbb{R}$. Together with the conditions $Q_{-1}(x) = 0$ and $Q_0(x) = 1$, all the $Q_n(x)$ can be determined based on this recurrence relation. The converse is also true and is known as the Favard theorem, see [6]. Moreover, they satisfy the following eigenvalue equation,

$$\mathcal{L}Q_n(x) = \lambda_n Q_n(x), \tag{1.27}$$

for $\mathcal{L}$ a second-order differential operator in the continuous case or a finite difference operator in the discrete case.

The reducibility condition in Definition 1 is mirrored by the following (inequivalent) one which refers to orthogonal polynomials in the continuous series as opposed to Gaussian hypergeometric functions:

Definition 3. *The problem of finding the transitional probability density in (1.14) and the Laplace transform in (1.15) is said to be* reducible to a spectral integral over orthogonal polynomials *if the two eigenvalue problems in (1.16) have the same orthogonal polynomials as eigenfunctions.*

We first restrict the framework to continuous orthogonal polynomials that have as generator a second-order differential operator,

$$\mathcal{L} = \frac{\sigma^2}{2} A(x) \frac{\partial^2}{\partial x^2} + (a - bx) \frac{\partial}{\partial x}, \tag{1.28}$$

acting on the Hilbert space $L^2(D, \rho)$, where $A(x) \in \{1, x, x(1-x)\}$, $a \in \mathbb{R}$ and $b, \sigma > 0$. This class consists of the Hermite, Laguerre and Jacobi polynomials up to diffeomorphism. Our main result concerning the latter continuous orthogonal polynomials can be stated as follows:

Theorem 2 (Second Classification Theorem). *The most general reducible diffusion process (up to diffeomorphism) in the sense of Definition 3, has infinitesimal generator $\mathcal{L}$ given by (1.28). Its transitional probability density can be expressed as*

$$p_{T-t}(x, y) = \sum_{n=0}^{\infty} \frac{e^{\lambda_n (T-t)}}{d_n^2} Q_n(x; a, b) Q_n(y; a, b) w(y). \tag{1.29}$$

Furthermore, for some parameters $\bar{a}, \bar{b}, C \in \mathbb{R}$,

$$\phi(x) = C + \big((a - bx) - (\bar{a} - \bar{b}x)\big) \frac{A'(x)}{2\vartheta A(x)} + \frac{(\bar{a} - \bar{b}x)^2 - (a - bx)^2}{2\vartheta \sigma^2 A(x)} \tag{1.30}$$

and the Laplace transform is given by the following convergent series:

$$L_{T-t}(x, \vartheta) = \exp\left(\int^x \frac{(\bar{a} - \bar{b}y) - (a - by)}{\sigma^2 A(y)} dy \right) \sum_{n=0}^{\infty} e^{\bar{\lambda}_n (T-t)} z_n Q_n(x; \bar{a}, \bar{b}). \tag{1.31}$$

The coefficients z_n are given by:

$$z_n = \frac{1}{\bar{d}_n^2} \int_{D_x} q(x) \exp\left(- \int^x \frac{(\bar{a} - \bar{b}y) - (a - by)}{\sigma^2 A(y)} dy \right) Q_n(x; \bar{a}, \bar{b}) \bar{\rho}(dx). \tag{1.32}$$

In Section 4 we present the proof of this alternative classification scheme based on orthogonal polynomials of the continuous series. This discussion sets the premise for the extension of the result to orthogonal polynomials in the discrete series. Discrete orthogonal polynomials are characterized by a finite difference generator on the Hilbert space $l^2(\Lambda_N, w)$ with Λ_N the set $\{0, \ldots, N\}$.

Definition 4. *Let Δ^h and ∇_+^h denote the difference operators defined as follows:*

$$\Delta^h y(x) = y(x+h) - 2y(x) + y(x-h) \quad , \quad \nabla_+^h y(x) = y(x+h) - y(x). \tag{1.33}$$

Using these operators, the finite difference generator $\mathcal{L}$ takes the form

$$\mathcal{L} = -D(x)\Delta^1 + \big(D(x) - B(x)\big)\nabla_+^1, \tag{1.34}$$

where $B(x)$ and $D(x)$ are rational functions of at most fourth order in the numerator and at most second order in the denominator. Our main result in the discrete case can be stated as follows:

Theorem 3 (Third Classification Theorem). *The most general reducible discrete Markov process in the sense of Definition 3, has infinitesimal generator $\mathcal{L}$ given by (1.34). Its transitional probability density can be expressed as*

$$p_{T-t}(x,y) = \sum_{n=0}^{N} \frac{e^{\lambda_n(T-t)}}{d_n^2} Q_n(x)Q_n(y)w(y). \tag{1.35}$$

$\phi(x)$ must be of the form

$$\phi(x) = B(x) + D(x) - \bar{B}(x) - \bar{D}(x), \tag{1.36}$$

where $\bar{B}(x)$ and $\bar{D}(x)$ are the same rational functions as $B(x)$ and $D(x)$ up to a multiplicative constant, but for different parameters, and satisfy the lattice condition

$$\bar{B}(x-1)\bar{D}(x) = B(x-1)D(x). \tag{1.37}$$

The Laplace transform is given by the convergent series ($\prod_{k=1}^{0} = 1$ by convention):

$$L_{T-t}(x,1) = \prod_{k=1}^{x} \frac{D(k)}{\bar{D}(k)} \sum_{n=0}^{N} e^{\bar{\lambda}_n(T-t)} z_n \bar{Q}_n(x). \tag{1.38}$$

The coefficients z_n are as follows:

$$z_n = \frac{1}{\bar{d}_n^2} \sum_{x\in\Lambda_N} \prod_{k=1}^{x} \frac{\bar{D}(k)}{D(k)}\, q(x)\bar{Q}_n(x)\bar{w}(x). \tag{1.39}$$

In our notation, $\{\bar{Q}_n(x)\}$ is the same set of orthogonal polynomials as $\{Q_n(x)\}$ expect for the value of the parameters. Section 5 gives a proof of the latter result, as well as explicit representations for processes based on the Meixner, the dual Hahn and the Racah polynomials.

This paper is organized as follows: Section 2 presents the proof of the first classification scheme in the diffusion case and goes in more detail to provide a spectral representation formula for transitional probability densities and the Laplace transform. Section 3 contains a discussion of the classical examples showing how the geometric Brownian motion, the Ornstein-Uhlenbeck and the CIR process fit in this classification scheme. Section 4 contains the proof of an alternative classification scheme based on orthogonal polynomials of the continuous series, along with interesting examples. This discussion sets the premise for Section 5 where we extend the result to orthogonal polynomials in the discrete series. Finally, in Section 6 we discuss limiting relation and establish connections between models corresponding to the discrete and the continuous series, which is a useful result to construct numerical very stable discretization schemes.

2. Classification Theorem for Diffusion Processes

In this section, we prove our first classification result, Theorem 1, in the diffusion case. We start by reviewing some background notions concerning Fuchsian differential equations and the so-called Bose invariants, and then proceed to the proof of the theorem.

2.1. Fuchsian Differential Equations

Consider the second order partial differential equations for the holomorphic function $F(z)$

$$F''(z) + p(z)F'(z) + q(z)F(z) = 0 \tag{2.1}$$

for some holomorphic functions $p(z), q(z)$.

Definition 5. *Let $\alpha \in \mathbb{C}$ be an isolated singularity for the holomorphic function $F(z)$. The singularity in α is called regular if there is an exponent $\rho \in \mathbb{C}$ for which the function $(z-\alpha)^{-\rho}F(z)$ admits a Laurent expansion with finitely many negative powers around $z = \alpha$, i.e.*

$$F(z) = (z-\alpha)^{\rho} \sum_{n=-m}^{0} (z-\alpha)^n \tag{2.2}$$

for some $m \in \mathbb{N}$. The point ∞ is a regular singularity of the function $F(z)$ if $z = 0$ is a regular singularity of the function $F\left(\frac{1}{z}\right)$.

In [18], Fuchs gives conditions on the coefficients $p(z)$ and $q(z)$ which ensure that solutions have only *regular* singularities.

Theorem 4 (Fuchs). *Let $F(z)$ be a solution of equation (2.1) with singularities in the points $\alpha_1, \ldots, \alpha_n$ and ∞. Then these singularities are all regular if and only if the functions $p(z)$ and $q(z)$ have the form*

$$p(z) = \frac{p_0(z)}{(z-\alpha_1)\ldots(z-\alpha_n)} \tag{2.3}$$

and

$$q(z) = \frac{q_0(z)}{(z-\alpha_1)^2\ldots(z-\alpha_n)^2} \tag{2.4}$$

where $p_0(z)$ is a polynomial of order $(n-1)$ and $q_0(z)$ is a polynomial of order $2n-2$.

An alternative expression for the coefficient $p(z)$ of an equation with only regular singularities is

$$p(z) = \sum_{i=1}^{n} \frac{\delta_i}{z-\alpha_i} \tag{2.5}$$

where the δ_i, $i = 1, ..n$, are constants. In particular, we have that

$$\exp\left(\frac{1}{2}\int^z p(w)dw\right) \;=\; C\prod_{i=1}^{n}(z-\alpha_i)^{\frac{\delta_i}{2}} \tag{2.6}$$

where C is a constant. The function

$$\bar{F}(y) = \prod_{i=1}^{n}(y-\alpha_i)^{\frac{\delta_i}{2}}F(y) \tag{2.7}$$

solves the equation

$$\bar{F}''(y) + I(y)\bar{F}(y) = 0 \tag{2.8}$$

where

$$I(y) = -\frac{1}{2}p'(y) + \frac{1}{4}p(y)^2 + q(y). \tag{2.9}$$

Definition 6. *The function $I(y)$ is called the* Bose invariant *of the equation (2.1). Notice that $I(y)$ has the form*

$$I(y) = \frac{I_0(y)}{(y-\alpha_1)^2 \dots (y-\alpha_n)^2} \tag{2.10}$$

where $I_0(y)$ is a polynomial of order $2n-2$, without restrictions on the coefficients.

Let us focus again on the case $n = 2$, assume that coefficients are real and that only *real* linear fractional transformations are allowed to move the regular singularities. In this situation we have to distinguish between two different cases for Bose invariants:

– Case I ($\alpha_1 = 0,\ \alpha_2 = 1$):

$$I(y) = \frac{s_0(1-y) + s_1 y + s_2 y(1-y)}{y^2(1-y)^2} \tag{2.11}$$

– Case II ($\alpha_1 = i,\ \alpha_2 = -i$):

$$I(y) = \frac{s_0 + s_1 y + s_2 y^2}{(y^2+1)^2} \tag{2.12}$$

These cases reduce to the Gaussian hypergeometric equation (2.18) for the function ${}_2F_1$ as is shown below. Furthermore, special cases for the Bose invariant occur in the limit when either α_1 or α_2 or both roots tend to ∞, i.e.

– Case III ($\alpha_1 = 0,\ \alpha_2 = \infty$):

$$I(y) = \frac{s_0 + s_1 y + s_2 y^2}{y^2} \tag{2.13}$$

– Case IV ($\alpha_1 = \infty,\ \alpha_2 = \infty$):

$$I(y) = s_0 + s_1 y + s_2 y^2 \tag{2.14}$$

Case III reduces to the confluent hypergeometric equation ${}_1F_1$ and Case IV corresponds to the case of triple confluence at infinity. Notice that the above four cases can all be captured by a single expression as stated in the following:

Proposition 1. *The Bose invariants corresponding to the Gaussian hypergeometric function ${}_2F_1$ and to its confluent limit can be reduced to the following normal form by means of a real valued linear fractional transformation:*

$$I(y) = \frac{Q(y)}{A(y)^2} \tag{2.15}$$

where $A(y) \in \{y(1-y), y^2+1, y, 1\}$ and Q is a polynomial in y with $\deg Q \le 2$.

The first two cases correspond to three regular singularities at distinct points. In these cases, solutions can be expressed through Gaussian hypergeometric functions ${}_2F_1$. Fractional linear transformations of the form

$$z \mapsto \frac{az+b}{cz+d} \tag{2.16}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, are one-to-one maps of the extended complex line $\mathbb{C} \cup \infty$ into itself and map regular singularities into regular singularities. By applying

a fractional linear transformation, one can map the singularities α_1 and α_2 to 0 and 1, respectively. Furthermore, we have transformations of the form

$$F(z) \mapsto (z-\alpha_1)^{\rho_1}(z-\alpha_2)^{\rho_2}F(z). \tag{2.17}$$

The combination of these two transformations allows one to reduce any Fuchsian differential equation with three regular singular points to the form

$$z(1-z)F''(z) + (\gamma - (1+\alpha+\beta)z)F'(z) - \alpha\beta F(z) = 0. \tag{2.18}$$

The function ${}_2F_1(\alpha,\beta;\gamma;z)$ is an elementary solution of this equation along with ${}_2F_1(\alpha,\beta;1+\alpha+\beta-\gamma;1-z)$.

Case III corresponds to the limit when a regular singularity merges with the regular singularity at ∞ while the other one stays at 0. This limit can be obtained starting from the equation corresponding to two coinciding singularities at 0, i.e. $\alpha_1 = \alpha_2 = 0$:

$$F''(z) + \frac{c_1 + c_2 z}{z^2}F'(z) + \frac{c_3 + c_4 z + c_5 z^2}{z^4}F(z) = 0. \tag{2.19}$$

By applying the coordinate transformation $z \mapsto \frac{1}{z}$ we find that

$$z^2F''(z) + ((2+c_2)z + c_1 z^2)F'(z) + (c_3 + c_4 z + c_5 z^2)F(z) = 0 \tag{2.20}$$

By rescaling the independent variable z and rescaling the function so that $F(z) \mapsto e^{\rho z}f(\omega z)$ this equation reduces to the Kummer differential equation

$$zF''(z) + (\gamma - z)F'(z) - \alpha F(z) = 0 \tag{2.21}$$

which admits ${}_1F_1(\alpha;\gamma;z)$ as a solution. In alternative, one can reduce equation (2.20) to the form

$$F''(z) + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right)F(z) = 0 \tag{2.22}$$

which is called the *Whittaker differential equation*. The case where all three singularities merge at ∞ is also interesting and can be solved by rescaled confluent hypergeometric functions.

