Spike-layer Solutions to Singularly Perturbed Semilinear Systems of Coupled Schrödinger Equations

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Outline

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1. Introduction

In this talk, we are concerned with two coupled Schrödinger systems

\[
\begin{aligned}
-\varepsilon^2 \Delta u + u &= \mu_1 u^3 + \beta uv^2, \ x \in \Omega, \\
-\varepsilon^2 \Delta v + v &= \mu_2 v^3 + \beta u^2 v, \ x \in \Omega,
\end{aligned}
\]

\((S_\varepsilon)\)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \leq 3)\), \(\mu_1, \mu_2\) are positive constants.
For $\Omega = \mathbb{R}^N$ and $\varepsilon = 1$, $(S_{\varepsilon})$ leads to investigate the following problems

\[
\begin{align*}
- \Delta u + u &= \mu_1 u^3 + \beta u v^2, \quad x \in \mathbb{R}^N, \\
- \Delta v + v &= \mu_2 v^3 + \beta v^2 u, \quad x \in \mathbb{R}^N, \\
u > 0, \quad v > 0, \\
u(x) \rightarrow 0, \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.
\end{align*}
\]

Problem (1.1) arises in the Hartree-Fock theory for a double condensate i.e. a binary mixture of Bose-Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$. Physically, $u$ and $v$ are the corresponding condensate amplitudes, $\mu_j$ and $\beta$ are the intraspecies and interspecies scattering lengths. The sign of the scattering length $\beta$ determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive. In contrast, when $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are attractive.
• The first authors to consider problem (1.1), as far as we know, is T.Lin and J.Wei[Comm.Math.Phys.,255(2005)629-653.]. In that paper, they proved that there exist $0 < \beta_0 < \sqrt{\mu_1\mu_2}$ sufficiently small, such that for $-\infty < \beta < \beta_0$, (1.1) has a least energy solution.

• In 2007, B.Sirakov[Comm.Math.Phys.,271(2007),199 - 221.] considered problem (1.1) for the whole $\beta \in \mathbb{R}$ and analyzed for which $\beta$ problem (1.1) assures a least energy solution and for which $\beta$ problem (1.1) has no least energy solution. More precisely, he proved that for $-\infty < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, (1.1) has a least energy solution but for $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, (1.1) has no solution.
  \( \beta > 0 \) sufficiently small, the solution of (1.1) is unique;

• More recently, W. Yao and J. Wei also studied the uniqueness of solution of problem
  (1.1), they proved that for \( \beta > 0 \) sufficiently small or \( \beta > \max\{\mu_1, \mu_2\} \), the solution
  of problem (1.1) is unique;

• but for \( 0 < \beta < \min\{\mu_1, \mu_2\} \), the uniqueness of the solution to problem (1.1) is still
  open.
When $\Omega$ is bounded domain in $\mathbb{R}^N (N \leq 3)$, under Dirichlet boundary condition, namely the following problem

\[
\begin{align*}
-\varepsilon^2 \Delta u + u &= \mu_1 u^3 + \beta uv^2, \quad x \in \Omega, \\
-\varepsilon^2 \Delta v + v &= \mu_2 v^3 + \beta u^2 v, \quad x \in \Omega, \\
u > 0, \quad v > 0, \quad &x \in \Omega, \\
u = 0, v = 0, \quad &x \in \partial \Omega.
\end{align*}
\]

There are also a lot of papers to consider ($S^1_\varepsilon$).
• The first paper, as far we know to consider $(S^1_\varepsilon)$ is T.Lin and J.Wei[Ann. Inst. H.Poincaré Anal. Non Linéaire 22(4)(2005)403-439.]. They obtained the existence of the least energy solution to $(S^1_\varepsilon)$ by minimizing the certain Nehari manifold for $-\infty < \beta < \beta_0$ and also discussed the asymptotic behavior as $\varepsilon$ goes to zero, where $0 < \beta_0 < \sqrt{\mu_1\mu_2}$ is a sufficiently small positive constant. More precisely, they pointed out that when $\beta < 0$, the maximum points of the two components of the least energy solution to $(S^1_\varepsilon)$ approach different points as $\varepsilon \to 0$ whereas when $0 < \beta < \beta_0$, the maximum points of the two components of the least energy solution to $(S^1_\varepsilon)$ go together as $\varepsilon \to 0$.

• In 2009, N.Dancer and J.Wei[Tran.Amer.Math.Soci.,361 No 3(2009)1189-1208] proved that there exists an interval $I = [a_0, b_0]$ and a sequence of numbers $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$ such that for any $\beta \in (0, \infty) \setminus (I \cup \{\beta_1, \beta_2, \cdots, \})$, $(S^1_\varepsilon)$ has a solution $(u_\varepsilon, v_\varepsilon)$ such that both $u_\varepsilon$ and $v_\varepsilon$ develop a spike layer at the innermost part of the domain. They also proved the nondegeneracy of radial solutions in $\mathbb{R}^N$. 
• In 2010, when $\Omega$ is a symmetric domain (possibly unbounded) T.Bartsch, N.Dancer and Z.Q.Wang [Calc. Var., 37(2010)345 – 361.] investigated the local and global bifurcation in terms of the parameter $\beta$ which provides a-priori bounds of solution branches to $\left( S_{\varepsilon}^1 \right)$.

• In the following two papers, J.Wei and T.Weth [Rend. Lincei Math. Appl., 18 (2007) 279-293.], J.Wei and T.Weth [Arch.Rat.Mech.Anal.,190(2008)83-106.], They considered the multiplicity properties of $\left( S_{\varepsilon}^1 \right)$ when $\varepsilon = 1$ and $\beta$ is negative.
2. Existence of least energy solution

In this talk, we consider problem \((S_\varepsilon)\) under Neumann boundary condition, namely the following problem

\[
\begin{aligned}
-\varepsilon^2 \Delta u + u &= \mu_1 u^3 + \beta u v^2, \quad x \in \Omega, \\
-\varepsilon^2 \Delta v + v &= \mu_2 v^3 + \beta u^2 v, \quad x \in \Omega,
\end{aligned}
\]

\[
\begin{aligned}
u &> 0, \quad v > 0, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega
\end{aligned}
\]

where \(\frac{\partial}{\partial n}\) denotes the external normal derivative on the boundary \(\partial \Omega\) and \(\Omega\) is bounded smooth domain in \(\mathbb{R}^N(N \leq 3)\).
A solution \((u, v)\) of \((S^2_{\varepsilon})\) which has a zero component \((u \equiv 0 \text{ or } v \equiv 0)\) will be called a standard solution. \((0,0)\) is referred as the trivial solution of \((S^2_{\varepsilon})\). We are concerned on the nonstandard solutions of \((S^2_{\varepsilon})\), namely solutions with both components are all nonzero.

