Structures and lower bounds for binary covering arrays

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Notations

- $B_q = \{0, 1, \ldots, q - 1\}$.
- For $u = (u_1, u_2, \ldots, u_n) \in B_q^n$,
  - $\text{supp}(u) = \{i \mid u_i \neq 0\}$.
  - $\text{wt}(u) = |\text{supp}(u)|$.
- $[n] = \{1, 2, \ldots, n\}$.
- For $C = (c_{ij})$ over $B_q$, $c^i$ is the $i$-th column of $C$. 
The covering array

**Definition**

An \( m \times n \) matrix \( C \) over \( B_q \) is called a \( t \)-covering array (or, a covering array of size \( m \), strength \( t \), degree \( n \), and order \( q \)) if, in any \( t \) columns of \( C \), all \( q^t \) possible \( q \)-ary \( t \)-vectors occur at least once. We denote such an array by \( CA(m; t, n, q) \).

**Example**

The following matrix is a 2-covering array over \( B_2 \).

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]
Definition

An \( m \times n \) matrix \( C \) over \( B_q \) is called a \( t \)-covering array (or, a covering array of size \( m \), strength \( t \), degree \( n \), and order \( q \)) if, in any \( t \) columns of \( C \), all \( q^t \) possible \( q \)-ary \( t \)-vectors occur at least once. We denote such an array by \( CA(m; t, n, q) \).

Applications

- circuit testing,
- intersecting codes,
- data compression.
The main problem is to optimize one of the parameters \( m \) and \( n \) for given value of the other:

- \((a)\) find the minimum size \( \text{CAN}(t, n, q) \) of a \( t \)-covering array of given degree \( n \) over \( B_q \);
- \((b)\) find the maximum degree \( \overline{\text{CAN}}(t, m, q) \) of a \( t \)-covering array of given size \( m \) over \( B_q \).

- \( q^t \leq \text{CAN}(t, n, q) \leq q^n \).

- Rènyi (for \( m \) even), and independently Katona, and Kleitman and Spencer (for all \( m \)) showed that \( \overline{\text{CAN}}(2, m, 2) = \left( \left\lfloor \frac{m-1}{2} \right\rfloor - 1 \right) \).

- Johnson and Entringer showed that \( \text{CAN}(n - 2, n, 2) = \left\lfloor \frac{2^n}{3} \right\rfloor \).

- Colbourn et al. give all the known upper and lower bounds for covering arrays up to degree 10, order 8 and all possible strengths, but their classification results are much more limited.
Theorem

(G. Roux 1987)

\[ \text{CAN}(t + 1, n + 1, q) \geq q \text{CAN}(t, n, q), \]
\[ \text{CAN}(3, 2n, 2) \leq \text{CAN}(3, n, 2) + \text{CAN}(2, n, 2). \]
Example

The following matrix is a 2-covering array over $B_2$.

$$
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
$$
### Example

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
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<tbody>
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<td>1</td>
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<td>0</td>
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<td>1</td>
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</table>

**Permutation of the rows**

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<table>
<thead>
<tr>
<th></th>
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<tbody>
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<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Permutation of the columns**

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<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Permutation of the values of any column**
The covering array

**Definition**

Two covering arrays $C$ and $C'$ are equivalent if one can be transformed into the other by a series of operations of the following types:

(a) permutation of the rows;
(b) permutation of the columns;
(c) permutation of the values of any column.

- Katona proved that maximal binary covering arrays of strength 2 are uniquely determined up to equivalence.
- Johnson and Entringer showed that $\left\lfloor \frac{2^n}{3} \right\rfloor \times n$ binary covering arrays of strength $n - 2$ are uniquely determined up to equivalence.
Goals

- Classify the structures of some optimal binary 2-covering arrays.
- Improve the lower bound of Roux on $CAN(t, n, q)$ when $t = 3, q = 2$. 
For $u \in B_2^n$, $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ where

$$\overline{u}_i = \begin{cases} 1, & \text{if } u_i = 0; \\ 0, & \text{if } u_i = 1. \end{cases}$$

- $u \in B_2^n \iff \text{supp}(u) \subseteq [n]$
- The following statements are equivalent.
  - $C$ is a binary $t$-covering array.
  - $\bigcap_{k=1}^{t} X_{i_k} \neq \emptyset$ for $\{i_1, \ldots, i_t\} \subseteq [n]$, where $X_k$ is either $\text{supp}(c^k)$ or $\overline{\text{supp}(c^k)}$. 
### Binary 2-covering array

A binary 2-covering array is a matrix $C$ with entries $c_i^j$ that satisfies the following properties:

- Each row of $C$ contains exactly two ones.
- Each column of $C$ contains exactly two ones.

Here is an example of a binary 2-covering array:

$$
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 \\
\end{array}
$$

The support of each column is defined as the set of rows containing a one in that column. For example, the support of $c^1$ is $\{1, 2\}$.