2.2. Proof of the First Classification Theorem 1

We start by presenting obvious facts about the transformations T_i in Definition 1.

Proposition 2. *The transformations T_i are invertible with respective inverse:*

$$T_Z^{-1} = T_X \quad , \quad T_h^{-1} = T_{\frac{1}{h}} \quad , \quad T_{\gamma^2}^{-1} = T_{\frac{1}{\gamma^2}} \tag{2.23}$$

where $X : D_z \to D_x$ is the inverse of $Z(x)$.

Proposition 3. *The transformations T_i commute with one another.*

The proof of the First Classification Theorem 1 for diffusion processes follows. The process X_t has infinitesimal generator

$$\mathcal{L} = \frac{A(x)^2}{R(x)}\frac{\partial^2}{\partial x^2} + 2\frac{h'(x)}{h(x)}\frac{A(x)^2}{R(x)}\frac{\partial}{\partial x}. \tag{2.24}$$

To solve the eigenvalue problem $\mathcal{L}f = \lambda f$, define the following left-multiplication T_{γ^2} and gauge transformation T_h:

$$T_{\gamma^2}\mathcal{L} = \frac{A(x)^2}{R(x)}\mathcal{L} \quad \text{and} \quad T_h\mathcal{L} = \frac{1}{h}\mathcal{L}h, \tag{2.25}$$

so that the eigenvalue equation then transforms into

$$T_{\gamma^2}^{-1}T_h^{-1} : \mathcal{L}f = \lambda f \mapsto \left(\frac{\partial^2}{\partial x^2} + \frac{S(x)}{A(x)^2}\right) f(x) = \lambda\frac{R(x)}{A(x)^2}f(x). \tag{2.26}$$

As $R(x), S(x)$ are second order polynomials and $A(x) \in \{1, x, x(1-x), 1+x^2\}$, the solution $f(x)$ can be expressed as a hypergeometric function with Bose invariant

$$I(x) = \frac{S(x) - \lambda R(x)}{A(x)^2}. \tag{2.27}$$

The same operations can be applied to the eigenvalue problem for the Feynman-Kac operator, $\tilde{\mathcal{L}}\bar{f} = \bar{\lambda}\bar{f}$ with

$$\tilde{\mathcal{L}} = \mathcal{L} - \vartheta\phi(x), \tag{2.28}$$

which leads to

$$T_{\gamma^2}^{-1}T_h^{-1} : \tilde{\mathcal{L}}\bar{f} = \bar{\lambda}\bar{f} \mapsto \left(\frac{\partial^2}{\partial x^2} + \frac{S(x)}{A(x)^2} - \vartheta\phi(x)\right) \bar{f}(x) = \bar{\lambda}\frac{R(x)}{A(x)^2}\bar{f}(x). \tag{2.29}$$

For $\bar{f}(x)$ to be expressed as a hypergeometric function, we require

$$\phi(x) = \frac{Q(x,\vartheta)}{\vartheta A(x)^2} \tag{2.30}$$

for $Q(x,\vartheta)$ a second order polynomial in x. The Bose invariant for the equation is

$$\tilde{I}(x) = \frac{S(x) - Q(x,\vartheta) - \bar{\lambda}R(x)}{A(x)^2}. \tag{2.31}$$

The converse follows from the facts that the transformations are invertible and the Bose invariants are both expressed in the most general form.

Remark 1. The first transformation (T_Y: change of variable) has not been used in the proof. X_t is therefore the most general reducible diffusion process only up to diffeomorphism.

2.3. Spectral Resolutions

Theorem 1 describes all processes with explicitly solvable transitional probability density and Laplace transform for the integral of the process. We wish now to give a closed form expression for both quantities. The following lemma allows one to determine the nature of the spectrum of the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$. The nature of the spectrum is indeed based on the shape of the Schrödinger potential, coming out of the eigenvalue problem once reduced to a Schrödinger equation. Theorem 5 shows how this transformation operates on the kernel of the semigroup generated by the Schrödinger operators and gives a general spectral resolution of the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

Lemma 1. *Let $T = T_g\ T_Z\ T_h^{-1}$ where the diffeomorphism $Z : D_x \to D_z$ is given by $Z'(x) = \dfrac{\sqrt{R(x)}}{A(x)}$ with inverse X and the gauge transformation T_g by $g(z) = \left(\dfrac{A(X(z))^2}{R(X(z))}\right)^{1/4}$. Then the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ reduce to the following Schrödinger operators:*

$$\begin{aligned} T\mathcal{L} &= \frac{\partial^2}{\partial z^2} - U_1(z) \equiv -\mathbb{H}_1 \\ T\tilde{\mathcal{L}} &= \frac{\partial^2}{\partial z^2} - U_2(z) \equiv -\mathbb{H}_2 \end{aligned} \tag{2.32}$$

where the potentials are given by

$$\begin{aligned} U(z) &= \left(\frac{g'}{g}\right)^2 - \left(\frac{g'}{g}\right)' - \frac{S(X(z))}{R(X(z))} \\ \tilde{U}(z) &= \left(\frac{g'}{g}\right)^2 - \left(\frac{g'}{g}\right)' - \frac{S(X(z)) - Q(X(z),\vartheta)}{R(X(z))} \end{aligned} \tag{2.33}$$

and $'$ denotes the derivative with respect to z.

Proof.

$$\begin{aligned} T_g\ T_Z\ T_h^{-1}\mathcal{L} &= T_g\ T_Z \left(\frac{A(x)^2}{R(x)}\frac{\partial^2}{\partial x^2} + \frac{S(x)}{R(x)}\right) \\ &= T_g \left(\frac{\partial^2}{\partial z^2} + \frac{Z''}{(Z')^2}\frac{\partial}{\partial z} + \frac{S(X(z))}{R(X(z))}\right) \\ &= \frac{\partial^2}{\partial z^2} + \frac{g''}{g} - 2\left(\frac{g'}{g}\right)^2 + \frac{S(X(z))}{R(X(z))} \end{aligned}$$

and

$$T_g\ T_Z\ T_h^{-1}\tilde{\mathcal{L}} = T_g\ T_Z\ T_h^{-1}\mathcal{L} - \vartheta\phi(X(z)).$$

The two Schrödinger operators $\mathbb{H}_1$ and $\mathbb{H}_2$ defined in the previous lemma have a spectral resolution,

$$\mathbb{H}_i \Phi_\rho(z) = \rho\ \Phi_\rho(z), \tag{2.34}$$

given by a complete set of normalized eigenfunctions $\Phi_\rho(z)$ for $\rho = -\lambda$ or $\rho = -\bar{\lambda}$, $i = 1, 2$ respectively. The spectrum is in general a combination of a pure-point spectrum $\sigma_{pp}(\mathbb{H}_i)$ and an absolutely continuous spectrum $\sigma_{ac}(\mathbb{H}_i)$. The kernel of the semigroup generated by the respective Schrödinger operators has the general form

$$e^{-(T-t)\mathbb{H}_i}(z_0, z_1) = \sum_{\rho\in\sigma_{pp}(\mathbb{H}_i)} e^{-(T-t)\rho}\Phi_\rho(z_0)\Phi_\rho^*(z_1) + \int_{\rho\in\sigma_{ac}(\mathbb{H}_i)} e^{-(T-t)\rho}\Phi_\rho(z_0)\Phi_\rho^*(z_1)dk(\rho) \tag{2.35}$$

with $dk(\rho) = \frac{d\rho}{2\sqrt{\rho - U_i^-}}$ and U_i^- the lowest limit of the potential $U_i(z)$ as z tends to the boundaries of the domain D_z.

Theorem 5. *The transitional probability density and the Laplace transform of any reducible process described in Theorem 1 by the operator $\mathcal{L}$ and $\tilde{\mathcal{L}}$ is related to the kernels of the semigroups generated by the respective Schrödinger operators as follows:*

$$\begin{aligned} e^{(T-t)\mathcal{L}}(x,y) &= \frac{h(y)}{h(x)}\ \left(\frac{A(x)^2}{R(x)}\right)^{1/4}\left(\frac{R(y)}{A(y)^2}\right)^{3/4} e^{-(T-t)\mathbb{H}_1}\big(Z(x), Z(y)\big), \\ e^{(T-t)\tilde{\mathcal{L}}}(x,y) &= \frac{h(y)}{h(x)}\ \left(\frac{A(x)^2}{R(x)}\right)^{1/4}\left(\frac{R(y)}{A(y)^2}\right)^{3/4} e^{-(T-t)\mathbb{H}_2}\big(Z(x), Z(y)\big). \end{aligned} \tag{2.36}$$

Proof. The transformation T defined in Lemma 1 extends to the kernels as follows:

$$\begin{aligned} e^{(T-t)\mathcal{L}}(x,y) &= e^{-(T-t)\ T_h T_Z^{-1} T_g^{-1} \mathbb{H}_1}(x,y) \\ &= \frac{h(y)}{h(x)}\ \frac{dZ}{dy}\ \frac{g(Z(x))}{g(Z(y))}\ e^{-(T-t)\mathbb{H}_1}\big(Z(x), Z(y)\big) \end{aligned}$$

and similarly for $\tilde{\mathcal{L}}$ and $\mathbb{H}_2$. Recall that $g(Z(x)) = \left(\frac{A(x)^2}{R(x)}\right)^{1/4}$ and the Jacobian of T_Z is given by $\frac{dZ}{dy} = \frac{\sqrt{R(y)}}{A(y)}$, which concludes the proof.

3. Examples of Solvable Diffusions

In this section, we show that the transitional probability density and the Laplace transform for the integral of the geometric Brownian motion, the Ornstein-Uhlenbeck process and the CIR process arise as corollaries of the First Classification Theorem 1.

3.1. The geometric Brownian motion

Definition 7. *The geometric Brownian motion is defined by the solution of the following stochastic differential equation:*

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{3.1}$$

with initial condition $X_{t=0} = x_0$.

Corollary 1. *The transitional probability density for the geometric Brownian motion is given by the following formula:*

$$p_{T-t}(x,y) = \frac{1}{y\sqrt{2\pi\sigma^2(T-t)}}\ \exp\left(-\frac{\left(\ln(\frac{y}{x}) - (\mu - \frac{\sigma^2}{2})(T-t)\right)^2}{2\sigma^2(T-t)}\right). \tag{3.2}$$

The Laplace transform is explicitly solvable if and only if

$$\phi(x) = \frac{\sigma^2}{2\vartheta}\left(\frac{\mu}{\sigma^2}\left(1 - \frac{\mu}{\sigma^2}\right) - t_0 - t_1 x + t_2 x^2\right), \tag{3.3}$$

where $t_0, t_1 \in \mathbb{R}$ *and* $t_2 > 0$ *could depend on* ϑ*. It is then given in terms of the Laguerre polynomials* $L_n^{(\delta)}$ *and the Whittaker function* $M_{\lambda,\mu}$ *(by convention,* $\sum_{n=0}^{N} = 0$ *if* $N < 0$*):*

$$\begin{aligned} L_{T-t}(x,\vartheta) = {} & x^{-\frac{\mu}{\sigma^2}+\frac{1}{2}}\ e^{-\sqrt{t_2}x}\ \sum_{n=0}^{N} z_n e^{(T-t)\lambda_n} x^{\frac{\delta_n}{2}} L_n^{(\delta_n)}(2\sqrt{t_2}x) \\ & + x^{-\frac{\mu}{\sigma^2}} \int_0^\infty z_k e^{-(T-t)(k^2+U_-)} M_{\frac{t_1}{2\sqrt{t_2}},\frac{\delta_k}{2}}(2\sqrt{t_2}x)dk \end{aligned} \tag{3.4}$$

where $N \equiv \lceil \frac{t_1}{\sqrt{t_2}} - \frac{1}{2} \rceil$ *(*$\lceil t \rceil$ *denotes the integer part of* t*),* $\delta_n \equiv -2n - 1 + \frac{t_1}{\sqrt{t_2}}$*,* $U_- \equiv \frac{\sigma^2}{2}(\frac{1}{4} - t_0)$ *and* $\delta_k \equiv i\sqrt{\frac{8}{\sigma^2}}k$*. For* $n = 0, 1, \ldots, N$*, the discrete eigenvalues are given by*

$$\lambda_n = \frac{\sigma^2}{2}\left(\left(n + \frac{1}{2} - \frac{t_1}{2\sqrt{t_2}}\right)^2 + t_0 - \frac{1}{4}\right). \tag{3.5}$$

The coefficients z_n *and* z_k *are respectively*

$$z_n = \frac{(2\sqrt{t_2})^{\delta_n}}{\Gamma(\delta_n)} \int_0^\infty q(x)\ x^{\frac{\mu}{\sigma^2}+\frac{\delta_n}{2}-\frac{3}{2}} e^{-\sqrt{t_2}x} L_n^{(\delta_n)}(2\sqrt{t_2}x)dx \tag{3.6}$$

and

$$z_k = \frac{1}{2\pi}\sqrt{\frac{2}{\sigma^2 t_2}} \int_0^\infty q(x)\ x^{\frac{\mu}{\sigma^2}-2} M_{\frac{t_1}{2\sqrt{t_2}},\frac{-\delta_k}{2}}(2\sqrt{t_2}x)dx. \tag{3.7}$$