The energy functional corresponding to \((S^2_{\varepsilon})\) is as follows:

\[
J_{\varepsilon}(u, v) := \frac{1}{2} \int_{\Omega} \left[ \varepsilon^2 |\nabla u|^2 + u^2 + \varepsilon^2 |\nabla v|^2 + v^2 \right] dx - \frac{1}{4} \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2) dx, \tag{2.1}
\]

for every \((u, v) \in H^1(\Omega) \times H^1(\Omega)\).
We consider the set

\[ \mathcal{N}(\varepsilon, \Omega) = \left\{ \begin{array}{l}
(u, v) \in H^1(\Omega)^2 : \\
\varepsilon^2 |\nabla u|^2 + u^2 = \int_{\Omega} \mu_1 u^4 + \beta u^2 v^2 \\
\varepsilon^2 |\nabla v|^2 + v^2 = \int_{\Omega} \mu_2 v^4 + \beta u^2 v^2 
\end{array} \right\} \]

and let

\[ c_\varepsilon = \inf_{(u,v) \in \mathcal{N}(\varepsilon, \Omega)} J_\varepsilon(u, v). \]
Our first results deal with the existence of least energy solutions of \((S^2_\varepsilon)\) which achieve \(c_\varepsilon\).

Our first statement is:

**Theorem 2.1** For any \(\varepsilon > 0\), if \(-\infty < \beta < \min\{\mu_1, \mu_2\}\) or \(\beta > \max\{\mu_1, \mu_2\}\), there exists a least energy solution \((u_\varepsilon, v_\varepsilon)\) to system \((S^2_\varepsilon)\) which achieves \(c_\varepsilon\). If \(\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}\), \((S^2_\varepsilon)\) has no solution. In fact, suppose \(\omega_\varepsilon\) is a least energy solution of the scalar case \(-\varepsilon^2 \Delta u + u = u^3, u > 0, x \in \Omega\) under Neumann boundary conditions. Then for \(0 \leq \beta < \min\{\mu_1, \mu_2\}\) or \(\beta > \max\{\mu_1, \mu_2\}\), \((\sqrt{k} \omega_\varepsilon, \sqrt{l} \omega_\varepsilon)\) is a least energy solutions of system \((S^2_\varepsilon)\) which achieves \(c_\varepsilon\), where \(k = \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}\) and \(l = \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}\).
For the following scalar equation

\[
\begin{aligned}
- \varepsilon^2 \Delta w + w &= w^p, \quad x \in \Omega, \\
w &> 0, \quad x \in \Omega, \\
\frac{\partial w}{\partial n} &= 0, \quad x \in \partial \Omega
\end{aligned}
\]  \tag{2.2}

where \(1 < p < 2^* - 1\) and \(2^* = \frac{2N}{N-2}\) for \(N \geq 3\) and \(2^* = +\infty\) for \(N = 1, 2\).

- Firstly, Ni and Takagi [Duke Math. J., 70(1993), 247 - 281] considered the existence of least energy solution to (2.2) and also asymptotic behavior of the least energy solution as \(\varepsilon\) goes to zero.


- More recently, J. Byeon [J. Diff. Equat., 244(2008), 2473-2497] extended the above results for almost optimal general nonlinearities.
Let $\omega_\varepsilon$ be a least energy solution of (2.2) with $p = 3$ and $N \leq 3$. Let

$$S_{\mu,\varepsilon} = \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2)\,dx}{\int_\Omega \mu u^4\,dx},$$

$$T_{\mu,\varepsilon} = \inf_{u \in \mathcal{M}_\varepsilon} \left\{ \frac{1}{2} \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2)\,dx - \frac{1}{4} \int_\Omega \mu u^4\,dx \right\},$$

where $\mathcal{M}_\varepsilon = \{ u \in H^1(\Omega), u \neq 0 : \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2)\,dx = \int_\Omega \mu u^4\,dx \}$. Then we have that $\omega_\varepsilon$ is a minimizer of $T_{1,\varepsilon}$ and $\mu^{-\frac{1}{2}} \omega_\varepsilon(x)$ is a minimizer of $T_{\mu,\varepsilon}$. In addition we have

$$T_{\mu,\varepsilon} = \frac{1}{4} S_{\mu,\varepsilon}^2, \quad S_{\mu,\varepsilon} = \mu^{-\frac{1}{2}} S_{1,\varepsilon}.$$
Sketch of the Proof of Theorem 2.1
Firstly consider the case of $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$.

Step 1  We consider the following linear system in $k, l \in \mathbb{R}$.

\[
\begin{align*}
\mu_1 k + \beta l &= 1, \\
\beta k + \mu_2 l &= 1.
\end{align*}
\]  

(2.3)

It is easy to check that for $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, system (2.3) has a unique solution $(k, l)$ such that $k > 0$ and $l > 0$. Indeed, by a direct calculation, we have that

\[
k = \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}, \quad l = \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}.
\]  

(2.4)
Step 2  We claim that the couple \((\sqrt{k} \omega \varepsilon, \sqrt{l} \omega \varepsilon)\) belongs to \(\mathcal{N}(\varepsilon, \Omega)\) with \(k, l\) is defined as in (2.4).

