Lower Bounds on the Complexity of MSO$_1$ Model-Checking

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Joint work with
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Algorithmic Meta Theorems

Theorems that identify classes of tractable problems, rather than a few isolated problems.

Examples

- All graph properties expressible in MSO\textsubscript{2} can be decided in linear time on graphs of bounded treewidth [Courcelle, 1990].
- All problems in MAX SNP have constant-factor approximation algorithms [Papadimitriou and Yannakakis, 1991].
- Compact parameterized problems expressible in CMSO admit polynomial kernels on graphs of bounded genus [Bodlaender et al, 2010].

Uses

- Quick way of checking whether a problem admits an algorithm of a particular kind.
Theorem (Courcelle, 1990)

Any graph property definable in $\text{MSO}_2$ can be decided in linear time on any class of graphs of bounded treewidth.

$\text{MSO}_2$ – monadic second-order logic with quantification over sets of vertices and/or edges

Expressible in $\text{MSO}_2$: $\text{HAMILTONIAN CYCLE}, \text{VERTEX COVER}, \ldots$

3-Colourability:

$$\exists V_1, V_2, V_3 \left[ \forall v \left( v \in V_1 \lor v \in V_2 \lor v \in V_3 \right) \land \land_{i=1,2,3} \forall v, w \left( v \notin V_i \lor w \notin V_i \lor \neg \text{adj}(v, w) \right) \right]$$
The theorem may also be stated as:

**Theorem (Courcelle, 1990)**

Let $\mathcal{C}$ be any class of graphs of bounded tree-width. Then $\text{MC}(\text{MSO}_2, \mathcal{C})$ is decidable in linear time.

$\text{MC}(\text{MSO}_2, \mathcal{C})$ – the $\text{MSO}_2$ model-checking problem on $\mathcal{C}$:
Given a $G \in \mathcal{C}$ and $\varphi \in \text{MSO}_2$ check whether $G \models \varphi$.

**Questions:**

- Are there classes of graphs of unbounded treewidth such that Courcelle’s Theorem still holds? **YES!** (in XP) [Makowsky and Mariño, 2004]
- How fast must the treewidth grow for Courcelle’s theorem to fail? *Poly-logarithmically* [Kreutzer and Tazari, 2010]
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Courcelle’s Theorem – Lower Bounds

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Unbounding Tree-Width

Definition (Kreutzer and Tazari, 2010)

The treewidth of a graph class $C$ is *strongly unbounded by* $f : \mathbb{N} \to \mathbb{N}$ if for all $n \in \mathbb{N}$ there exists $G_n \in C$ with

- $f(|G_n|) \leq tw \ G_n$ *unbounded*
- $n \leq tw \ G_n \leq n^\gamma$, for some fixed $\gamma$ *dense*
- $G_n$ can be constructed in time $2^{n^\epsilon}$, for some fixed $\epsilon < 1$ *constructable*

strongly unbounded poly-logarithmically:

$$\log^c(|G_n|) \leq tw \ G_n \text{ for all } c \geq 1$$
Recent known results

**Theorem (Kreutzer and Tazari, SODA’10)**

Let $\mathcal{C}$ be a graph class with the following properties:

1. the treewidth of $\mathcal{C}$ is strongly unbounded poly-logarithmically
2. $\mathcal{C}$ is closed under $\Gamma$-colourings

Then $\text{MC}(\text{MSO}_2\Gamma, \mathcal{C})$ is not in $\text{XP}$ ($|G|^f(|\varphi|)$ for any computable $f$), unless the Exponential-Time Hypothesis (ETH) fails.

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Let $\mathcal{C}$ be a graph class with the following properties:

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Our Results I

**Theorem**

Assume a (suitable but fixed) finite label set $L$. Let $\mathcal{C}$ be a graph class with the following properties:

1. The tree-width of $\mathcal{C}$ is densely unbounded poly-logarithmically.
2. $\mathcal{C}$ is closed under taking subgraphs.

Then $\text{MC}(\text{MSO}_1^{-L}, \mathcal{C}^L)$ is not in XP unless the nonuniform Exponential-Time Hypothesis (nonuniform ETH) fails.

**MSO$_1$** – no quantification over sets of edges

**MSO$_1^{-L}$** – extension of MSO$_1$ with vertex-label predicates for a finite set of labels $L$

**$\mathcal{C}^L$** – the class of all $L$-vertex labelled graphs from $\mathcal{C}$

**nonuniform ETH** – SAT not in $2^{o(n)}$ with subexponential advice
Main Improvements

1. Our definition of “densely unbounded” avoids the constructability requirement in the definition of “strongly unbounded”. We “pay” for this by using a stronger complexity assumption: Nonuniform ETH.