Proof. The infinitesimal generator is

$$\mathcal{L} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} \tag{3.8}$$

whereas the Feynman-Kac operator has the form

$$\tilde{\mathcal{L}} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} + \vartheta\phi(x). \tag{3.9}$$

This case fits the classification scheme in Theorem 1 if one selects

$$A(x) = x \quad , \quad R(x) = \frac{2}{\sigma^2} \quad , \quad \frac{h'}{h} = \frac{\mu}{\sigma^2 x}. \tag{3.10}$$

This choice sets the shape of the polynomial $S(x)$, from (1.20) in Theorem 1, to be

$$S(x) = \frac{1}{\sigma^2}\left(2\xi + \mu\big(1 - \frac{\mu}{\sigma^2}\big)\right), \tag{3.11}$$

which in turn defines the Bose invariant as

$$I(x) = \frac{2(\xi - \lambda) + \mu\big(1 - \frac{\mu}{\sigma^2}\big)}{\sigma^2 x^2}. \tag{3.12}$$

The Schrödinger potential, given by

$$U(z) = \frac{\sigma^2}{8} - \xi - \frac{\mu}{2}\left(1 - \frac{\mu}{\sigma^2}\right), \tag{3.13}$$

is constant for all z. Hence the spectrum is absolutely continuous. The normalized eigenfunctions for the Schrödinger equations (2.34) are

$$\Phi_\rho(z) = \frac{1}{\sqrt{2\pi}}\, e^{\pm ik(\rho)z} \tag{3.14}$$

where $k^2 = \rho - \frac{\sigma^2}{8} + \frac{\mu}{2}\left(1 - \frac{\mu}{\sigma^2}\right)$. Theorem 5 yields the kernel for the semigroup generated by the operator $\mathcal{L}$:

$$e^{(T-t)\mathcal{L}}(x,y) = \frac{1}{y\sqrt{2\pi\sigma^2(T-t)}}\, \exp\left(-\frac{\Big(\ln(\frac{y}{x}) - (\mu - \frac{\sigma^2}{2})(T-t)\Big)^2}{2\sigma^2(T-t)}\right) \tag{3.15}$$

which is $p_{T-t}(x,y)$ and in which one recognizes the transitional probability density of the geometric Brownian motion.

The kernel of the semigroup generated by $\tilde{\mathcal{L}}$ is more general, since there is no restrictions on the polynomial $Q(x,\vartheta)$, which, for $t_0, t_1, t_2 \in \mathbb{R}$, can be written in the form

$$Q(x,\vartheta) = S(x) - t_0 - t_1 x + t_2 x^2. \tag{3.16}$$

The Bose invariant in this case is

$$\tilde{I}(x) = -t_2 + \frac{t_1}{x} + \frac{t_0 - \frac{2\lambda}{\sigma^2}}{x^2} \tag{3.17}$$

which gives rise to two independent solutions to the eigenvalue problem $u'' + Iu = 0$:

$$\begin{aligned} u_\pm(x) &= M_{\frac{t_1}{2\sqrt{t_2}}, \pm\frac{\delta_0}{2}}\left(2\sqrt{t_2}x\right) \\ &= (2\sqrt{t_2}x)^{\frac{\pm\delta_0+1}{2}} e^{-\sqrt{t_2}x}\, {}_1F_1\left(\begin{matrix}\frac{\pm\delta_0+1}{2} - \frac{t_1}{2\sqrt{t_2}} \\ \pm\delta_0 + 1\end{matrix}\,\middle|\, 2\sqrt{t_2}x\right) \end{aligned} \tag{3.18}$$

where $\frac{\delta_0^2}{4} = \frac{1}{4} - t_0 + \frac{2\lambda}{\sigma^2}$. The Schrödinger potential, given by

$$\tilde{U}(z) = \frac{\sigma^2}{2}\left(\frac{1}{4} - t_0 - t_1 e^{\sqrt{\frac{\sigma^2}{2}}z} + t_2 e^{2\sqrt{\frac{\sigma^2}{2}}z}\right), \tag{3.19}$$

has no singularities and is bounded from below if $t_2 > 0$. It indicates that the spectrum is not strictly discrete, since $\tilde{U}(z) \to \infty$ as $z \to \infty$ but $\tilde{U}(z) \to U_- \equiv \frac{\sigma^2}{2}(\frac{1}{4} - t_0)$ as $z \to -\infty$. Hence, the description of the spectrum separates in two cases:

- If $t_1 \leq 0$, then $\tilde{U}(z)$ is monotonously increasing on $D_z = \mathbb{R}$ and the spectrum is continuous: $\rho = -\lambda \geq U_-$.
- If $t_1 > 0$, then $\tilde{U}(z)$ has a minimum, $U_0 \equiv U_- - \frac{\sigma^2}{2}\frac{t_1^2}{4t_2}$, at $z = \sqrt{\frac{2}{\sigma^2}}\ln\frac{t_1}{2t_2}$ and the spectrum is discrete for $U_0 < \rho < U_-$ and continuous for $\rho \geq U_-$.

Both cases can however be solved simultaneously. The solution to the Schrödinger equation, $\Phi_\rho(z)$, is related to $u_\pm(x)$, via the diffeomorphism $x = X(z) = e^{\sqrt{\frac{\sigma^2}{2}}z}$ and the gauge transformation $g(z) = (\frac{\sigma^2}{2})^{\frac{1}{4}} e^{\sqrt{\frac{\sigma^2}{8}}z}$, as follows:

$$\Phi_\rho(z) = g^{-1}(z) u_\pm(X(z)). \tag{3.20}$$

The asymptotic behavior as $z \to -\infty$, $\Phi_\rho(z) \approx \frac{1}{\sqrt{\pi}} e^{\pm ikz}$ with $k = \sqrt{\rho - U_-} \geq 0$, enforces the normalization condition for $\Phi_\rho(z)$ to the following:

$$\Phi_\rho(z) = \frac{1}{\sqrt{\pi}}(2\sqrt{t_2})^{-\frac{1\pm\delta_0}{2}} e^{-\sqrt{\frac{\sigma^2}{8}}z} u_\pm(X(z)). \tag{3.21}$$

The continuous spectrum appears for ρ greater than the lowest limit U_-, implying that $\delta_k \equiv \delta_0(k) = i\sqrt{\frac{8}{\sigma^2}}k$ is imaginary. Whether or not the discrete part of the spectrum has an infinite number of discrete levels depends on the asymptotic behavior of the potential. As $z \to -\infty$, $x \to 0$ and $\tilde{U}(Z(x))$ develops up to second order:

$$\tilde{U}(Z(x)) = \frac{\sigma^2}{2}\left(\frac{1}{4} - t_0 - t_1 x + t_2 x^2\right). \tag{3.22}$$

Since $t_1 > 0$ for the discrete spectrum, the convergence to the limit U_- as $x \to 0$ is faster than x^2, which implies that there is only a finite number of bound states.

The hypergeometric functions in the solutions $u_\pm(x)$ reduce to polynomials for $x \in [0, \infty)$ if respectively

$$\frac{\pm\delta_0(n) + 1}{2} - \frac{t_1}{2\sqrt{t_2}} = -n \in -\mathbb{N}. \tag{3.23}$$

We set $\delta_n \equiv \delta_0(n)$ to emphasize the dependance on n. The solution $u_-(x)$ is not L^2-normalizable, whereas $u_+(x)$ is given in terms of the Laguerre polynomials

$$u_n(x) = C(2\sqrt{t_2}x)^{\frac{\delta_n+1}{2}} e^{-\sqrt{t_2}x} L_n^{(\delta_n)}(2\sqrt{t_2}x) \tag{3.24}$$

with normalization constant C. The normalization condition $\int_{-\infty}^{\infty} |\Phi_n(z)|^2 dz = 1$ fixes the constant $C = \sqrt{\frac{\sigma^2}{4\sqrt{t_2}\Gamma(\delta_n)}}$ (cf. [28], p. 462). The discrete eigenvalues $\rho = -\lambda_n$ are given by (3.5), where n is restricted to $\{0, 1, \ldots, N\}$ by the condition $-\lambda_n > U_0$, and $N \equiv \lceil \frac{t_1}{\sqrt{t_2}} - \frac{1}{2} \rceil$.

Hence, from Theorem 5, the kernel of the semigroup generated by $\tilde{\mathcal{L}}$ is given by the following spectral resolution:

$$\begin{aligned} e^{(T-t)\tilde{\mathcal{L}}}(x,y) &= x^{-\frac{\mu}{\sigma^2}+\frac{1}{2}} y^{\frac{\mu}{\sigma^2}-\frac{3}{2}}\ e^{-\sqrt{t_2}(x+y)} \\ &\quad \cdot \sum_{n=0}^{N} e^{(T-t)\lambda_n} \frac{(2\sqrt{t_2})^{\delta_n}}{\Gamma(\delta_n)} (xy)^{\frac{\delta_n}{2}} L_n^{(\delta_n)}(2\sqrt{t_2}x) L_n^{(\delta_n)}(2\sqrt{t_2}y) \\ &\quad + \frac{1}{2\pi}\sqrt{\frac{2}{\sigma^2 t_2}} x^{-\frac{\mu}{\sigma^2}} y^{\frac{\mu}{\sigma^2}-2} \\ &\quad \cdot \int_0^{\infty} e^{-(T-t)(k^2+U_-)} M_{\frac{t_1}{2\sqrt{t_2}},\frac{\delta_k}{2}}(2\sqrt{t_2}x) M_{\frac{t_1}{2\sqrt{t_2}},\frac{-\delta_k}{2}}(2\sqrt{t_2}y) dk. \end{aligned} \tag{3.25}$$

Finally the Laplace transform of the latter kernel is given by integration over $D_x = [0,\infty)$ and yields (3.4).

3.2. The Ornstein-Uhlenbeck process

In this subsection, we restrict the framework to the affine models, i.e. we set

$$q(x) = \exp\left(\omega\phi(x)\right) \tag{3.26}$$

for some $\omega \in \mathbb{R}$. We show that in the special case of the Ornstein-Uhlenbeck process, both the transitional probability density and the Laplace transform can be expressed as summations over Hermite polynomials.

Definition 8. *The Ornstein-Uhlenbeck process is defined by the solution of the following stochastic differential equation:*

$$dX_t = (a - bX_t)dt + \sigma dW_t \tag{3.27}$$

with $b > 0$ and initial condition $X_{t=0} = x_0$.

Corollary 2. *The transitional probability density is given by the following formula:*

$$p_{T-t}(x,y) = \sqrt{\frac{b}{\sigma^2\pi}}\left(1 - e^{-2b(T-t)}\right)^{-1/2} \exp\left[-\frac{\left(z(y) - z(x)e^{-b(T-t)}\right)^2}{1 - e^{-2b(T-t)}}\right] \tag{3.28}$$

where $z(x) = \sqrt{\frac{b}{\sigma^2}}\left(x - \frac{a}{b}\right)$. The Laplace transform is explicitly solvable if and only if

$$\phi(x) = \frac{\sigma^2}{2\vartheta}\left(\frac{b}{\sigma^2} - \frac{a^2}{\sigma^4} - t_0 + 2\frac{ab - \bar{a}\bar{b}}{\sigma^4}x + \frac{\bar{b}^2 - b^2}{\sigma^4}x^2\right), \tag{3.29}$$

with $t_0, \bar{a}, \bar{b} \in \mathbb{R}$ and could depend on ϑ. It is of the quadratic form

$$L_{T-t}(x,\vartheta,\omega) = e^{m(T-t) - n(T-t)x - l(T-t)x^2} \tag{3.30}$$

where the functions $m(\tau), n(\tau)$ and $l(\tau)$ are as follows

$$\begin{aligned} m(\tau) &= \frac{1}{2}\ln\left(\frac{\bar{b}}{\sigma^2}\right) - \frac{1}{2}\ln\left(p - (p - \frac{\bar{b}}{\sigma^2})e^{-2\bar{b}\tau}\right) - \frac{\bar{a}^2}{\sigma^2\bar{b}} - \frac{\tau}{2}\left(\bar{b} - \frac{\bar{a}^2}{\sigma^2} - \sigma^2 t_0\right) \\ &\quad + \frac{\omega}{2\vartheta}\left(b - \frac{a^2}{\sigma^2} - \sigma^2 t_0\right) + p\left(q + \frac{\bar{a}}{\bar{b}}\right)^2 - \frac{2pq\frac{\bar{a}}{\sigma^2}e^{-\bar{b}\tau} + \left((p - \frac{\bar{b}}{\sigma^2})\frac{\bar{a}^2}{\bar{b}\sigma^2} + pq^2\frac{\bar{b}}{\sigma^2}\right)e^{-2\bar{b}\tau}}{p - (p - \frac{\bar{b}}{\sigma^2})e^{-2\bar{b}\tau}} \\ n(\tau) &= -\frac{\bar{a} - a}{\sigma^2} - \frac{2pq\frac{\bar{b}}{\sigma^2}e^{-\bar{b}\tau} + (p - \frac{\bar{b}}{\sigma^2})2\frac{\bar{a}}{\sigma^2}e^{-2\bar{b}\tau}}{p - (p - \frac{\bar{b}}{\sigma^2})e^{-2\bar{b}\tau}} \\ l(\tau) &= \frac{\bar{b} - b}{2\sigma^2} + \frac{(p - \frac{\bar{b}}{\sigma^2})\frac{\bar{b}}{\sigma^2}e^{-2\bar{b}\tau}}{p - (p - \frac{\bar{b}}{\sigma^2})e^{-2\bar{b}\tau}} \end{aligned}$$

with

$$p = \frac{(\bar{b}+b)(\omega(b-\bar{b})+\vartheta)}{2\sigma^2\vartheta} \quad , \quad q = \frac{\vartheta(a+\bar{a})+\omega(ab-\bar{a}\bar{b})}{(\bar{b}+b)(\vartheta-\omega(\bar{b}-b))} - \frac{\bar{a}}{\bar{b}}.$$