Thus we have

\[
c_\varepsilon \leq J_\varepsilon(\sqrt{k} \omega \varepsilon, \sqrt{l} \omega \varepsilon) = \frac{1}{4} (k + l) S_{1,\varepsilon}^2 = \frac{1}{4} \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1 \mu_2 - \beta^2} S_{1,\varepsilon}^2 \tag{2.5}
\]
Step 3  We prove that the inverse inequality of (2.5) also holds, namely we have

\[ c_\varepsilon \geq \frac{1}{4}(k + l)S_{1,\varepsilon}^2 = \frac{1}{4}\frac{\mu_1 + \mu_2 - 2\beta}{\mu_1\mu_2 - \beta^2}S_{1,\varepsilon}^2. \] (2.6)

Suppose \((u_m, v_m) \in \mathcal{N}(\varepsilon, \Omega)\) is a minimizing sequence of \(c_\varepsilon\), namely

\[ J_\varepsilon(u_m, v_m) = \frac{1}{4} \left[ \int_{\Omega} (\varepsilon^2|\nabla u_m|^2 + u_m^2) dx + \int_{\Omega} (\varepsilon^2|\nabla u_m|^2 + v_m^2) dx \right] \]

\[ = \frac{1}{4} \int_{\Omega} (\mu_1 u_m^4 + \mu_2 v_m^4 + 2\beta u_m^2 v_m^2) dx \rightarrow c_\varepsilon \]

as \(m \rightarrow +\infty\). It follows that \((u_m, v_m)\) is bounded in \(H^1(\Omega) \times H^1(\Omega)\).
Set

\[ \xi_m = \left( \int_{\Omega} u_m^4 \, dx \right)^{1/2}, \quad \zeta_m = \left( \int_{\Omega} v_m^4 \, dx \right)^{1/2}. \]

It follows from the definition of \( S_{1,\varepsilon} \) that

\[ S_{1,\varepsilon} \xi_m \leq \int_{\Omega} (\varepsilon^2 |\nabla u_m|^2 + u_m^2) \, dx = \int_{\Omega} (\mu_1 u_m^4 + \beta u_m^2 v_m^2) \, dx \leq \mu_1 \xi_m^2 + \beta \xi_m \zeta_m, \tag{2.7} \]

\[ S_{1,\varepsilon} \zeta_m \leq \int_{\Omega} (\varepsilon^2 |\nabla v_m|^2 + v_m^2) \, dx = \int_{\Omega} (\mu_2 v_m^4 + \beta u_m^2 v_m^2) \, dx \leq \mu_2 \zeta_m^2 + \beta \xi_m \zeta_m, \tag{2.8} \]

Adding up (2.7) and (2.8) results in

\[ S_{1,\varepsilon} (\xi_m + \zeta_m) \leq \int_{\Omega} (\mu_1 u_m^4 + 2\beta u_m^2 v_m^2 + \mu_2 v_m^4) \, dx = 4c_\varepsilon + o(1), \tag{2.9} \]

where \( o(1) \to 0 \) as \( m \to \infty \).
Set \( \xi_1^m = \frac{\xi_m}{S_{1,\varepsilon}} \) and \( \zeta_1^m = \frac{\zeta_m}{S_{1,\varepsilon}} \). Then combining (2.7), (2.8) and (2.9), we obtain the following inequalities

\[
\begin{align*}
\xi_1^1 + \zeta_1^1 &\leq k + l + o(1), \\
\mu_1 \xi_1^1 + \beta \zeta_1^1 &\geq 1, \\
\beta \xi_1^1 + \mu_2 \zeta_1^1 &\geq 1.
\end{align*}
\] (2.10)

where \( k, l \) as in (2.4). Let \( \xi_2^m = \xi_1^m - k \) and \( \zeta_2^m = \zeta_1^m - l \), then we have from the inequality (2.10) that

\[
\begin{align*}
\xi_2^2 + \zeta_2^2 &\leq o(1), \\
\mu_1 \xi_2^2 + \beta \zeta_2^2 &\geq 0, \\
\beta \xi_2^2 + \mu_2 \zeta_2^2 &\geq 0.
\end{align*}
\] (2.11)
Now it is easy to check that when

\[ 0 \leq \beta < \min\{\mu_1, \mu_2\} \text{ or } \beta > \max\{\mu_1, \mu_2\}, \]

the three half spaces \( \{t = (t_1, t_2) : t_1 + t_2 < o(1)\} \), \( \{t = (t_1, t_2) : \mu_1 t_1 + \beta t_2 \geq 0\} \), \( \{t = (t_1, t_2) : \beta t_1 + \mu_2 t_2 \geq 0\} \) meet at most in a triangle in the \((t_1, t_2)\)-plane, and this triangle shrinks to \( t_1 = t_2 = 0 \) as \( m \to \infty \), which imply \( \xi^1_m \to k \) and \( \zeta^1_m \to l \) as \( m \to \infty \). Thus by passing to the limit in (2.9), we obtain that

\[ c_\varepsilon \geq \frac{1}{4}(k + l)S^2_{1,\varepsilon} = J_\varepsilon(\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon) \]

which is (2.6).

After \textbf{Step 1-Step 3} we proved that the couple \((\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon)\) is indeed a least energy solutions of system \((S^2_\varepsilon)\) which achieves \( c_\varepsilon \) for the case of \( 0 \leq \beta < \min\{\mu_1, \mu_2\} \) or \( \beta > \max\{\mu_1, \mu_2\} \).
Secondly we consider the case of $\beta < 0$.

Suppose \( \{(u_m, v_m)\}_{m=1}^{\infty} \subset \mathcal{N}(\varepsilon, \Omega) \) is a minimizing sequence of $c_{\varepsilon}$ such that $J_{\varepsilon}(u_m, v_m) \to c_{\varepsilon}$ as $m \to \infty$.

We firstly claim that there is a constant $c_0 > 0$ which is independent of $m > 0$ such that

\[
\int_{\Omega} |u_m|^4 \, dx \geq c_0 \varepsilon^N, \quad \int_{\Omega} |v_m|^4 \, dx \geq c_0 \varepsilon^N.
\]

(2.12)
In fact, we recall that \( u_m \neq 0, v_m \neq 0 \) for each \( m \), as the argument in the case of \( \beta > 0 \), we still denote

\[
\xi_m = \left( \int_{\Omega} |u_m|^4 \, dx \right)^{1/2}, \quad \zeta_m = \left( \int_{\Omega} |v_m|^4 \, dx \right)^{1/2}.
\]

From the definition of \( S_{1,\varepsilon} \), we know that there is constant \( c > 0 \) such that \( S_{1,\varepsilon} > c\varepsilon^{N/2} \). Using the Sobolev and Hölder inequalities, we have

\[
S_{1,\varepsilon} \xi_m \leq \int_{\Omega} \left[ \varepsilon^2 |\nabla u_m|^2 + u_m^2 \right] \, dx = \int_{\Omega} \left[ \mu_1 u_m^4 + \beta u_m^2 v_m^2 \right] \, dx \leq \mu_1 \xi_m^2, \tag{2.13}
\]

\[
S_{1,\varepsilon} \zeta_m \leq \int_{\Omega} \left[ \varepsilon^2 |\nabla v_m|^2 + v_m^2 \right] \, dx = \int_{\Omega} \left[ \mu_2 v_m^4 + \beta u_m^2 v_m^2 \right] \, dx \leq \mu_2 \zeta_m^2. \tag{2.14}
\]

To the last inequalities in (2.13) and (2.14), we have used the fact that \( \beta < 0 \). From (2.13) and (2.14), it is easy to see that (2.12) holds for some \( c_0 > 0 \).
On the other hand, we prove that \( \{(u_m, v_m)\}_{m=1}^{\infty} \) is bounded in \( H^1(\Omega) \times H^1(\Omega) \). and without loss of generality, we assume that \( u_m \rightharpoonup u, v_m \rightharpoonup v \) weakly in \( H^1(\Omega) \) and \( u_m \to u, v_m \to v \) strongly in \( L^4(\Omega) \). Thus by (2.12), we know that \( u \not\equiv 0, v \not\equiv 0 \).