2. Kreutzer and Tazari: MSO\textsubscript{2} on unlabelled graphs
   
   Our result: MSO\textsubscript{1}-L.

3. Much simpler, streamlined proof.
MSO$_1$-$L$ is much weaker than MSO$_2$:

- HAMILTONIANPATH cannot be expressed in MSO$_1$-$L$
- Extending [Courcelle, Makowski, Rotics, 2000] from MSO$_1$ to MSO$_2$ would mean EXP=NEXP

Labels do not matter:

- Many results concerning MSO$_1$ can be formulated with or without labels.
- Both Courcelle’s theorem and [CMR00] can be extended with labels.
At a high level, our proof technique is similar to Kreutzer and Tazari: a *multi-step reduction from SAT*.

Proof *shorter* mainly because we do not need to tediously construct a “skeleton” in the class $\mathcal{C}$ suitable for reduction. It comes “for free” from the *oracle advice function* which comes with the nonuniform computing model.

We *avoid* the need for $\text{MSO}_2$ by using *strong edge colourings* to simulate certain edge-sets inside $\text{MSO}_1$-$L$. 
High-level Proof Description

Reduce SAT to MC(MSO$_2$, $\mathcal{C}$).

- **Input:** A SAT formula $F$ of length $n$.
- **Question:** Is $F$ satisfiable?

**Reduction**

1. Construct $G_n \in \mathcal{C}$ of treewidth $n^d$ s.t. $\log^c(|G_n|) < tw \ G_n$ and $c > d$.
   - Strongly poly-logarithmically unbounded.
2. Encode $F$ in a subgraph of $G_n$.
   - Using closure under subgraphs.
3. Define an MSO-formula $\varphi$ (independent of $F$) s.t. $F$ satisfiable iff $G_n \models \varphi$.
   - Deciding $G_n \models \varphi$ takes time $2^{n^{c/d} \cdot f(|\varphi|)}$, subexponential in $|F|$. 
Grid-like graphs (minors)
A pair $(G, \mathcal{P})$ such that:

1. $G$ is the union of all the paths in $\mathcal{P}$,
2. each path in $\mathcal{P}$ has at least two vertices, and
3. the *intersection graph* $I(\mathcal{P})$ of the path collection is bipartite.

**Theorem (Reed and Wood, 2008)**

Every graph with tree-width at least $c \ell^4 \sqrt{\log \ell}$ contains a subgraph which is grid-like of order $\ell$, for some constant $c$.

**Order of a grid-like graph**: the maximum integer $\ell$ such that the intersection graph $I(\mathcal{P})$ contains a $K_{\ell}$-minor.
Strong Edge Colourings
An assignment of colours to the edges of a graph such that no path of length three contains the same colour twice.

Theorem (Cranston, 2006)
Every graph of maximum degree 4 has a strong edge-colouring using at most 22 colours. This colouring can be found with a polynomial-time algorithm.
**MSO$_1$ interpretation in $\{1,3\}$-regular graphs**

**Theorem (Ganian et al, 2010)**

The MSO$_1$ theory of all simple graphs has an efficient interpretation in the MSO$_1$ theory of all simple $\{1,3\}$-regular graphs. Furthermore, this efficient interpretation $I$ can be chosen such that, for every MSO$_1$ formula $\psi$, the resulting property $\psi^I$ is invariant under subdivisions of edges.
Our main theorem can be strengthened by taking a stricter assumption:

**Theorem**

Let $\mathcal{C}$ be a graph class with the following properties:

1. the tree-width of $\mathcal{C}$ is densely unbounded poly-logarithmically
2. $\mathcal{C}$ is closed under taking subgraphs

Then $\text{MC}(\text{MSO}_1\cdot L, \mathcal{C}_L)$ for every finite set of labels $L$ such that $|L| = (|\varphi|)$ is not in XP unless $\text{PH} \subseteq \text{DTIME}(2^{o(n)})/\text{SUBEXP}$. 
This is an extension of [Ganian et al, 2010]

**Theorem**

Let $L$ be a finite set of labels, $|L| \geq 47$. Unless the nonuniform Exponential-Time Hypothesis fails, there exists no directed width measure $\delta$ satisfying following three properties:

1. $\delta$ is monotone under taking subdigraphs;
2. $\delta$ largely surpasses the tree-width of underlying undirected graphs; and
3. for all $L$-vertex-labelled digraphs $D$ and all formulas $\varphi \in \text{MSO}_1$-$L$, the problem of deciding whether $D \models \varphi$ is solvable in time $O(|D|^{f(\delta(D), |\varphi|)})$ for some computable $f$. 
Food!