Proof. The infinitesimal generator of the Ornstein-Uhlenbeck process is

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x} \tag{3.31}$$

and the Feynman-Kac operator has the form

$$\tilde{\mathcal{L}} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x} - \vartheta\phi(x). \tag{3.32}$$

This case fits the classification scheme in Theorem 1 if one selects

$$A(x) = 1 \quad , \quad R(x) = \frac{2}{\sigma^2} \quad , \quad h(x) = e^{\frac{a}{\sigma^2}x - \frac{b}{2\sigma^2}x^2}. \tag{3.33}$$

This choice implies that the shape of the polynomial $S(x)$, from (1.20) in Theorem 1, must be

$$S(x) = \frac{1}{\sigma^2}(2\xi + b - \frac{a^2}{\sigma^2}) + \frac{2ab}{\sigma^4}x - \frac{b^2}{\sigma^4}x^2, \tag{3.34}$$

which in turn defines the Bose invariant as

$$I(x) = \frac{1}{\sigma^2}(2\xi - 2\lambda + b - \frac{a^2}{\sigma^2}) + \frac{2ab}{\sigma^4}x - \frac{b^2}{\sigma^4}x^2. \tag{3.35}$$

The Schrödinger potential, given by

$$U(z) = -\xi - \frac{b}{2} + \frac{a^2}{2\sigma^2} - \frac{ab}{\sqrt{2\sigma^2}}z + \frac{b^2}{4}z^2 \tag{3.36}$$

with $z = \sqrt{\frac{2}{\sigma^2}}x$, goes to ∞ as $z \to \pm\infty$. Hence, the spectrum displays discrete eigenvalues of the form $\lambda_n = -bn$ for $n \in \mathbb{N}$. The corresponding normalized eigenfunctions that satisfy the Schrödinger equation (2.34) are

$$\Phi_n(Z(x)) = \frac{1}{\sqrt{n!2^n}}\left(\frac{b}{2\pi}\right)^{1/4} e^{-\frac{b}{2\sigma^2}\left(x-\frac{a}{b}\right)^2} H_n\left(\sqrt{\frac{b}{\sigma^2}}(x-\frac{a}{b})\right). \tag{3.37}$$

Theorem 5 yields the kernel for the semigroup generated by $\mathcal{L}$ as a summation over Hermite polynomials. Using Mehler's formula (cf. [28], p. 710) and the notation $z(x) = \sqrt{\frac{b}{\sigma^2}}\,(x - \frac{a}{b})$, the kernel re-sums into

$$e^{(T-t)\mathcal{L}}(x,y) = \sqrt{\frac{b}{\sigma^2\pi}}\left(1-e^{-2b(T-t)}\right)^{-1/2} \exp\left[-\frac{\left(z(y)-z(x)e^{-b(T-t)}\right)^2}{1-e^{-2b(T-t)}}\right], \tag{3.38}$$

which is $p_{T-t}(x,y)$ and in which one recognizes the probability density of the Ornstein-Uhlenbeck process.

For convenience and without loss of generality, we set the form of the polynomial $Q(x,\vartheta)$ to

$$Q(x,\vartheta) = S(x) - t_0 - \frac{2\bar{a}\bar{b}}{\sigma^4}x + \frac{\bar{b}^2}{\sigma^4}x^2. \tag{3.39}$$

where $t_0, \bar{a}, \bar{b}$ could depend on ϑ. The Bose invariant in this case is

$$\tilde{I}(x) = t_0 - \frac{2\lambda}{\sigma^2} + \frac{2\bar{a}\bar{b}}{\sigma^4}x - \frac{\bar{b}^2}{\sigma^4}x^2. \tag{3.40}$$

The Schrödinger potential, given by

$$\tilde{U}(z) = -\frac{t_0}{2} - \frac{\bar{a}\bar{b}}{\sigma^2}\sqrt{\frac{\sigma^2}{2}}z + \frac{\bar{b}^2}{4}z^2 \tag{3.41}$$

is very similar to $U(z)$ and indicates that the spectrum is again discrete since $\tilde{U}(z) \to \infty$ as $z \to \pm\infty$. The solution $\Phi_n(y)$ is given in terms of the Hermite polynomials

$$\Phi_n(Z(x)) = \frac{1}{\sqrt{n!2^n}}\left(\frac{\bar{b}}{2\pi}\right)^{1/4} e^{-\frac{\bar{b}}{2\sigma^2}\left(x-\frac{\bar{a}}{\bar{b}}\right)^2} H_n\left(\sqrt{\frac{\bar{b}}{\sigma^2}}\left(x-\frac{\bar{a}}{\bar{b}}\right)\right). \tag{3.42}$$

for the eigenvalues $\lambda_n = -\bar{b}n - \frac{\bar{b}}{2} + \frac{\bar{a}^2}{2\sigma^2} + \frac{\sigma^2}{2}t_0$. From Theorem 5, the kernel of the semigroup generated by $\tilde{\mathcal{L}}$ is given by the following spectral resolution:

$$\begin{aligned} e^{(T-t)\tilde{\mathcal{L}}}(x,y) = \sqrt{\frac{\bar{b}}{\sigma^2\pi}}\exp\left(-\frac{\bar{a}^2}{\sigma^2\bar{b}} + \frac{y}{\sigma^2}(\bar{a}+a) - \frac{y^2}{2\sigma^2}(\bar{b}+b) + \frac{x}{\sigma^2}(\bar{a}-a) - \frac{x^2}{2\sigma^2}(\bar{b}-b)\right) \\ \cdot \sum_{n=0}^{\infty}\frac{e^{\lambda_n(T-t)}}{2^n n!}H_n\left(\sqrt{\frac{\bar{b}}{\sigma^2}}\left(x-\frac{\bar{a}}{\bar{b}}\right)\right)H_n\left(\sqrt{\frac{\bar{b}}{\sigma^2}}\left(y-\frac{\bar{a}}{\bar{b}}\right)\right). \end{aligned} \tag{3.43}$$

The integration of the latter yields the Laplace transform for $q(x) = \exp(\omega x)$ as a convergent series in terms of the Hermite polynomials which re-sums to the formula (3.30) (cf. [28], p. 488(16) and p. 710(1)).

Remark 2. Computing the Laplace transform L in the case where $\phi(x)$ is affine, i.e. $\phi(x) = x$, is a direct consequence of the previous corollary. Setting the parameters to

$$t_0 = \frac{b}{\sigma^2} - \left(\frac{a}{\sigma^2}\right)^2 \quad , \quad \bar{a} = a - \frac{\sigma^2\vartheta}{b} \quad , \quad \bar{b} = b \tag{3.44}$$

proves the following proposition:

Proposition 4. *The Laplace transform for the affine Ornstein-Uhlenbeck process is as follows:*

$$L_{T-t}(x,\vartheta,\omega) = e^{m(T-t)-n(T-t)x} \tag{3.45}$$

where

$$\begin{aligned} n(\tau) &= \frac{\vartheta - (\vartheta+\omega b)e^{-b\tau}}{b} \\ m(\tau) &= \frac{\left(n(\tau)+\omega-\vartheta\tau\right)\left(ab-\vartheta\frac{\sigma^2}{2}\right)}{b^2} - \frac{\sigma^2}{4b}\left(n(\tau)^2-\omega^2\right). \end{aligned} \tag{3.46}$$

3.3. The CIR process

In this subsection, we focuss again on the affine models, i.e. we set

$$q(x) = \exp(\omega x) \tag{3.47}$$

for some $\omega \in \mathbb{R}$. We show how to derive from the First Classification Theorem 1, the transitional probability density for the CIR process and the Laplace transform in the affine case. The main tool is the use of the Laguerre polynomials as eigenfunctions for both the infinitesimal generator and the Feynman-Kac operator.

Definition 9. *The CIR process is defined as the solution of the following stochastic differential equation on $D_x = \mathbb{R}_+$:*

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t \tag{3.48}$$

with $a, b > 0$ and initial condition $X_{t=0} = x_0$.

Corollary 3. *The transitional probability density of the CIR process is given in terms of the modified Bessel function I_α as follows:*

$$p_{T-t}(x,y) = c\left(\frac{ye^{b(T-t)}}{x}\right)^{\frac{1}{2}\left(\frac{2a}{\sigma^2}-1\right)} \exp\left[-c\left(y + xe^{-b(T-t)}\right)\right] I_{\frac{2a}{\sigma^2}-1}\left(2c\sqrt{xye^{-b(T-t)}}\right) \tag{3.49}$$

with $c \equiv c(T-t) = \frac{2b}{\sigma^2}(1 - e^{-b(T-t)})^{-1}$. The Laplace transform is computable in closed form if and only if

$$\phi(x) = \left(\frac{a}{2}(1 - \frac{a}{\sigma^2}) - \frac{\sigma^2 t_0}{2}\right)\frac{1}{x} + \left(\frac{ab}{\sigma^2} - \frac{\sigma^2 t_1}{2}\right) + \left(\frac{\sigma^2 t_2}{2} - \frac{b^2}{2\sigma^2}\right)x, \tag{3.50}$$

where $t_0, t_1, t_2 \in \mathbb{R}$ could depend on ϑ.

In the affine case, where $\phi(x) = x$, the Laplace transform is of the closed form

$$L_{T-t}(x, \vartheta, \omega) = e^{m(T-t) - n(T-t)x} \tag{3.51}$$

where

$$\begin{aligned} m(\tau) &= \frac{2a}{\sigma^2}\ln\left[\frac{\bar{b}e^{b\tau/2}}{\bar{b}\cosh(\frac{\bar{b}\tau}{2}) + (b - \omega\sigma^2)\sinh(\frac{\bar{b}\tau}{2})}\right] \\ n(\tau) &= -\omega + \frac{\bar{b}^2 - (b - \omega\sigma^2)^2}{\sigma^2}\,\frac{\sinh(\frac{\bar{b}\tau}{2})}{\bar{b}\cosh(\frac{\bar{b}\tau}{2}) + (b - \omega\sigma^2)\sinh(\frac{\bar{b}\tau}{2})} \end{aligned} \tag{3.52}$$

and $\bar{b} = \sqrt{2\vartheta\sigma^2 + b^2}$.

Proof. The infinitesimal generator of the CIR process is

$$\mathcal{L} = \frac{\sigma^2}{2}x\frac{\partial^2}{\partial x^2} + (a - bx)\frac{\partial}{\partial x} \tag{3.53}$$

and the Feynman-Kac operator has the form

$$\tilde{\mathcal{L}} = \frac{\sigma^2}{2}x\frac{\partial^2}{\partial x^2} + (a - bx)\frac{\partial}{\partial x} - \vartheta\phi(x). \tag{3.54}$$

This case fits the classification scheme in Theorem 1 if one selects

$$A(x) = x \quad , \quad R(x) = \frac{2x}{\sigma^2} \quad , \quad h(x) = x^{a/\sigma^2} e^{-\frac{b}{\sigma^2}x}. \tag{3.55}$$

This choice sets the form of the polynomial $S(x)$, from (1.20) in Theorem 1, to be

$$S(x) = \frac{a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right) + \frac{2}{\sigma^2}\left(\xi + \frac{ab}{\sigma^2}\right)x - \frac{b^2}{\sigma^4}x^2, \tag{3.56}$$

which in turn defines the Bose invariant as

$$I(x) = -\frac{b^2}{\sigma^4} + \frac{2}{\sigma^2}\left(\xi - \lambda + \frac{ab}{\sigma^2}\right)\frac{1}{x} + \frac{a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right)\frac{1}{x^2}. \tag{3.57}$$

The Schrödinger potential, given for $z = \sqrt{\frac{8x}{\sigma^2}}$ by

$$U(z) = \left(\frac{3}{4} - \frac{4a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right)\right)\frac{1}{z^2} - \xi - \frac{ab}{\sigma^2} + \frac{b^2}{16}z^2 \tag{3.58}$$

goes to ∞ as $z \to \infty$ and has a singularity at $z = 0$. The asymptotic behavior as $z \to 0$ implies that the Schrödinger operator has a self-adjoint closure with completely discrete spectrum. We have assumed $\frac{4a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right) \leq 1$ which is always satisfied. Hence, the spectrum presents discrete eigenvalues of the form $\lambda_n = -bn$ for $n \in \mathbb{N}$. The corresponding normalized eigenfunctions that satisfy the Schrödinger equation (2.34) are given in terms of the Laguerre polynomials

$$\Phi_n(Z(x)) = \sqrt{\frac{n!}{\Gamma(n + \frac{2a}{\sigma^2})}}\left(\frac{\sigma^2}{2x}\right)^{1/4}\left(\frac{2bx}{\sigma^2}\right)^{a/\sigma^2} e^{-\frac{b}{\sigma^2}x} L_n^{(\frac{2a}{\sigma^2}-1)}\left(\frac{2b}{\sigma^2}x\right). \tag{3.59}$$

Theorem 5 yields the kernel for the semigroup generated by the operator $\mathcal{L}$ as a summation over Laguerre polynomials, which can be re-summed into (cf. [28], p. 705(7)):

$$e^{(T-t)\mathcal{L}}(x,y) = c\left(\frac{ye^{b(T-t)}}{x}\right)^{\frac{1}{2}\left(\frac{2a}{\sigma^2}-1\right)} \exp\left[-c\left(y + xe^{-b(T-t)}\right)\right] I_{\frac{2a}{\sigma^2}-1}\left(2c\sqrt{xye^{-b(T-t)}}\right) \tag{3.60}$$

with $c \equiv c(T-t) = \frac{2b}{\sigma^2}(1 - e^{-b(T-t)})^{-1}$ and in which one recognizes the transitional probability density of the CIR process.