Furthermore

\[
\int_\Omega \left[ \varepsilon^2 |\nabla u|^2 + u^2 + \varepsilon^2 |\nabla v|^2 + v^2 \right] dx \\
\leq \liminf_{m \to \infty} \int_\Omega \left[ \varepsilon^2 |\nabla u_m|^2 + u_m^2 + \varepsilon^2 |\nabla v_m|^2 + v_m^2 \right] dx \\
= 4 \liminf_{m \to \infty} J_\varepsilon(u_m, v_m) = 4c_\varepsilon
\] (2.15)

In addition, we have

\[
\int_\Omega \left[ \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right] dx \\
\leq \liminf_{m \to \infty} \int_\Omega \left[ \mu_1 u_m^4 + \mu_2 v_m^4 + 2\beta u_m^2 v_m^2 \right] dx = 4 \liminf_{m \to \infty} J_\varepsilon(u_m, v_m) = 4c_\varepsilon
\] (2.16)
On the other hand, let $m \to \infty$ in (2.13) and (2.14), we obtain that

$$S_{1,\varepsilon} \left( \int_{\Omega} u^4 dx \right)^{1/2} \leq \int_{\Omega} [\mu_1 u^4 + \beta u^2 v^2] dx$$

and

$$S_{1,\varepsilon} \left( \int_{\Omega} v^4 dx \right)^{1/2} \leq \int_{\Omega} [\mu_2 v^4 + \beta u^2 v^2] dx,$$

which imply that

$$\int_{\Omega} \mu_1 u^4 dx > -\beta \int_{\Omega} u^2 v^2 dx \quad \text{and} \quad \int_{\Omega} \mu_2 v^4 dx > -\beta \int_{\Omega} u^2 v^2 dx.$$
From above two inequalities, we can easily find that the following matrix

\[
\begin{pmatrix}
\mu_1 \int_{\Omega} u^4 dx & \beta \int_{\Omega} u^2 v^2 dx \\
\beta \int_{\Omega} u^2 v^2 dx & \mu_2 \int_{\Omega} v^4 dx
\end{pmatrix}
\]

is positively definite.
Let $s_1, s_2$ is the unique solution of the following system

\[
\begin{align*}
\left\{
\begin{array}{ll}
(\mu_1 \int_\Omega u^4 \, dx) s_1 + (\beta \int_\Omega u^2 v^2 \, dx) s_2 &= \int_\Omega [\varepsilon^2 |\nabla u|^2 + u^2] \, dx, \\
(\beta \int_\Omega u^2 v^2 \, dx) s_1 + (\mu_2 \int_\Omega v^4 \, dx) s_2 &= \int_\Omega [\varepsilon^2 |\nabla v|^2 + v^2] \, dx
\end{array}
\right.
\end{align*}
\]

(2.17)

If $s_1 = 1, s_2 = 1$, then $(u, v) \in \mathcal{N}(\varepsilon, \Omega)$, and by (2.15) and (2.16) we have that $(u, v)$ is minimizer of $c_\varepsilon$, furthermore by a standard argument, we can prove that $(u, v)$ is a nonstandard solution of $(S^1_\varepsilon)$ and we denote it by $(u_\varepsilon, v_\varepsilon)$. On the other hand by the strong convergence of $u_m, v_m$ in $L^4(\Omega)$ and also from (2.12), we can easily get the following estimate

\[
\int_\Omega |u_\varepsilon|^4 \, dx \geq c_0 \varepsilon^n, \quad \int_\Omega |v_\varepsilon|^4 \, dx \geq c_0 \varepsilon^n,
\]

(2.18)

In the end, we prove that the unique solutions $(s_1, s_2)$ of the linear system (2.17) is indeed $(1, 1)$ and thus we complete the proof of Theorem 2.1.
3. Asymptotic behavior of the least energy solutions

In this section, we consider the asymptotic behavior of the least energy solution \((u_\varepsilon, v_\varepsilon)\) as \(\varepsilon\) goes to zero. Let \(P_\varepsilon\) be the maximum point of \(u_\varepsilon\) in \(\overline{\Omega}\), \(Q_\varepsilon\) be the maximum point of \(v_\varepsilon\) in \(\overline{\Omega}\).

Suppose \(U(x)\) is the unique least energy solution of the following problem

\[
\begin{cases}
-\Delta u(x) + u(x) = u(x)^3, u(x) > 0, & x \in \mathbb{R}^N \\
\max_{x \in \mathbb{R}^N} u(x), u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\] (3.1)
It is well known that $U(x)$ is radially symmetric and strictly decreasing as $|x| \to \infty$. Furthermore as $|x| \to \infty$

$$|D^\alpha U(x)| \sim |x|^{-\frac{N-1}{2}} e^{-|x|} \text{ for } |\alpha| \leq 2.$$ 

The energy functional of $(3.1)$ is as follows

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} [|
abla u|^2 + u^2] \, dx - \frac{1}{4} \int_{\mathbb{R}^N} u^4 \, dx, \text{ for every } u \in H^1(\mathbb{R}^N). \quad (3.2)$$

It is also well known that

$$c^* := I(U(x)) = \inf_{v \in H^1(\mathbb{R}^N), v \neq 0} \sup_{t > 0} I(tv). \quad (3.3)$$
Let $\omega_\varepsilon, T_{1,\varepsilon}$ be defined as in Section 2, we assume that $x_\varepsilon$ is the global maximum point of $\omega_\varepsilon$ in $\bar{\Omega}$, then as proved in Ni and Takagi, for sufficient small $\varepsilon$, $x_\varepsilon \in \partial \Omega$ and

(i) $H(x_\varepsilon) \to H(\bar{x}) = \max_{x \in \partial \Omega} H(x)$, where $H(x)$ denotes mean curvature of the boundary.