<table>
<thead>
<tr>
<th>Column</th>
<th>$c^1$</th>
<th>$c^2$</th>
<th>$c^3$</th>
<th>$c^4$</th>
<th>supp($c^1$)</th>
<th>supp($c^2$)</th>
<th>supp($c^3$)</th>
<th>supp($c^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>${1, 2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>${1, 3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>${1, 4}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>${1, 5}$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Binary 2-covering array

Let $\mathbf{C} = \mathbf{c}_1 \mathbf{c}_2 \mathbf{c}_3 \mathbf{c}_4$ be a binary covering array.

$$
\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 1 \\
\end{array}
$$

- $\text{supp}(\mathbf{c}_1) = \{1, 2\}$
- $\text{supp}(\mathbf{c}_2) = \{1, 3\}$
- $\text{supp}(\mathbf{c}_3) = \{1, 4\}$
- $\text{supp}(\mathbf{c}_4) = \{1, 5\}$

![Venn diagram showing the support of $\mathbf{c}_1$ and $\mathbf{c}_2$](image_url)
**Definition**

The standard maximal binary 2-covering array $C$ of size $m$ is an $m \times \left(\left\lfloor \frac{m-1}{2} \right\rfloor - 1\right)$ matrix with

1. the first row of $C$ is all 1 row,
2. the columns of the remaining matrix is the family of all vectors of $(\left\lfloor \frac{m}{2} \right\rfloor - 1)$ 1's and $(m - \left\lfloor \frac{m}{2} \right\rfloor)$ 0's.

**Example**

$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
$$
Theorem

(E. W. Hall 1935)
Suppose we have a bipartite graph $G$ with two vertex sets $V_1$ and $V_2$. Suppose that

$$|\Gamma(S)| \geq |S| \quad \text{for every } S \subset V_1.$$

Then $G$ contains a complete matching.
Lemma

Let $C$ be a 2-covering array of size $m$ and degree $n$ with $\text{wt}(c^i) \leq \lfloor \frac{m}{2} \rfloor$ for all $1 \leq i \leq n$. Put $s = \min_{1 \leq i \leq n} \text{wt}(c^i)$. For any integer $s'$ satisfying $s < s' \leq \lfloor \frac{m}{2} \rfloor$, there is a 2-covering array $C'$ of size $m$ and degree $n$ with $s' \leq \text{wt}(c'^i) \leq \lfloor \frac{m}{2} \rfloor$ such that $\text{supp}(c^i) \subseteq \text{supp}(c'^i)$ for all $i \in [n]$.

Corollary

Let $C$ be a 2-covering array of size $m$ and degree $n$ with $\text{wt}(c^i) \leq \lfloor \frac{m}{2} \rfloor$ for all $i \in [n]$ and $\text{wt}(c^j) < \lfloor \frac{m}{2} \rfloor$. Then there is a 2-covering array $C'$ of size $m$ and degree $n$ with $\text{wt}(c'^j) = \lfloor \frac{m}{2} \rfloor - 1$ and $\text{wt}(c'^i) = \lfloor \frac{m}{2} \rfloor$ for all $i \in [n]$ and $i \neq j$ such that $\text{supp}(c^i) \subseteq \text{supp}(c'^i)$ for all $i \in [n]$. 
Theorem


Let $2 \leq k \leq \frac{m}{2}$. Let $C$ be a binary 2-covering array of size $m$ such that $\text{wt}(c^i) \leq k$ for any column of $C$ and $\bigcap_{1 \leq i \leq n} \text{supp}(c^i) = \emptyset$. Then

$$n \leq d = 1 + \binom{m - 1}{k - 1} - \binom{m - k - 1}{k - 1}.$$ 

There is strict inequality if $\text{wt}(c^i) < k$ for some $i \in [n]$. 
Theorem

Let $m \geq 4$, $k = \lfloor \frac{m}{2} \rfloor$, and $(\binom{m-1}{k-1}) + m - 3k + 1 \leq n \leq (\binom{m-1}{k-1})$.

If an $m \times n$ matrix $C$ over $B_2$ is a 2-covering array, then $C$ is equivalent to the matrix made from deleting columns of standard binary 2-covering.
Corollary

Every maximal binary 2-covering arrays is equivalent to the standard maximal 2-covering array.

Corollary

If \( m \geq 6 \) and \( n = \left( \left\lfloor \frac{m-1}{2} \right\rfloor - 1 \right) - 1 \), then every \( m \times n \) binary 2-covering array \( C \) is made from deleting a column of the standard maximal 2-covering array.

- 10 \( \times \) 5, 12 \( \times \) 11 binary optimal 3-covering and 24 \( \times \) 12 binary optimal 4-covering arrays are unique.
- There is no 48 \( \times \) 13 binary 5-covering array.
Theorem

If $m \geq 7$, $k = \lfloor \frac{m}{2} \rfloor$, and $(\binom{m-1}{k-1}) + m - 3k + 1 \leq n \leq (\binom{m-1}{k-1})$, then

$$\text{CAN}(3, n + 1, 2) \geq \begin{cases} 2\text{CAN}(2, n, 2) + 1 & \text{if } m \text{ is odd} \\ 2\text{CAN}(2, n, 2) + 2 & \text{if } m \text{ is even} \end{cases}$$
Table 1: The number of covering arrays $CA(6; 2, n, 2)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CA(6; 2, n, 2)$</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: The number of covering arrays $CA(8; 2, n, 2)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CA(8; 2, n, 2)$</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 3: Tables of \( \text{CAN}(3, n, 2) \).
Thank you for your attention!