For the Laplace transform L, we specialize to the case where $\phi(x)$ is affine, i.e. $\phi(x) = x$. Therefore, (1.22) sets the shape of the polynomial $Q(x, \vartheta)$ to

$$Q(x, \vartheta) = S(x) + \frac{2\vartheta}{\sigma^2}x^2. \tag{3.61}$$

The Bose invariant in this case is

$$\tilde{I}(x) = -\frac{b^2}{\sigma^4} - \frac{2\vartheta}{\sigma^2} + \frac{2}{\sigma^2}\left(\xi - \lambda + \frac{ab}{\sigma^2}\right)\frac{1}{x} + \frac{a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right)\frac{1}{x^2}. \tag{3.62}$$

whereas the Schrödinger potential, given by

$$\tilde{U}(z) = \left(\frac{3}{4} - \frac{4a}{\sigma^2}\left(1 - \frac{a}{\sigma^2}\right)\right)\frac{1}{z^2} - \xi - \frac{ab}{\sigma^2} + \frac{b^2 + 2\vartheta\sigma^2}{16}z^2 \tag{3.63}$$

corresponds to a discrete spectrum, following the same reasoning as for $U(z)$. The solution $\Phi_n(z)$ is given in terms of the Laguerre polynomials

$$\Phi_n(Z(x)) = \sqrt{\frac{n!}{\Gamma(n+\frac{2a}{\sigma^2})}} \left(\frac{\sigma^2}{2x}\right)^{1/4} \left(\frac{2\bar{b}x}{\sigma^2}\right)^{a/\sigma^2} e^{-\frac{\bar{b}}{\sigma^2}x} L_n^{(\frac{2a}{\sigma^2}-1)}\left(\frac{2\bar{b}}{\sigma^2}x\right) \tag{3.64}$$

with $\bar{b} = \sqrt{2\vartheta\sigma^2 + b^2}$ and for the corresponding eigenvalues $\lambda_n = -\bar{b}n - \frac{a}{\sigma^2}(\bar{b} - b)$. The Laplace transform L in (1.15) is easily integrated and yields a convergent series in terms of the Laguerre polynomials which re-sums to the famous formula (3.51) (cf. [28], p. 462(3) and p. 705(7)).

4. Processes related to continuous orthogonal polynomials

4.1. Proof of the Second Classification Theorem

We give a constructive proof of Theorem 2, independent of Theorem 1. In the following remark, we show that Theorem 2 can actually be regarded as a corollary of Theorem 1.

The reducibility condition implies that the infinitesimal generator $\mathcal{L}$ must be of the form

$$\mathcal{L} = \frac{\sigma^2}{2} A(x) \frac{\partial^2}{\partial x^2} + (a - bx) \frac{\partial}{\partial x} \tag{4.1}$$

for coefficients $a \in \mathbb{R}$ and $b > 0$. The transitional probability density of the process, $p_{T-t}(x, y)$, satisfies the backward Kolmogorov equation:

$$\frac{\partial p}{\partial t} + (a - bx) \frac{\partial p}{\partial x} + \frac{\sigma^2}{2} A(x) \frac{\partial^2 p}{\partial x^2} = 0 \tag{4.2}$$

with final time condition

$$\lim_{t \to T} p_{T-t}(x, y) = \delta(x - y). \tag{4.3}$$

By the reducibility assumption, a general solution to this equation is given by the following eigenfunction expansion in terms of the orthogonal polynomials $Q_n(x; a, b)$

$$p = \sum_{n=0}^{\infty} h_n(t) Q_n(x; a, b). \tag{4.4}$$

According to (1.27), the functions of time $h_n(t)$ satisfy the ordinary differential equations

$$\dot{h}_n + \lambda_n h_n = 0 \tag{4.5}$$

which admits the general solution, for $t \leq T$,

$$h_n(t) = z_n \; e^{\lambda_n (T-t)}. \tag{4.6}$$

The coefficients z_n are given by the final time condition (4.3):

$$\sum_{n=0}^{\infty} z_n Q_n(x; a, b) = \delta(x - y). \tag{4.7}$$

Hence, multiplying on both sides by $Q_m(x; a, b)$ and the invariant measure $\rho(dx)$, before integrating over the domain of x, leads to the result (1.29) by the orthogonality property (1.23).

The Laplace transform $L_{T-t}(x, \vartheta)$ satisfies the backward Kolmogorov equation with differential operator given by (1.13):

$$\frac{\partial L}{\partial t} + (a - bx) \frac{\partial L}{\partial x} + \frac{\sigma^2}{2} A(x) \frac{\partial^2 L}{\partial x^2} = \vartheta \phi(x) L \tag{4.8}$$

with final time condition

$$\lim_{t \to T} L_{T-t}(x, \vartheta) = q(x). \tag{4.9}$$

For the sake of having a clearer constructive proof, consider the following ansatz for the Laplace transform

$$L = V(x) \bar{L}. \tag{4.10}$$

The previous equation then reads

$$V\frac{\partial \bar{L}}{\partial t} + (a-bx)(V'\bar{L} + V\bar{L}') + \frac{\sigma^2}{2}|A(x)|(V''\bar{L} + 2V'\bar{L}' + V\bar{L}'') = \vartheta\phi V\bar{L}, \tag{4.11}$$

where the symbol $'$ denotes differentiation in the x-variable. The function V is chosen to satisfy

$$(a-bx)V + 2\frac{\sigma^2}{2}A(x)V' = (\bar{a}-\bar{b}x)V \tag{4.12}$$

for some parameters $\bar{a} \in \mathbb{R}$ and $\bar{b} > 0$, which implies

$$V(x) = \exp\left(\int^x \frac{(\bar{a}-\bar{b}y) - (a-by)}{\sigma^2|A(y)|}dy\right). \tag{4.13}$$

The function $\phi(x)$ is specified as follows:

$$\vartheta\phi(x) = (a-bx)\frac{V'}{V} + \frac{\sigma^2}{2}A(x)\frac{V''}{V}, \tag{4.14}$$

which is equivalent to (1.30) with the assumption that C regroups all the constant terms. This choice yields the following partial differential equation for the function $\bar{L}$:

$$\frac{\partial \bar{L}}{\partial t} + (\bar{a}-\bar{b}x)\frac{\partial \bar{L}}{\partial x} + \frac{\sigma^2}{2}A(x)\frac{\partial^2 \bar{L}}{\partial x^2} = 0. \tag{4.15}$$

Following the same reasoning as for the transitional probability denstiy, a general solution to this equation is given by the following eigenfunction expansion in terms of the same orthogonal polynomials $Q_n(x;\bar{a},\bar{b})$, but with different coefficients,

$$\bar{L} = \sum_{n=0}^{\infty} h_n(t)Q_n(x;\bar{a},\bar{b}). \tag{4.16}$$

According to (1.27), the functions of time $h_n(t)$ satisfy the ordinary differential equations

$$\dot{h}_n + \bar{\lambda}_n h_n = 0 \tag{4.17}$$

where $\bar{\lambda}_n$ are the eigenvalues corresponding to $Q_n(x;\bar{a},\bar{b})$. The general solution, for $t \le T$, is

$$h_n(t) = z_n\ e^{\bar{\lambda}_n(T-t)} \tag{4.18}$$

where the z_n are constants. The latter equation for $h_n(t)$ with the explicit form of $V(x)$ in (4.13) and the expression of $\bar{L}$ in (4.16) gives the expected result (1.31) for the Laplace transform.

The coefficients z_n are given by the final time condition (4.9):

$$\sum_{n=0}^{\infty} z_n Q_n(x;\bar{a},\bar{b}) = q(x)\ \exp\left(-\int^x \frac{(\bar{a}-\bar{b}y) - (a-by)}{\sigma^2|A(y)|}dy\right). \tag{4.19}$$

Hence, multiplying on both sides by $Q_m(x)$ and the invariant measure $\rho(dx)$, before integrating over the domain of x, leads to the final result (1.32) by the orthogonality property (1.23) and concludes the proof of Theorem 2.

Remark 3. The reducibility condition implies that the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ have the form

$$\begin{aligned}\mathcal{L} &= \frac{\sigma^2}{2}A(x)\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x}\\ \tilde{\mathcal{L}} &= \frac{\sigma^2}{2}A(x)\frac{\partial^2}{\partial x^2} + (\bar{a}-\bar{b}x)\frac{\partial}{\partial x}\end{aligned} \tag{4.20}$$

for possibly different coefficients $a, \bar{a} \in \mathbb{R}$ and $b, \bar{b} > 0$. Hence, setting

$$R(x) = \frac{2A(x)}{\sigma^2} \quad \text{and} \quad h(x) = \exp\left(\int^x \frac{a-by}{\sigma^2 A(y)}dy\right) \tag{4.21}$$

in the First Classification Theorem 1 yields the Second Classification Theorem 2. Moreover, consider the gauge transformation $T_{\bar{h}}$ defined by

$$\bar{h}(x) = \exp\left(\int^x \frac{\bar{a}-\bar{b}y}{\sigma^2 A(y)}dy\right). \tag{4.22}$$

Then we have the following relation:

$$T_h T_{\bar{h}}^{-1}\tilde{\mathcal{L}} = \frac{\sigma^2}{2}A(x)\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{Q(x,\vartheta)}{\vartheta A(x)}. \tag{4.23}$$

Notice further that the function $V(x)$ of the previous proof is related to the latter two gauge transformations by $V(x) = \frac{\bar{h}(x)}{h(x)}$.

4.2. The Ornstein-Uhlenbeck process

Corollary 4. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to Hermite polynomials. Assume also that the function $\phi(x) = x$. Then the transitional probability density of the Ornstein-Uhlenbeck process is given by:*

$$p_{T-t}(x,y) = \sqrt{\frac{b}{\sigma^2\pi}}e^{-\frac{b}{\sigma^2}\left(y-\frac{a}{b}\right)^2}\sum_{n=0}^{\infty}\frac{e^{-bn(T-t)}}{n!2^n}H_n\big(z(x)\big)H_n\big(z(y)\big) \tag{4.24}$$

and the Laplace transform is given by the following convergent series:

$$L_{T-t}(x,\vartheta) = e^{\frac{\vartheta}{b}x}\sum_{n=0}^{\infty}e^{-bn(T-t)}z_n H_n\big(\bar{z}(x)\big), \tag{4.25}$$

where $z(x) = \sqrt{\frac{b}{\sigma^2}}\left(x-\frac{a}{b}\right)$ and $\bar{z}(x) = \sqrt{\frac{b}{\sigma^2}}\left(x-\frac{\bar{a}}{b}\right)$. The coefficients z_n are given by:

$$z_n = \frac{1}{n!2^n}\sqrt{\frac{b}{\sigma^2\pi}}\int_{-\infty}^{\infty}q(x)\; e^{-\frac{\vartheta}{b}x}H_n\big(\bar{z}(x)\big)e^{-\frac{b}{\sigma^2}\left(x-\frac{\bar{a}}{b}\right)^2}dx. \tag{4.26}$$

Proof. This case fits the classification scheme in Theorem 2 if one selects $A(x) = 1$. This choice implies

$$\phi(x) = C + \frac{1}{2\vartheta\sigma^2}\big((\bar{a}-\bar{b}x)^2 - (a-bx)^2\big). \tag{4.27}$$

In order to reduce $\phi(x)$ to an affine function, we choose $\bar{b} = b$, $\bar{a} = a - \frac{\vartheta\sigma^2}{b}$ and $C = \frac{a^2-\bar{a}^2}{2\vartheta\sigma^2}$. The operator $\mathcal{L}$, which has the form

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x}, \tag{4.28}$$

has the Hermite polynomials $H_n\big(z(x)\big)$ as eigenfunctions with eigenvalues $\lambda_n = -bn$. The invariant measure density and the normalization factor are respectively

$$w(x) = \sqrt{\frac{b}{\sigma^2\pi}}\; e^{-\frac{b}{\sigma^2}\left(x-\frac{a}{b}\right)^2} \quad \text{and} \quad d_n^2 = n!2^n \tag{4.29}$$

which leads, by Theorem 2, to the formulation of the kernel of the semigroup generated by $\mathcal{L}$ as the convergent series in (4.24). The latter series re-sums using Mehler's formula to give (3.28). By Theorem 2, the Laplace transform is given by (4.25). The coefficients z_n are given by (4.26) and lead to the result (3.45), once integrated and re-summed.