(ii) $T_{1,\varepsilon} = \varepsilon^N \left\{ \frac{c^*}{2} - \gamma \varepsilon H(\bar{x}) + o(\varepsilon) \right\}$, \hspace{1cm} (3.4)

where $\gamma$ is constant depending only on $N$.

Furthermore let $\tilde{\omega}_\varepsilon(y) = \omega_\varepsilon(x_\varepsilon + \varepsilon y)$, then as $\varepsilon \to 0$, $\tilde{\omega}_\varepsilon(y) \to U(y)$ strongly in $H^1(\mathbb{R}^N)$. 
Firstly, we consider the case of $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, we have the following theorem.

**Theorem 3.1** For $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, suppose $(u_\varepsilon, v_\varepsilon)$ is a least energy solution of $(S_\varepsilon^2)$ and let $P_\varepsilon$ and $Q_\varepsilon$ be the maximum points of $u_\varepsilon$ and $v_\varepsilon$ in $\tilde{\Omega}$ respectively. Then as $\varepsilon$ small enough, both $P_\varepsilon$ and $Q_\varepsilon$ locate on the boundary of $\Omega$.

Moreover,

(i) as $\varepsilon \to 0$, $\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to 0$ and for $N = 2$ and $N = 3$,

$$H(P_\varepsilon) \to \max_{P \in \partial \Omega} H(P), \quad H(Q_\varepsilon) \to \max_{P \in \partial \Omega} H(P). \quad (3.5)$$

Furthermore, $u_\varepsilon, v_\varepsilon \to 0$ in $C^1_{loc}(\tilde{\Omega} \setminus \{P_\varepsilon, Q_\varepsilon\})$ and let

$$U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) = v_\varepsilon(Q_\varepsilon + \varepsilon y)$$

then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ which is a least energy solution of the following problem defined in $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$.

$$\begin{cases} 
- \Delta U_0 + U_0 = \mu_1 U_0^3 + \beta U_0 V_0^2, \\
- \Delta V_0 + V_0 = \mu_2 V_0^3 + \beta U_0^2 V_0, \\
U_0, V_0 > 0, U_0, V_0 \to 0 \quad \text{as} \ |x| \to \infty, \\
\frac{\partial U_0}{\partial n} = 0, \frac{\partial V_0}{\partial n} = 0 \quad \text{on} \ \partial \mathbb{R}^N_+, 
\end{cases} \quad (3.6)$$
Remark 3.2 As proved in Theorem 2.1, For $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, the couple $(\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon)$ is a least energy solutions of system $(S^2_\varepsilon)$ which achieves $c_\varepsilon$, where $k = \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}$ and $l = \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}$. Thus $c_\varepsilon = (k + l)T_{1,\varepsilon}$ and since the asymptotic behavior of $\omega_\varepsilon$ as $\varepsilon$ goes to zero is well known as described above, so the conclusion of Theorem 3.1 as to $(k\omega_\varepsilon, l\omega_\varepsilon)$ is quite clear. However, since we do not know wether the least energy solution of $(S^2_\varepsilon)$ is unique or not, so here we are concerned about the asymptotic behavior of the general least energy solution of system $(S^2_\varepsilon)$ which might not be $(\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon)$.
Sketch of the proof of Theorem 3.1   By Remark 3.2, for $0 \leq \beta \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} c_{\varepsilon} = \lim_{\varepsilon \to 0} \varepsilon^{-N} \left( \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1 \mu_2 - \beta^2} \right) T_{1,\varepsilon} = \left( \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1 \mu_2 - \beta^2} \right) \frac{c^*}{2}$$

$$> \lim_{\varepsilon \to 0} \max \varepsilon^{-N} \left\{ \frac{1}{\mu_1}, \frac{1}{\mu_2} \right\} T_{1,\varepsilon} = \max \left\{ \frac{1}{\mu_1}, \frac{1}{\mu_2} \right\} \frac{c^*}{2}. \quad (3.7)$$

Using (3.7), we indeed can prove that

$$\frac{|P_{\varepsilon} - Q_{\varepsilon}|}{\varepsilon} \to 0, \text{ as } \varepsilon \to 0. \quad (3.8)$$
Secondly, let \( \tilde{U}_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y) \), \( \tilde{V}_\varepsilon(y) = v_\varepsilon(P_\varepsilon + \varepsilon y) \), and for any open set \( \Gamma \), we denote

\[
J_\Gamma(u, v) = \frac{1}{2} \int_\Gamma (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2) \, dx - \frac{1}{4} \int_\Gamma [\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2] \, dx.
\]

For any set \( D \subset \mathbb{R}^N \), we set \( D_\varepsilon = \{ y \in \mathbb{R}^N : x = P_\varepsilon + \varepsilon y \in D \} \), we define \( U^1_\varepsilon \) and \( V^1_\varepsilon \) on \( \mathbb{R}^N_+ \cap H_\varepsilon \) as \( U^1_\varepsilon(y', y_N) = \tilde{U}_\varepsilon(y', G_\varepsilon(\varepsilon y')/\varepsilon) \), \( V^1_\varepsilon(y', y_N) = \tilde{V}_\varepsilon(y', G_\varepsilon(\varepsilon y')/\varepsilon) \) if \( G_\varepsilon(\varepsilon y') > 0 \), \( 0 < y_N \leq G_\varepsilon(\varepsilon y')/\varepsilon \) and \( U^1_\varepsilon(y', y_N) = \tilde{U}_\varepsilon(y', y_N) \), \( V^1_\varepsilon(y', y_N) = \tilde{V}_\varepsilon(y', y_N) \) in other cases.