4.3. The CIR process

Corollary 5. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to Laguerre polynomials. Assume also that the function $\phi(x) = x$. Then the transitional probability density of the CIR process is given by:*

$$p_{T-t}(x,y) = \left(\frac{2b}{\sigma^2}\right)^{\frac{2a}{\sigma^2}}\; y^{\frac{2a}{\sigma^2}-1}\; e^{-\frac{2b}{\sigma^2}y} \sum_{n=0}^{\infty} \frac{n!e^{-bn(T-t)}}{\Gamma(n+\frac{2a}{\sigma^2})} L_n^{(\frac{2a}{\sigma^2}-1)}\big(\frac{2b}{\sigma^2}x\big) L_n^{(\frac{2a}{\sigma^2}-1)}\big(\frac{2b}{\sigma^2}y\big) \tag{4.30}$$

and the Laplace transform is given by the following convergent series:

$$L_{T-t}(x,\vartheta) = e^{\frac{b-\bar{b}}{\sigma^2}x} \sum_{n=0}^{\infty} e^{-\bar{b}n(T-t)} z_n L_n^{(\frac{2a}{\sigma^2}-1)}\big(\frac{2\bar{b}}{\sigma^2}x\big). \tag{4.31}$$

The coefficients z_n are given by:

$$z_n = \frac{n!}{\Gamma(n+\frac{2a}{\sigma^2})}\left(\frac{2\bar{b}}{\sigma^2}\right)^{\frac{2a}{\sigma^2}} \int_0^{\infty} q(x)\; x^{\frac{2a}{\sigma^2}-1}\; e^{-\frac{b+\bar{b}}{\sigma^2}x} L_n^{(\frac{2a}{\sigma^2}-1)}\big(\frac{2\bar{b}}{\sigma^2}x\big)dx. \tag{4.32}$$

Proof. This case also fits the classification scheme in Theorem 2 if one selects $A(x) = x$. This choice implies

$$\phi(x) = C + \frac{1}{2\vartheta x}\big((a-bx) - (\bar{a}-\bar{b}x)\big) + \frac{1}{2\vartheta\sigma^2 x}\big((\bar{a}-\bar{b}x)^2 - (a-bx)^2\big). \tag{4.33}$$

In order to reduce $\phi(x)$ to an affine function, we choose $\bar{a} = a$, $\bar{b} = \sqrt{2\vartheta\sigma^2 + b^2}$ and $C = \frac{\bar{b}-b}{\vartheta}\left(\frac{a}{\sigma^2} - \frac{1}{2}\right)$. The operator $\mathcal{L}$, which has the form

$$\mathcal{L} = \frac{\sigma^2 x}{2}\,\frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x}, \tag{4.34}$$

has the Laguerre polynomials $L_n^{(\frac{2a}{\sigma^2}-1)}\big(\frac{2b}{\sigma^2}x\big)$ as eigenfunctions with eigenvalues $\lambda_n = -bn$ if $a > 0$. The invariant measure density and the normalization factor are respectively

$$w(x) = \left(\frac{2b}{\sigma^2}\right)^{\frac{2a}{\sigma^2}}\; x^{\frac{2a}{\sigma^2}-1}\; e^{-\frac{2b}{\sigma^2}x} \quad \text{and} \quad d_n^2 = \frac{\Gamma(n+\frac{2a}{\sigma^2})}{n!} \tag{4.35}$$

which leads, by Theorem 2, to the formulation of the kernel of the semigroup generated by $\mathcal{L}$ as the convergent series in (4.30) which re-sums to give (3.49). The Laplace transform is given by (4.31) and the coefficients z_n are given by (4.32). The latter results lead to (3.51), once integrated and re-summed.

4.4. The Jacobi process

Definition 10. *The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by the following Gaussian hypergeometric function for $x \in [-1,1]$:*

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1 \\ \alpha+1\end{matrix}\middle|\ \frac{1-x}{2}\right),\ n=0,1,2,\ldots \tag{4.36}$$

Definition 11. *The Jacobi process is solution to the following equation:*

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t(1-X_t)}dW_t \tag{4.37}$$

with initial condition $X_{t=0} = x_0 \in (0,1)$.

Corollary 6. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to Jacobi polynomials. Then:*

$$\phi(x) = \frac{\sigma^2}{8\vartheta}\left(\frac{\bar{\alpha}^2-\alpha^2}{x} + \frac{\bar{\beta}^2-\beta^2}{1-x}\right), \tag{4.38}$$

for $\alpha = \frac{2a}{\sigma^2} - 1 > -1, \beta = \frac{2}{\sigma^2}(b-a) - 1 > -1$ and $\bar{\alpha} = \frac{2\bar{a}}{\sigma^2} - 1 > -1, \bar{\beta} = \frac{2}{\sigma^2}(\bar{b}-\bar{a}) - 1 > -1$. The transitional probability density of the Jacobi process is given by:

$$p_{T-t}(x,y) = y^{\alpha}\ (1-y)^{\beta} \sum_{n=0}^{\infty} \frac{e^{-\frac{\sigma^2}{2}n(n+\alpha+\beta+1)(T-t)}}{d_n^2} P_n^{(\alpha,\beta)}(1-2x) P_n^{(\alpha,\beta)}(1-2y) \tag{4.39}$$

with normalization constant

$$d_n^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}. \tag{4.40}$$

The Laplace transform is given by the following convergent series:

$$L_{T-t}(x,\vartheta) = x^{\frac{\bar{\alpha}-\alpha}{2}}(1-x)^{\frac{\bar{\beta}-\beta}{2}} \sum_{n=0}^{\infty} e^{-\frac{\sigma^2}{2}n(n+\bar{\alpha}+\bar{\beta}+1)(T-t)} z_n P_n^{(\bar{\alpha},\bar{\beta})}(1-2x). \tag{4.41}$$

The coefficients z_n are given by:

$$z_n = \frac{1}{\bar{d}_n^2}\int_0^1 q(x)\ P_n^{(\bar{\alpha},\bar{\beta})}(1-2x)\ x^{\frac{\alpha+\bar{\alpha}}{2}}(1-x)^{\frac{\beta+\bar{\beta}}{2}} dx, \tag{4.42}$$

with normalization constant

$$\bar{d}_n^2 = \frac{\Gamma(n+\bar{\alpha}+1)\Gamma(n+\bar{\beta}+1)}{(2n+\bar{\alpha}+\bar{\beta}+1)\Gamma(n+\bar{\alpha}+\bar{\beta}+1)n!}. \tag{4.43}$$

Proof. This case also fits the classification scheme in Theorem 2 if one selects $A(x) = x(1-x)$. This choice implies

$$\phi(x) = C + \frac{\sigma^2}{8\vartheta}\left(\frac{\bar{\alpha}^2-\alpha^2}{x} + \frac{\bar{\beta}^2-\beta^2}{1-x} + (\alpha+\beta)^2 - (\bar{\alpha}+\bar{\beta})^2\right). \tag{4.44}$$

Since C is an arbitrary constant, we set it to $C = (\bar{\alpha}+\bar{\beta})^2 - (\alpha+\beta)^2$. The infinitesimal generator $\mathcal{L}$, which has the form

$$\mathcal{L} = \frac{\sigma^2}{2}\ x(1-x)\ \frac{\partial^2}{\partial x^2} + (a-bx)\frac{\partial}{\partial x}, \tag{4.45}$$

has the Jacobi polynomials $P_n^{(\alpha,\beta)}(1-2x)$ as eigenfunctions with eigenvalues $\lambda_n = -\frac{\sigma^2}{2}n(n+\alpha+\beta+1)$ if $\alpha > -1$ and $\beta > -1$. The invariant measure density and the normalization factor are respectively

$$w(x) = x^\alpha (1-x)^\beta \quad \text{and} \quad d_n^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}, \tag{4.46}$$

which concludes the proof by Theorem 2.

4.5. The Dual Jacobi process

We introduce the dual Jacobi polynomials by applying the transformation $x \mapsto Z(x) = x(2-x)$ to the Jacobi polynomials defined in the previous subsection.

Definition 12. *The dual Jacobi polynomials $D_n^{(\alpha,\beta)}(x) \equiv P_n^{(\alpha,\beta)}(1-2Z(x))$ are defined as follows:*

$$D_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1 \\ \alpha+1\end{matrix}\,\middle|\, x(2-x)\right), \quad n = 0,1,2,\dots \tag{4.47}$$

They also satisfy an orthogonality relation (1.23) on $(0,1)$ with normalization constants and continuous measure density:

$$w(x) = 2\big(x(2-x)\big)^\alpha (1-x)^{2\beta+1} \quad , \quad d_n^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}. \tag{4.48}$$

They are solutions to the eigenvalue problem (1.27) with generator:

$$\mathcal{L} = \frac{\sigma^2}{8}x(2-x)\frac{\partial^2}{\partial x^2} + \frac{2a - (2b - \frac{\sigma^2}{2})x(2-x)}{4(1-x)}\frac{\partial}{\partial x} \tag{4.49}$$

and eigenvalues $\lambda_n = -\frac{\sigma^2}{2}n(n+\alpha+\beta+1)$, still conditioned to $\alpha = \frac{2a}{\sigma^2} - 1 > -1$ and $\beta = \frac{2}{\sigma^2}(b-a) - 1 > -1$. Hence, we have the definition of the dual Jacobi process as follows:

Definition 13. *The dual Jacobi process is solution to the following equation:*

$$dX_t = \frac{2a - (2b - \frac{\sigma^2}{2})X_t(2-X_t)}{4(1-X_t)}dt + \frac{\sigma}{2}\sqrt{X_t(2-X_t)}dW_t \tag{4.50}$$

with initial condition $X_{t=0} = x_0 \in (0,1)$.

We obtain yet another corollary to Theorem 2:

Corollary 7. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to the dual Jacobi polynomials. Then:*

$$\phi(x) = \frac{\sigma^2}{8\vartheta}\left(\frac{\bar\alpha^2 - \alpha^2}{x(2-x)} + \frac{\bar\beta^2 - \beta^2}{(1-x)^2}\right), \tag{4.51}$$

for $\bar\alpha = \frac{2\bar a}{\sigma^2} - 1 > -1, \bar\beta = \frac{2}{\sigma^2}(\bar b - \bar a) - 1 > -1$. The transitional probability density of the dual Jacobi process is given by:

$$p_{T-t}(x,y) = 2\big(y(2-y)\big)^\alpha\ (1-y)^{2\beta+1}\sum_{n=0}^{\infty}\frac{e^{\lambda_n(T-t)}}{d_n^2}D_n^{(\alpha,\beta)}(x)D_n^{(\alpha,\beta)}(y). \tag{4.52}$$

The Laplace transform can be expressed by the following convergent series:

$$L_{T-t}(x,\vartheta) = 2\big(x(2-x)\big)^{\frac{\bar{\alpha}-\alpha}{2}}(1-x)^{\bar{\beta}-\beta}\ \sum_{n=0}^{\infty} e^{\lambda_n(T-t)} z_n D_n^{(\bar{\alpha},\bar{\beta})}(x) \tag{4.53}$$

where the coefficients z_n are given by:

$$z_n = \frac{1}{\bar{d}_n^2}\int_0^1 q(x)\ D_n^{(\bar{\alpha},\bar{\beta})}(x)\ \big(x(2-x)\big)^{\frac{\alpha+\bar{\alpha}}{2}}(1-x)^{\beta+\bar{\beta}+1}dx. \tag{4.54}$$

with normalization constant

$$\bar{d}_n^2 = \frac{\Gamma(n+\bar{\alpha}+1)\Gamma(n+\bar{\beta}+1)}{(2n+\bar{\alpha}+\bar{\beta}+1)\Gamma(n+\bar{\alpha}+\bar{\beta}+1)n!}. \tag{4.55}$$

Proof. The proof follows from Corollary 6 and the transformation $x \mapsto Z(x) = x(2-x)$.

5. Processes related to discrete orthogonal polynomials

The proof of Theorem 3 follows a very similar reasoning as the proof of Theorem 2, its continuous version.

5.1. Proof of the Third Classification Theorem

The reducibility condition implies that the infinitesimal generator $\mathcal{L}$ must be of the form

$$\mathcal{L} = -D(x)\Delta^1 + \big(D(x) - B(x)\big)\nabla^1_+. \tag{5.1}$$

The transitional probability density $p_{T-t}(x,y)$ satisfies the backward Kolmogorov equation with generator given by (1.12):

$$\frac{\partial p}{\partial t} - D(x)\Delta^1 p + \big(D(x) - B(x)\big)\nabla^1_+ p = 0 \tag{5.2}$$

with final time condition

$$\lim_{t\to T} p_{T-t}(x,y) = \delta(x-y). \tag{5.3}$$

By the reducibility assumption, a general solution to this equation is given by the following eigenfunction expansion in terms of the discrete orthogonal polynomials $Q_n(x)$

$$p = \sum_{n=0}^{\infty} h_n(t) Q_n(x). \tag{5.4}$$

According to (1.27), the functions of time $h_n(t)$ satisfy the ordinary differential equations

$$\dot{h}_n + \lambda_n h_n = 0 \tag{5.5}$$

which admits the general solution, for $t \le T$,

$$h_n(t) = z_n \; e^{\lambda_n (T-t)}. \tag{5.6}$$

The coefficients z_n are given by the final time condition (5.3):

$$\sum_{n=0}^{\infty} z_n Q_n(x) = \delta(x-y). \tag{5.7}$$

Hence, multiplying on both sides by $Q_m(x)$ and the weight $w(x)$, before summing over the lattice Λ_N, leads to the result (1.35) by the orthogonality property (1.23).