By the definition of \( (u_\varepsilon, v_\varepsilon) \), it is easy to check that

\[
c_\varepsilon = J_\varepsilon(u_\varepsilon, v_\varepsilon) = \max_{t>0, s>0} J_\varepsilon(tu_\varepsilon, sv_\varepsilon) = \varepsilon^N \max_{t>0, s>0} J_{\Omega_\varepsilon}(t\tilde{U}_\varepsilon, s\tilde{V}_\varepsilon).
\]

On the other hand we have

\[
J_{\Omega_\varepsilon}(t\tilde{U}_\varepsilon, s\tilde{V}_\varepsilon) = J_{\mathbb{R}^N_+ \cap H_\varepsilon}(tU^1_\varepsilon, sV^1_\varepsilon) + J_{(\Omega_\varepsilon \cap H_\varepsilon) \setminus \mathbb{R}^N_+}(t\tilde{U}_\varepsilon, s\tilde{V}_\varepsilon) - J_{\mathbb{R}^N_+ \cap H_\varepsilon \setminus \Omega_\varepsilon}(tU^1_\varepsilon, sV^1_\varepsilon)
\]
Finally we can prove that
\[
c_\varepsilon \geq \varepsilon^N \left( (k + l) \frac{c_*}{2} - \varepsilon \gamma H(\bar{P}) + o(\varepsilon) \right), \tag{3.9}
\]
where \( \gamma \) is a positive constant. On the other hand, for any \( P \in \partial \Omega \), it is easy to see that there exists \( t^{\varepsilon}, s^{\varepsilon} \) such that \((t^{\varepsilon} U_0(\frac{P-x}{\varepsilon}), s^{\varepsilon} V_0(\frac{P-x}{\varepsilon})) \in \mathcal{N}(\Omega, \varepsilon)\) and \( t^{\varepsilon} \to 1, s^{\varepsilon} \to 1 \) as \( \varepsilon \to 0 \). By a direct computation, we can easily obtain that
\[
c_\varepsilon \leq \max_{t>0, s>0} J_\varepsilon(t U_0(\frac{P-x}{\varepsilon}), s V_0(\frac{P-x}{\varepsilon}))
= J_\varepsilon(t^{\varepsilon} U_0(\frac{P-x}{\varepsilon}), s^{\varepsilon} V_0(\frac{P-x}{\varepsilon}))
\leq \varepsilon^N \left( (k + l) \frac{c_*}{2} - \varepsilon \gamma H(P) + o(\varepsilon) \right). \tag{3.10}
\]
Combining (3.9) and (3.10) we conclude that \( H(\bar{P}) \geq H(P) \) for any \( P \in \partial \Omega \). which implies (3.5). This completes the proof of the Theorem.
Now we come to consider the case when $\beta < 0$, we have the following theorem

**Theorem 3.3** For $\beta < 0$, suppose $(u_\varepsilon, v_\varepsilon)$ is a least energy solution of $(S_\varepsilon^2)$ such that

$$
\int_\Omega |u_\varepsilon|^4 dx \geq c_0 \varepsilon^N, \int_\Omega |v_\varepsilon|^4 dx \geq c_0 \varepsilon^N,
$$

(3.11)

where $c_0$ is constant independent of $\varepsilon$. Let $P_\varepsilon$ and $Q_\varepsilon$ are the maximum points of $u_\varepsilon$ and $v_\varepsilon$ in $\bar{\Omega}$ respectively. Then as $\varepsilon$ small enough, both $P_\varepsilon$ and $Q_\varepsilon$ locate on the boundary of $\Omega$. Moreover

(i) as $\varepsilon \to 0$, \( \frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to \infty \) and for $N = 2$ and $N = 3$,

$$
\lim_{\varepsilon \to 0} H(P_\varepsilon) = \max_{P \in \partial \Omega} H(P), \quad \lim_{\varepsilon \to 0} H(Q_\varepsilon) = \max_{P \in \partial \Omega} H(P).
$$

(3.12)

Furthermore, $u_\varepsilon, v_\varepsilon \to 0$ in $C^1_{\text{loc}}(\bar{\Omega} \setminus \{P_\varepsilon, Q_\varepsilon\})$ and let

$$
U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) = v_\varepsilon(Q_\varepsilon + \varepsilon y)
$$

then as $\varepsilon \to 0$,

$$
U_\varepsilon \to \omega_1(y), \quad V_\varepsilon \to \omega_2(y)
$$

where $\omega_i (i = 1, 2)$ is the unique least energy solution of

$$
\begin{cases}
- \Delta \omega_i + \omega_i = \mu_i \omega_i^3, & x \in \mathbb{R}^N_+,
\omega_i > 0 \text{ in } \mathbb{R}^N_+, \omega_i \to 0, \text{ as } |x| \to \infty,
\frac{\partial \omega_i}{\partial n} = 0, & x \in \partial \mathbb{R}^N_+.
\end{cases}
$$

(3.13)
Sketch of the proof of Theorem 3.3: Let $U(x)$ be the unique solution of (3.1), then

$$\left(\frac{1}{\mu_i}\right)^{1/2} U(x)|_{\mathbb{R}^N_+} (i = 1, 2)$$

is the unique solution of (3.13). Let $P, Q \in \partial \Omega$ and $P \neq Q$, we define

$$U_{P, \varepsilon}(x) := \left(\frac{1}{\mu_1}\right)^{1/2} U\left(\frac{P - x}{\varepsilon}\right)|_{\mathbb{R}^N_+}, U_{Q, \varepsilon}(x) := \left(\frac{1}{\mu_2}\right)^{1/2} U\left(\frac{Q - x}{\varepsilon}\right)|_{\mathbb{R}^N_+}.$$

Let $t_\varepsilon, s_\varepsilon$ are such that

$$J_\varepsilon(t_\varepsilon U_{P, \varepsilon}(x), s_\varepsilon U_{Q, \varepsilon}(x)) = \max_{t>0, s>0} J_\varepsilon(tU_{P, \varepsilon}(x), sU_{Q, \varepsilon}(x)).$$

Then we can prove that $(t_\varepsilon U_{P, \varepsilon}(x), s_\varepsilon U_{Q, \varepsilon}(x)) \in \mathcal{N}(\varepsilon, \Omega)$, furthermore as $\varepsilon \to 0$ and $t_\varepsilon \to 1$ and $s_\varepsilon \to 1$.

By a direct computation, we have

$$c_\varepsilon \leq J_\varepsilon(t_\varepsilon U_{P, \varepsilon}(x), s_\varepsilon U_{Q, \varepsilon}(x)) = \varepsilon^N \left(\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^{1/2} c^*_2 - \gamma_1 H(P)\varepsilon - \gamma_2 H(Q)\varepsilon + o(\varepsilon)\right),$$

(3.14)

where $\gamma_i = \eta \int_0^\infty \left(\frac{1}{2}[(U')^2 + U^2] - \frac{1}{4} \mu_i U^4\right) dr (i = 1, 2)$ and $\eta$ is defined as above.
On the other hand, we can show that as $\varepsilon \to 0$ small enough, both $\frac{\text{dist}\{P_\varepsilon, \partial \Omega\}}{\varepsilon}$ and $\frac{\text{dist}\{Q_\varepsilon, \partial \Omega\}}{\varepsilon}$ remain bounded. Moreover as $\varepsilon \to 0$,

$$\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to \infty.$$ (3.15)

Let $U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, $\tilde{V}_\varepsilon(y) = v_\varepsilon(P_\varepsilon + \varepsilon y)$. Since (3.15), with a rotation we may assume that $(U_\varepsilon(y), \tilde{V}_\varepsilon(y)) \to (U_0, \tilde{V}_0)$ with $\tilde{V}_0 \equiv 0$ and thus $U_0$ satisfies

$$\begin{cases}
-\Delta U_0 + U_0 = \mu_1 U_0^3 & x \in \mathbb{R}_0^N, \\
U_0 > 0, x \in \mathbb{R}_0^N, U_0 \to 0 & \text{as } |x| \to \infty, \\
\frac{\partial U_0}{\partial n} = 0, \frac{\partial V_0}{\partial n} = 0 & x \in \partial \mathbb{R}_0^N,
\end{cases}$$

where $\delta_0 = \lim_{\varepsilon \to 0} \text{dist}\{P_\varepsilon, \partial \Omega\} \geq 0$. 


By a translation and using the property of $\omega_1$, we indeed can obtain that $\delta_0 = 0$ and $U_0 = \omega_1$. Similarly we also can prove that as $\varepsilon \to 0$, $V_\varepsilon(y) = v_\varepsilon(Q_\varepsilon + \varepsilon y) \to \omega_2$, where $\omega_i (i = 1, 2)$ is the unique solution of (3.13). Moreover using again the property of $\omega_i$, it is easy to see that both $P_\varepsilon$ and $Q_\varepsilon$ locate on the boundary of $\Omega$, otherwise we can get a degenerate maximum point of $\omega_i$. We assume that as $\varepsilon \to 0$, $P_\varepsilon \to \bar{P} \in \partial \Omega$ and $Q_\varepsilon \to \bar{Q} \in \partial \Omega$. 
Since $\beta < 0$ we have

$$c_\varepsilon = J_\varepsilon(u_\varepsilon, v_\varepsilon)$$

$$= \int_\Omega \frac{1}{2}[\varepsilon^2|\nabla u_\varepsilon^2 + u_\varepsilon^2 - \frac{1}{2}\mu_1 u_\varepsilon^4]dx + \int_\Omega \frac{1}{2}[\varepsilon^2|\nabla v_\varepsilon^2 + v_\varepsilon^2 - \frac{1}{2}\mu_2 v_\varepsilon^4]dx - \frac{1}{2} \int_\Omega \beta u_\varepsilon v_\varepsilon^2 dx$$

$$\geq \int_\Omega \frac{1}{2}[\varepsilon^2|\nabla u_\varepsilon^2 + u_\varepsilon^2 - \frac{1}{2}\mu_1 u_\varepsilon^4]dx + \int_\Omega \frac{1}{2}[\varepsilon^2|\nabla v_\varepsilon^2 + v_\varepsilon^2 - \frac{1}{2}\mu_2 v_\varepsilon^4]dx$$

$$=: I_1 + I_2,$$

with the same arguments as in Del Pino and Felmer[Indiana Univ.Math.J., 48(1999),63-97], it is easy to see that

$$I_1 = \frac{1}{2} \int_\Omega [\varepsilon^2|\nabla u_\varepsilon^2 + u_\varepsilon^2]dx - \frac{1}{4} \int_\Omega \mu_1 u_\varepsilon^4 dx$$

$$= \varepsilon^N \left( \frac{1}{\mu_1} \frac{c^*_1}{2} + \varepsilon \gamma_1 H(\bar{P}) + O(e^{-\delta_0/\varepsilon}) \right),$$

Similarly we also have

$$I_2 = \frac{1}{2} \int_\Omega [\varepsilon^2|\nabla v_\varepsilon^2 + v_\varepsilon^2]dx - \frac{1}{4} \int_\Omega \mu_2 v_\varepsilon^4 dx$$

$$= \varepsilon^N \left( \frac{1}{\mu_2} \frac{c^*_1}{2} + \varepsilon \gamma_2 H(\bar{Q}) + O(e^{-\delta_0/\varepsilon}) \right),$$

where $\gamma_i (i = 1, 2)$ is defined as in (3.14).
Combining with (3.14), we obtained that for any $P, Q \in \partial \Omega$ and $P \neq Q$,

$$\gamma_1 H(P) + \gamma_2 H(Q) \leq \gamma_1 H(\bar{P}) + \gamma_2 H(\bar{Q})$$

which implies $H(\bar{P}) = \max_{P \in \partial \Omega} H(P)$ and $H(\bar{Q}) = \max_{P \in \partial \Omega} H(P)$. This completes the proof of Theorem 3.3.
Remark 3.4 It is easy to calculate that for \(-\sqrt{\mu_1\mu_2} < \beta < 0\), \((\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon)\) is a solution of system \((S^2_\varepsilon)\), where \(k, l, \omega_\varepsilon\) are defined as in Theorem 2.1 and

\[
J_\varepsilon(\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon) = \frac{1}{4} \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1\mu_2 - \beta^2} S^2_{1,\varepsilon}
\]

\[
\Rightarrow \frac{\mu_1 + \mu_2 - 2\beta c_*}{\mu_1\mu_2 - \beta^2} \frac{2}{2}
\]

\[
> \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \frac{c_*}{2}
\]

Thus, by (3.14), it is easy to see that as \(\varepsilon\) small enough, \((\sqrt{k}\omega_\varepsilon, \sqrt{l}\omega_\varepsilon)\) is not the least energy solution of system \((S^2_\varepsilon)\).
4. Some remarks on Dirichlet problems

In this section, we give some remarks to \((S^1_\varepsilon)\)