The Laplace transform $L_{T-t}(x,1)$ satisfies this time a finite difference version of the Backward Kolmogorov equation

$$\frac{\partial L}{\partial t} - D(x)\Delta^1 L + \big(D(x) - B(x)\big)\nabla^1_+ L = \vartheta\phi L \tag{5.8}$$

with the same final time condition

$$\lim_{t\to T} L_{T-t}(x,\vartheta) = q(x). \tag{5.9}$$

Consider the ansatz for the Laplace transform

$$L = V(x)\bar{L}. \tag{5.10}$$

The latter finite difference equation reads

$$\frac{\partial \bar{L}}{\partial t} - \underbrace{B(x)\frac{V(x+1)}{V(x)}}_{\bar{B}(x)}\bar{L}(x+1) + \underbrace{\big(B(x)+D(x)-\vartheta\phi\big)}_{\bar{B}(x)+\bar{D}(x)}\bar{L}(x) - \underbrace{D(x)\frac{V(x-1)}{V(x)}}_{\bar{D}(x)}\bar{L}(x-1) = 0. \tag{5.11}$$

$\bar{B}(x)$ and $\bar{D}(x)$ are defined such that they satisfy the relations

$$\frac{V(x)}{V(x-1)} = \frac{\bar{B}(x-1)}{B(x-1)} = \frac{D(x)}{\bar{D}(x)} \tag{5.12}$$

which implies condition (1.37) and solves iteratively to give

$$V(x) = \prod_{k=1}^{x} \frac{D(k)}{\bar{D}(k)}. \tag{5.13}$$

The function $\phi(x)$ is specified as follows:

$$\phi(x) = \frac{1}{\vartheta}\big(B(x)+D(x)-\bar{B}(x)-\bar{D}(x)\big). \tag{5.14}$$

But $\phi(x), B(x), D(x)$ are all by definition independent of the parameter ϑ, so we are bound to set $\vartheta = 1$. This choice yields the following finite difference equation for the function $\bar{L}$:

$$\frac{\partial \bar{L}}{\partial t} - \bar{D}(x)\Delta^1\bar{L} + \big(\bar{D}(x)-\bar{B}(x)\big)\nabla_+^1\bar{L} = 0. \tag{5.15}$$

A general solution to this equation is given by the following eigenfunction expansion in terms of discrete orthogonal polynomials:

$$\bar{L} = \sum_{n=0}^{N} h_n(t)\bar{Q}_n(x). \tag{5.16}$$

The functions of time $h_n(t)$ satisfy the ordinary differential equations

$$\dot{h}_n + \bar{\lambda}_n\, h_n = 0 \tag{5.17}$$

which admits the general solution

$$h_n(t) = z_n\, e^{\bar{\lambda}_n(T-t)} \tag{5.18}$$

where the z_n are constants. The latter equation for $h_n(t)$ with the explicit form of $V(x)$ in (5.13) and the expression of $\bar{L}$ in (5.16) yields the expression (1.38) for the Laplace transform.

The coefficients z_n are given by the final time condition (5.9):

$$q(x)\prod_{k=1}^{x}\frac{\bar{D}(x)}{D(x)} = \sum_{n=0}^{N} z_n\bar{Q}_n(x). \tag{5.19}$$

Finally, multiplying on both sides by $\bar{Q}_m(x)$ and the weight $\bar{w}(x)$, before summing over Λ_N, gives the final result (1.39) by orthogonality of the polynomials and concludes the proof of Theorem 3.

5.2. The Meixner process

The Meixner polynomials provide a discrete lattice approximation to the Laguerre polynomials.

Definition 14. *The Meixner polynomials are defined as follows in case x is integer:*

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix}\,\middle|\, 1-\frac{1}{c}\right),\quad n=0,1,2,\dots \tag{5.20}$$

The Meixner polynomials satisfy an orthogonality relation with respect to the discrete measure supported on $\mathbb{Z}_+$. Namely,

$$\sum_{x=0}^{\infty} M_m(x;\beta,c)\; M_n(x;\beta,c)\; w(x) = \frac{c^{-n}n!}{(\beta)_n(1-c)^\beta}\delta_{nm}. \tag{5.21}$$

where the weight is

$$w(x) = \frac{(\beta)_x}{x!}c^x. \tag{5.22}$$

The Meixner polynomials are solutions to the eigenvalue problem (1.27) with generator

$$\mathcal{L} = \frac{\sigma^2}{2}\; x\; \Delta^1 + (a-bx)\nabla^1_+, \tag{5.23}$$

for $x\in\mathbb{Z}_+$, with $a,b>0$ and eigenvalues $\lambda_n=-bn$. The latter can be recast in the form (1.34) using the functions

$$\begin{aligned} B(x) &= -\frac{\sigma^2}{2}c(x+\beta)\\ D(x) &= -\frac{\sigma^2}{2}x, \end{aligned} \tag{5.24}$$

for $a=\frac{\sigma^2}{2}\beta$ and $b=\frac{\sigma^2}{2}(1-c)$. The parameters are conditioned to $\beta>0$ and $0<c<1$ which insures the Markov property. The Meixner process is the discrete Markov process generated by $\mathcal{L}$. It is a discrete version of the CIR process.

The following statement is a corollary to Theorem 3.

Corollary 8. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to the Meixner polynomials. Assume also that the function $\phi(x)$ is given by $\phi(x)=\varrho x+\zeta$,*

$$\varrho = \frac{\sigma^2}{2}\big(c(e^{\varphi}-1)+(e^{-\varphi}-1)\big)\quad,\quad \zeta = \frac{\sigma^2}{2}\beta c(e^{\varphi}-1) \tag{5.25}$$

with the real parameter $\varphi<-\frac{1}{2}\ln c$. Then the transitional probability density for the Meixner process is as follows:

$$\begin{aligned} p_{T-t}(x,y) = (1-c)^\beta\; &\frac{(1-e^{(T-t)(c-1)})^{x+y}}{(1-ce^{(T-t)(c-1)})^{x+y+\beta}}\;\frac{(\beta)_y c^y}{y!}\\ &\cdot\; {}_2F_1\left(\begin{matrix}-x,-y\\ \beta\end{matrix}\,\middle|\, \frac{e^{(T-t)(c-1)}(1-c)^2}{c(1-e^{(T-t)(c-1)})^2}\right). \end{aligned} \tag{5.26}$$

For $q(x)=\exp\big(\omega\phi(x)\big)$, the Laplace transform is affine:

$$L_{T-t}(x,1,\omega) = e^{m(T-t;\omega)x+n(T-t;\omega)}. \tag{5.27}$$

The functions of time $m(\tau;\omega)$ and $n(\tau;\omega)$ are as follows:

$$m(\tau;\omega) = \log\left(e^{\varphi}\ \frac{1-ce^{\varrho\omega+\varphi}-e^{(\bar{c}-1)e^{-\varphi}\tau}(1-e^{\varrho\omega-\varphi})}{1-ce^{\varrho\omega+\varphi}-\bar{c}e^{(\bar{c}-1)e^{-\varphi}\tau}(1-e^{\varrho\omega-\varphi})}\right)$$

$$n(\tau;\omega) = -\beta\log\left(e^{\frac{\sigma^2}{2}\omega c(1-e^{\varphi})}\ \frac{1-ce^{\varrho\omega+\varphi}-\bar{c}e^{(\bar{c}-1)e^{-\varphi}\tau}(1-e^{\varrho\omega-\varphi})}{1-\bar{c}}\right).$$

where $\bar{c} = ce^{2\varphi}$.

Proof. The transitional probability density follows from equation (1.35) in the discrete classification theorem. The definition of $\phi(x)$ suggests that we set $\bar{B}(x) = B(x)e^{\varphi}$ and $\bar{D}(x) = D(x)e^{-\varphi}$ in order to satisfy condition (1.37) in Theorem 3. The generator defined by $\bar{B}(x)$ and $\bar{D}(x)$ has eigenfunctions $M_n(x;\beta,\bar{c})$ with eigenvalues $\bar{\lambda}_n = \frac{\sigma^2}{2}e^{-\varphi}n(\bar{c}-1)$. Now from (1.39), we have

$$\begin{aligned} z_n &= \frac{(\beta)_n\bar{c}^n(1-\bar{c})^{\beta}}{n!}\sum_{x=0}^{\infty}e^{\omega\phi(x)-\varphi x}\ M_n(x;\beta,\bar{c})\ \frac{(\beta)_x\bar{c}^x}{x!} \\ &= e^{\omega\zeta}\ \left(\frac{1-\bar{c}}{1-\bar{c}e^{\varrho\omega-\varphi}}\right)^{\beta}\ \frac{(\beta)_n\bar{c}^n}{n!}\ \left(\frac{1-e^{\varrho\omega-\varphi}}{1-\bar{c}e^{\varrho\omega-\varphi}}\right)^{n}. \end{aligned}$$

The Laplace transform, given by (1.38), is as follows:

$$\begin{aligned} L_{T-t}(x,1,\omega) &= \left(\frac{1-\bar{c}}{1-\bar{c}e^{\varrho\omega-\varphi}}\right)^{\beta}\ e^{\omega\zeta}\ e^{\varphi x} \\ &\quad\cdot\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\ \left(\bar{c}e^{\frac{\sigma^2}{2}(\bar{c}-1)e^{-\varphi}(T-t)}\frac{1-e^{\varrho\omega-\varphi}}{1-\bar{c}e^{\varrho\omega-\varphi}}\right)^{n}\ M_n(x;\beta,\bar{c}) \\ &= \left(e^{\frac{\sigma^2}{2}\omega c(1-e^{\varphi})}\ \frac{1-ce^{\varrho\omega+\varphi}-\bar{c}(1-e^{\varrho\omega-\varphi})e^{\frac{\sigma^2}{2}(\bar{c}-1)e^{-\varphi}(T-t)}}{1-\bar{c}}\right)^{-\beta} \\ &\quad\cdot\left(e^{\varphi}\ \frac{1-ce^{\varrho\omega+\varphi}-(1-e^{\varrho\omega-\varphi})e^{\frac{\sigma^2}{2}(\bar{c}-1)e^{-\varphi}(T-t)}}{1-ce^{\varrho\omega+\varphi}-\bar{c}(1-e^{\varrho\omega-\varphi})e^{\frac{\sigma^2}{2}(\bar{c}-1)e^{-\varphi}(T-t)}}\right)^{x}. \end{aligned}$$

Note that the re-summation formula used to find the last two results is the generating function for the Meixner polynomials which can be found in [20]. Also notice that $M_n(x;\beta,\bar{c}) = M_x(n;\beta,\bar{c})$ by definition.

5.3. The Racah Process

Definition 15. *The Racah polynomials $R_n(\lambda(x)) := R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ are defined as follows:*

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}\right|1\right),\ n=0,1,2,\ldots,N \tag{5.28}$$

where $\lambda(x) = x(x+\gamma+\delta+1)$ and either $\alpha = -N-1$ or $\beta+\delta = -N-1$ or $\gamma = -N-1$.

The Racah polynomials satisfy an orthogonality relation with respect to the discrete measure supported on the set Λ_N. Namely,

$$\sum_{x\in\Lambda_N} R_m(\lambda(x))R_n(\lambda(x))w(x) = d_n^2\delta_{nm} \tag{5.29}$$

where the weight is

$$w(x) := w(x;\alpha,\beta,\gamma,\delta) = \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x x!} \tag{5.30}$$

and the normalization factor is

$$d_n^2 = M\,\frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n n!}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n} \tag{5.31}$$

with

$$M = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if } \alpha = -N-1 \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if } \beta+\delta = -N-1 \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if } \gamma = -N-1. \end{cases}$$

The Racah polynomials are solutions to the eigenvalue problem (1.27) with generator (1.34) given by the functions

$$\begin{aligned} B(x) &= \frac{\sigma^2}{2}\frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}, \\ D(x) &= \frac{\sigma^2}{2}\frac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}, \end{aligned} \tag{5.32}$$

and eigenvalues $\lambda_n = -\frac{\sigma^2}{2}n(n+\alpha+\beta+1)$. The Markov property is insured if $B(x) \le 0$ and $D(x) \le 0$, $\forall x \in \Lambda_N$. The process generated by the latter generator is called the *Racah process*. Also define the corresponding functions

$$\begin{aligned} \bar{B}(x) &= \frac{\sigma^2}{2}\frac{(x+\bar\alpha+1)(x+\bar\beta+\bar\delta+1)(x+\bar\gamma+1)(x+\bar\gamma+\bar\delta+1)}{(2x+\bar\gamma+\bar\delta+1)(2x+\bar\gamma+\bar\delta+2)}, \\ \bar{D}(x) &= \frac{\sigma^2}{2}\frac{x(x-\bar\alpha+\bar\gamma+\bar\delta)(x-\bar\beta+\bar\gamma)(x+\bar\delta)}{(2x+\bar\gamma+\bar\delta)(2x+\bar\gamma+\bar\delta+1)} \end{aligned} \tag{5.33}$$

for $\bar\alpha, \bar\beta, \bar\gamma, \bar\delta \in \mathbb{R}$.

Definition 16. *The set of parameters $\{\bar\alpha, \bar\beta, \bar\gamma, \bar\delta\}$ will be called acceptable with respect to $\{\alpha, \beta, \gamma, \delta\}$ if it satisfies condition (1.37) in Theorem 3, i.e. if $\bar{B}(x-1)\bar{D}(x) = B(x-1)D(x)$, and if $\bar{B}(x) \le 0$ and $\bar{D}(x) \le 0$, $\forall x \in \Lambda_N$.*

The following statement is a corollary to Theorem 3.

Corollary 9. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to Racah polynomials. Assume also that for an acceptable set of parameters $\{\bar\alpha, \bar\beta, \bar\gamma, \bar\delta\}$:*

$$\phi(x) = B(x) + D(x) - \bar{B}(x) - \bar{D}(x). \tag{5.34}$$

Then the transitional probability density for the Racah process is given by

$$p_{T-t}(x,y) = \sum_{n=0}^{N} \frac{e^{-\frac{\sigma^2}{2}n(n+\alpha+\beta+1)(T-t)}}{d_n^2} R_n(\lambda(x))R_n(\lambda(y))w(y). \tag{5.35}$$

while the Laplace transform is given by the following convergent series:

$$L_{T-t}(x,1) = \prod_{k=1}^{x} \frac{D(k)}{\bar{D}(k)} \sum_{n=0}^{N} e^{-\frac{\sigma^2}{2} n(n+\bar{\alpha}+\bar{\beta}+1)(T-t)} z_n R_n(\bar{\lambda}(x); \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}). \tag{5.36}$$

The coefficients z_n are as follows:

$$z_n = \frac{1}{\bar{d}_n^2} \sum_{x \in \Lambda_N} \prod_{k=1}^{x} \frac{\bar{D}(k)}{D(k)} \, q(x) R_n(\bar{\lambda}(x); \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) \bar{w}(x) \tag{5.37}$$

where $\bar{d}_n = d_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$, $\bar{w}(x) = w(x; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ and $\bar{\lambda}(x) = \lambda(x; \bar{\gamma}, \bar{\delta})$.