The energy functional corresponding to \((S^1_\varepsilon)\) is as follows:

\[
J^1_\varepsilon(u, v) := \frac{1}{2} \int_\Omega \left[ \varepsilon^2 |\nabla u|^2 + u^2 + \varepsilon^2 |\nabla v|^2 + v^2 \right] dx - \frac{1}{4} \int_\Omega \left( \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right) dx, \quad (4.1)
\]

for every \((u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)\).
As in Section 1, we also consider the set
\[ \mathcal{N}^1(\varepsilon, \Omega) = \left\{ (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega), \begin{array}{l}
u \geq 0, v \geq 0
\end{array} : \begin{array}{l}
\int_{\Omega} [\varepsilon^2 |\nabla u|^2 + u^2] \, dx = \int_{\Omega} [\mu_1 u^4 + \beta u^2 v^2] \, dx
\end{array} \right\} \]
and let
\[ c_1^\varepsilon = \inf_{(u, v) \in \mathcal{N}^1(\varepsilon, \Omega)} J_1^\varepsilon(u, v). \]
Using the same arguments in proving Theorem 2.1, we indeed can prove the following results

**Theorem 4.1** For any $\varepsilon > 0$, if $-\infty < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, there exists a least energy solution $(u_\varepsilon, v_\varepsilon)$ to system $(S_\varepsilon^1)$ which achieves $c_\varepsilon^1$. Furthermore, there exists two positive constant $c_1, c_2$ such that

$$c_1 \varepsilon^N \leq \int_\Omega |u_\varepsilon|^4 \, dx \leq c_2 \varepsilon^N, \quad c_1 \varepsilon^N \leq \int_\Omega |v_\varepsilon|^4 \, dx \leq c_2 \varepsilon^N.$$  \hspace{1cm} (4.2)

Moreover, for $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, $(S_\varepsilon^1)$ has no solution. In fact, suppose $\omega_\varepsilon^1$ is the least energy solution of the scalar case $-\varepsilon^2 \Delta u + u = u^3, u > 0, x \in \Omega$ under Dirichlet boundary conditions. Then for $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, $(\sqrt{k}\omega_\varepsilon^1, \sqrt{l}\omega_\varepsilon^1)$ is the least energy solutions of system $(S_\varepsilon^1)$ which achieves $c_\varepsilon^1$, where $k = \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}$ and $l = \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}$. 
Remark 4.2 To prove the main results in Lin and Wei[Ann.Inst.H.Poincaré Anal. Non Linéaire 22(4)(2005)403-439.] in the case of $\beta > 0$, technically the authors need to confine $0 < \beta < \beta_0 < \sqrt{\mu_1 \mu_2}$. However, In our results Theorem 4.1, we indeed considered the existence and nonexistence of least energy solutions of $(S_1^1)$ in all cases of $\beta \in \mathbb{R}$ and thus our result is a essential extension of Lin and Wei.

In fact, from Theorem 4.1, for $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$,

$$c_1^1 = J_1^1(\sqrt{k}\omega_1^1, \sqrt{l}\omega_1^1) = (k + l)T_\varepsilon,$$

where

$$T_\varepsilon = \inf_{u \in \mathcal{M}_\varepsilon^1} \left\{ \frac{1}{2} \int_{\Omega} (\varepsilon^2 |\nabla u|^2 + u^2) dx - \frac{1}{4} \int_{\Omega} \mu u^4 dx \right\},$$

where $\mathcal{M}_\varepsilon^1 = \left\{ u \in H_0^1(\Omega), u \neq 0 : \int_{\Omega} (\varepsilon^2 |\nabla u|^2 + u^2) dx = \int_{\Omega} \mu u^4 dx \right\}$. Then we have that $\omega_1^1$ is a minimizer of $T_\varepsilon$. 
By Del Pino and Felmer[Indiana Univ.Math.J., 48(1999),63-97], we indeed have

\[ T_\varepsilon = \varepsilon^N \left( c^* + e^{-2d_\ast + o(1)} \right), \quad (4.3) \]

where \( d_\ast = \max_{x \in \Omega} \text{dist}\{x, \partial \Omega\} \), \( c^* \) is defined as in (3.3) in Section 3. and \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

Combining (4.3), using the same arguments to prove the asymptotic behavior of the least energy solutions in Lin and Wei[Ann.Inst.H.Poincaré Anal. Non Linéaire 22(4)(2005)403-439.] in the case of \( 0 < \beta < \beta_0 < \sqrt{\mu_1 \mu_2} \), we can indeed extend the main results of Lin and Wei to the case of \( 0 < \beta < \min\{\mu_1, \mu_2\} \) or \( \beta > \max\{\mu_1, \mu_2\} \).
Namely, we have the following results

**Theorem 4.3** Suppose \((u_\varepsilon, v_\varepsilon)\) is a least energy solution of system \((S^1_\varepsilon)\) in the case of \(0 < \beta < \min\{\mu_1, \mu_2\}\) or \(\beta > \max\{\mu_1, \mu_2\}\). \(P_\varepsilon, Q_\varepsilon\) are the maximum points of \(u_\varepsilon, v_\varepsilon\) respectively. Then as \(\varepsilon \to 0\), \(\left| \frac{P_\varepsilon - Q_\varepsilon}{\varepsilon} \right| \to 0\) and

\[
\text{dist}\{P_\varepsilon, \partial\Omega\} \to \max_{P \in \Omega} \text{dist}\{P, \partial\Omega\}, \quad \text{dist}\{P_\varepsilon, \partial\Omega\} \to \max_{P \in \Omega} \text{dist}\{P, \partial\Omega\}.
\]

Moreover, \(u_\varepsilon, v_\varepsilon \to 0\) in \(C^{1}_{\text{loc}}(\bar{\Omega} \setminus \{P_\varepsilon, Q_\varepsilon\})\) and let

\[
U_\varepsilon(x) = u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(x) = v_\varepsilon(Q_\varepsilon + \varepsilon y)
\]

then as \(\varepsilon \to 0\), \((U_\varepsilon, V_\varepsilon) \to (U_0, V_0)\) such that \(U_0, V_0 > 0\) in \(\mathbb{R}^N\) and \((U_0, V_0)\) is a least energy solution of the following problem in whole space \(\mathbb{R}^N\)

\[
\begin{cases}
- \Delta U_0 + U_0 = \mu_1 U_0^3 + \beta U_0 V_0^2, \\
- \Delta V_0 + V_0 = \mu_2 V_0^3 + \beta U_0^2 V_0, \\
U_0(0) = \max_{y \in \mathbb{R}^N} U_0(y), \quad V_0(0) = \max_{y \in \mathbb{R}^N} V_0(y), \\
U_0, V_0 \to 0.
\end{cases}
\]

\[(4.4)\]
Thanks for your attention
Introduction
Existence of least . . .
Asymptotic behavior of . . .
Some remarks on . . .