Proof. The restrictions imposed on the set of parameters $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}\}$ ensures that condition (1.37) in Theorem 3 is satisfied. This is a necessary condition. The rest of the corollary is a direct application of Theorem 3.

5.4. The Dual Hahn Process

Definition 17. *The dual Hahn polynomials $R_n(\lambda(x)) := R_n(\lambda(x); \gamma, \delta, N)$ are defined as follows:*

$$R_n(\lambda(x); \gamma, \delta, N) = {}_3F_2\left(\left. \begin{matrix} -n, -x, x+\gamma+\delta+1 \\ \gamma+1, -N \end{matrix} \right| 1 \right), \quad n = 0, 1, 2, \ldots, N \tag{5.38}$$

where $\lambda(x) = x(x+\gamma+\delta+1)$.

For $\gamma > -1$ and $\delta > -1$ or for $\gamma < -N$ and $\delta < -N$, the dual Hahn polynomials satisfy an orthogonality relation with respect to the discrete measure supported on the set Λ_N. Namely,

$$\sum_{x \in \Lambda_N} R_m(\lambda(x)) R_n(\lambda(x)) w(x) = d_n^2 \delta_{nm} \tag{5.39}$$

where the weight is

$$w(x) := w(x; \gamma, \delta, N) = \frac{(2x+\gamma+\delta+1)(\gamma+1)_x (-N)_x N!}{(-1)^x (x+\gamma+\delta+1)_{N+1} (\delta+1)_x x!} \tag{5.40}$$

and the normalization factor is

$$d_n^2 = \frac{1}{\binom{\gamma+n}{n} \binom{\delta+N-n}{N-n}}. \tag{5.41}$$

The dual Hahn polynomials are solutions to the eigenvalue problem (1.27) with generator (1.34) given by the functions

$$\begin{aligned} B(x) &= -\frac{\sigma^2}{2} \frac{(x+\gamma+1)(x+\gamma+\delta+1)(N-x)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}, \\ D(x) &= -\frac{\sigma^2}{2} \frac{x(x+\gamma+\delta+N+1)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}, \end{aligned} \tag{5.42}$$

and eigenvalues $\lambda_n = -\frac{\sigma^2}{2} n$. The discrete Markov process generated by the latter generator is called the *dual Hahn process*.

The following statement is another corollary to Theorem 3.

Corollary 10. *Assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are reducible in the sense of Definition 3 to dual Hahn polynomials. Let $\delta > \gamma > -1$ or $\delta < \gamma < -N$ such that:*

$$\phi(x) = \frac{\sigma^2}{2}(\delta - \gamma)\left[\frac{x(x+\gamma+\delta+N+1)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)} - \frac{(x+\gamma+\delta+1)(N-x)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}\right]. \tag{5.43}$$

Then the transitional probability density for the dual Hahn process is given by

$$p_{T-t}(x,y) = \sum_{n=0}^{N} \frac{e^{-\frac{\sigma^2}{2}n(T-t)}}{d_n^2} R_n(\lambda(x))R_n(\lambda(y))w(y). \tag{5.44}$$

while the Laplace transform can be expressed as the following convergent series:

$$L_{T-t}(x,1) = \frac{(\delta+1)_x}{(\gamma+1)_x} \sum_{n=0}^{N} e^{-\frac{\sigma^2}{2}n(T-t)} z_n R_n(\lambda(x);\delta,\gamma,N). \tag{5.45}$$

The coefficients z_n are as follows:

$$z_n = \frac{1}{\bar{d}_n^2} \sum_{x\in\Lambda_N} \frac{(\gamma+1)_x}{(\delta+1)_x}\, q(x) R_n(\lambda(x);\delta,\gamma,N) w(x;\delta,\gamma,N) \tag{5.46}$$

where $\bar{d}_n = d_n(\delta,\gamma,N)$.

Proof. For $\bar{\gamma} = \delta$ and $\bar{\delta} = \gamma$, define the functions

$$\begin{aligned}
\bar{B}(x) &= -\frac{\sigma^2}{2}\frac{(x+\bar{\gamma}+1)(x+\bar{\gamma}+\bar{\delta}+1)(N-x)}{(2x+\bar{\gamma}+\bar{\delta}+1)(2x+\bar{\gamma}+\bar{\delta}+2)}, \\
\bar{D}(x) &= -\frac{\sigma^2}{2}\frac{x(x+\bar{\gamma}+\bar{\delta}+N+1)(x+\bar{\delta})}{(2x+\bar{\gamma}+\bar{\delta})(2x+\bar{\gamma}+\bar{\delta}+1)}.
\end{aligned} \tag{5.47}$$

The corollary then follows from Theorem 3.

6. Limit relations

Limit relations between orthogonal polynomials are well-known, see [20] and [26]. In this section, we show that the transitional probability densities and Laplace transforms obtained from the various corollaries of Theorems 2 and 3 are similarly connected to each other. We start by rigorously stating the relation between the Jacobi process and its dual.

Proposition 5. *The dual Jacobi process is obtained from the Jacobi process with the change of variable transformation*

$$Z(x) = x(2-x) \tag{6.1}$$

applied to the underlying process X_t. The same holds for its transitional probability density and Laplace transform.

For the discrete X_t process, the limit relation between the Racah process and the dual Hahn process is not as obvious.

Proposition 6. *The Racah process converges to the dual Hahn process in three different ways, corresponding to the three families of Racah polynomials:*

1. *$\alpha = -N-1$ with the acceptable set of parameters $\{-N-1, \beta+\delta-\gamma, \delta, \gamma\}$, conditioned to either*
$$\begin{cases} \beta \geq \gamma + N \\ \delta > \gamma > -1 \end{cases} \quad or \quad \begin{cases} \beta \geq -\delta - 1 \\ \delta < \gamma < -N \end{cases}$$
in the limit as $\beta \to \infty$.
2. *$\beta = -\delta - N - 1$ with the acceptable set $\{\alpha, -\gamma - N - 1, \delta, \gamma\}$, conditioned to either*
$$\begin{cases} \alpha \geq \gamma + \delta + N \\ \delta > \gamma > -1 \end{cases} \quad or \quad \begin{cases} \alpha \geq -1 \\ \delta < \gamma < -N \end{cases}$$
in the limit as $\alpha \to \infty$.
3. *$\gamma = -N - 1$ with the acceptable set $\{-\alpha + \delta - N - 1, \beta, -N - 1, \delta\}$, conditioned to either*
$$\begin{cases} \alpha > -1 \\ \beta \geq -1 \\ \delta > 2\alpha + N + 1 \end{cases} \quad or \quad \begin{cases} \alpha < -N \\ \beta \geq -\delta - 1 \\ \delta < 2\alpha + N + 1 \end{cases},$$
with first the mapping $\delta \mapsto \delta + \alpha + N + 1$ and then the limit $\beta \to \infty$. The dual Hahn parameters are in this third case (α, δ).

The result extends to the transitional probability densities and Laplace transforms.

The next proposition states that the Meixner process converges to the CIR process in the affine case.

Proposition 7. *Under the transformations $\varrho \mapsto \varrho(1-c)$ and $x \mapsto \dfrac{x}{1-c}$ in the Meixner process, the limit $c \to 1$ yields the CIR process with parameter $\alpha = \beta - 1$. The Laplace transform is affine in this case.*

Proof. The proof follows from the limit relation

$$\lim_{c\to 1} M_n\left(\frac{x}{1-c};\alpha+1,c\right)=\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$

Both the transitional probability density and the Laplace transform in the Meixner case then converge to the affine CIR case, since

$$\lim_{c\to 1}\phi\left(\frac{x}{1-c}\right)=\varrho\, x+\zeta.$$

The last proposition demonstrates the connection between a discrete and a continuous underlying process X_t by showing that the dual Jacobi process is actually a limiting case of the Racah process.

Proposition 8. *Consider the acceptable set of parameters* $\{\alpha,\bar\beta,\gamma,\delta\}$ *where*

$$\begin{aligned}\bar\beta&=-\beta-\delta-N-1\\ \gamma&=-N-1.\end{aligned}\tag{6.2}$$

Assume furthermore the inequalities:

$$\begin{aligned}&\alpha>-1\\ &\bar\beta>\beta\ >\ -1\\ &\gamma>\delta.\end{aligned}\tag{6.3}$$

Then, applying the transformation $x\mapsto xN$ *to the Racah process such that* $x\in[0,1]$*, yields the dual Jacobi process in the limit* $N\to\infty$ *for the special case of* $\bar\alpha=\alpha$ *and* $\bar\beta>|\beta|$*. This result applies to both the transitional probability density and the Laplace transform.*

Proof. The inequalities in the assumption insure that the process X_t satisfies the Markov property, as both $B(x)$ and $D(x)$ are negative for $x\in\Lambda_N$. Since on top of $\bar\alpha=\alpha,\bar\gamma=\gamma,\bar\delta=\delta$, we have $-\beta+\gamma=\bar\beta+\bar\delta$ and $\beta+\delta=-\bar\beta+\bar\gamma$, it is immediate that $\{\alpha,\bar\beta,\gamma,\delta\}$ is acceptable. The function $\phi(x)$ reduces to

$$\phi(x)=\frac{\sigma^2}{2}(\bar\beta-\beta)\left[\frac{x(x-\alpha+\gamma+\delta)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}-\frac{(x+\alpha+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}\right]\tag{6.4}$$

or equivalently,

$$\begin{aligned}\phi(x)=\frac{\sigma^2}{2}(\bar\beta-\beta)\ \Bigg[&\frac{x(x-2N-\alpha-\bar\beta-\beta-2)(x-N-\bar\beta-\beta-1)}{(2x-2N-\bar\beta-\beta-2)(2x-2N-\bar\beta-\beta-1)}\\ &-\frac{(x+\alpha+1)(x-N)(x-2N-\bar\beta-\beta-1)}{(2x-2N-\bar\beta-\beta-1)(2x-2N-\bar\beta-\beta)}\Bigg].\end{aligned}\tag{6.5}$$

$\phi(x)$ is bounded from below, from the inequality $\bar\beta>\beta>-1$ and is even monotonously increasing. Thus the assumptions of Corollary 9 are satisfied. So we have defined a proper Racah process with a Laplace transform that can be expressed as a convergent series over the Racah polynomials. With the transformation $x\mapsto Nx$, the function $\phi(Nx)$ converges, in the limit $N\to\infty$, to its continuous counterpart of Corollary 7 for $\vartheta=1$.

Moreover, the finite difference infinitesimal generator $\mathcal{L}$ in (1.34) converges to the dual Jacobi diffusion generator in (4.49), under the transformation $x \mapsto Nx$:

$$-\bar{D}(Nx)\Delta^h + \big(\bar{D}(Nx) - \bar{B}(Nx)\big)\nabla_+^h \longrightarrow \frac{\sigma^2}{2}\frac{2(\alpha+1)(1-x)^2 - (2\beta+1)x(2-x)}{4(1-x)}\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{x(2-x)}{4}\frac{\partial^2}{\partial x^2} \tag{6.6}$$

in the limit $h = 1/N \to 0$.

Furthermore, the Racah polynomials, solutions to the finite difference equation generated by (1.34), converge to the dual Jacobi polynomials up to some factor $\dfrac{n!}{(\alpha+1)_n}$, since

$$ {}_4F_3\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -Nx, Nx-2N-1-\beta-\bar{\beta} \\ \alpha+1, -\beta-N, -N\end{matrix}\right| 1\right) \longrightarrow {}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1 \\ \alpha+1\end{matrix}\right| Z(x)\right) \tag{6.7}$$

as $N \to \infty$, with $Z(x) = x(2-x)$. We also have as $N \to \infty$,

$$\prod_{k=1}^{Nx}\frac{D(k)}{\bar{D}(k)} = \prod_{k=1}^{Nx}\left(1 + \frac{\beta-\bar{\beta}}{N(1-\frac{k}{N})+\bar{\beta}+1}\right) \longrightarrow \exp\left(\int_0^x \frac{\beta-\bar{\beta}}{1-y}dy\right) = (1-x)^{\bar{\beta}-\beta}. \tag{6.8}$$

Finally, from all the above limit relations, the Laplace transform for the integral of the Racah process, given by (5.36) with the appropriate assumptions, provides an extension with underlying process X_t on the lattice to the Laplace transform (4.53) for the integral of the dual Jacobi process in the particular case $\bar{\alpha} = \alpha$, $\beta > |\bar{\beta}|$ and $\bar{\beta} \geq -1$.

7. Conclusion

We have given a complete classification scheme for diffusion processes for which Laplace transforms for integrals of stochastic processes and transitional probability densities can be expressed as integrals of hypergeometric functions against the spectral measure for certain self-adjoint operators. The known models such as the Ornstein-Uhlenbeck process, the CIR process and the geometric Brownian motion fit into this classification scheme. We have also presented extensions to these models in the quadratic Ornstein-Uhlenbeck process and the Jacobi process. An extension of the framework towards finite-state Markov processes related to hypergeometric polynomials in the discrete series of the Askey classification tree has been derived. Finally, we have explicitly computed some limit relations between discrete and continuous processes.

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