Introduction to Knot theory

Minhoon Kim

POSTECH

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What is the Knot theory?

Knot : circle in $\mathbb{R}^3$ or $S^3$  e.g.  

We want to say that they are same!
When two knots are same?

\( K_0, K_1 : \text{Knots i.e. } K_i = f_i(S^1), \text{ where } f_i : S^1 \rightarrow S^3 \text{ is 1-1 (i=0,1)} \)

We say \( K_0 = K_1 \) if there is \( f : S^1 \times [0,1] \rightarrow S^3 \) satisfying (1),(2)

1. \( f_0(x) = f(x,0) \) and \( f_1(x) = f(x,1) \) for all \( x \) in \( S^1 \)

2. \( f_t : S^1 \rightarrow S^3 \) : is 1-1 for all \( t \) in \( [0,1] \), where \( f_t(x) = f(x,t) \)
Reidemeister moves (1, 2, 3)

Reidemeister move 1

Reidemeister move 2

Reidemeister move 3

Theorem (Reidemeister) 3 moves are enough!
Example

\[\text{i.e.} \quad =\]
Seifert Surface

Definition $F$ in $S^3$: Seifert surface if $F$ is (orientable) surface, $\partial F = K$

Examples

Figure eight knot

Trefoil knot
Classification of closed surfaces

\[ F^2 : \text{compact, orientable, connected surface} \ (\partial F = 0) \]
\[ \Rightarrow F^2 = S_g \text{ for some } g \geq 0 \]

\[ g(F) := \text{genus of } F \]

\[ X(F) = 2 - 2g(F), \text{ where } X(F) : \text{Euler characteristic of } F \]
Genus of Knot

F: Seifert surface of K, K: knot

\[ g(F) := \frac{(1 - X(F))}{2} \iff X(F) = 1 - 2g(F) \]

**Note**: \( \partial F \) is not empty!

\[ g(K) := \min \{ g(F) | F \text{ in } S^3, \partial F = K \} \]

\( g(K) \): genus of a knot \( K \)

\( g(K) = 0 \iff K \) is trivial knot
Seifert form

F : Seifert surface of K with orientation (normal direction)

Define $\Theta : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$ (Seifert form) as follows

$\Theta(x,y) = \text{lk}(x,y^+)$, $y^+$: parallel of $y$ along (+) normal direction.

$\text{lk}(x,y)$ is linking number of $x$ and $y$ given by

\[
\begin{align*}
\text{lk}(x,y) &= 1 \\
\text{lk}(x,y) &= -1
\end{align*}
\]
Calculating Seifert form (Example)

1. Choose generator of \( H_1(F) = \mathbb{Z}^4 \), \( \{x_1, x_2, x_3, x_4\} \)

2. Fix orientation of \( F \) and calculate!

\[
\begin{pmatrix}
-4 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Matrix representation of \( \Theta \) with basis \( \{x_1, x_2, x_3, x_4\} \)
Slice knot, Alexander polynomial

\[ K : \text{slice if } \exists \text{ } D(\text{disk}) \text{ in } B^4 \text{ with } \partial D = K, \text{ } \partial B^4 = S^3 \]

Let \[ \Delta_K(t) = \det(t^{1/2} \Theta - t^{-1/2} \Theta^T) \]

\[ \Delta_K(t) := \text{Alexander polynomial of } K \]

\[ K: \text{slice } \Delta_K(t) = f(t)f(t^{-1}) \text{ for some polynomial } f \]
Signature of Knot

Signature of a matrix is defined by
# of positive eigenvalues - # of negative eigenvalues

Signature of a knot: Signature of $\Theta + \Theta^T$

Slice knot: zero signature
Slice genus and knot invariants

\[ g_s(K) = \min \{ g(F) | F \text{ in } B^4, \partial F = K, \partial B^4 = S^3 \} \]

\[ g_s(K) := \text{sieve genus.} \]

\[ K: \text{sieve} \iff g_s(K) = 0 \]

1. Ozsváth-szabó \( \tau \) -invariant (from Knot Floer homology)

2. Rasmussen \( s \) -invariant (from Khovanov homology)

\[ \tau (K) \leq g_s(K), s(K) \leq 2g_s(K) \text{ (bound for } g_s(K)) \]
Interplay with 4-manifold theory

**Conjecture** (Smooth Poincare Conjecture in 4 dimension)

\[ M^4 \text{ with } \pi_*(M^4) = \pi_*(S^4) \Rightarrow M^4 = S^4(\text{diffeomorphic})? \]

By Freedman’s work, this conjecture \(\Leftrightarrow\)

Are there \( M^4 \) with \( M^4 = S^4(\text{homeo.}) \) \& \( M^4 \neq S^4(\text{diffeo.})? \)

**Theorem** (Freedman, Gompf, Morrison, Walker)

If ‘some’ knot \( K \) satisfies \( s(K) \neq 0 \), then SPC4 is false.

4 manifold problem \(\Rightarrow\) Slice knot problem
Addendum

Question: Homeomorphic but not Diffeomorphic?

Answer: Possible!

J. Milnor’s 1st example: 28 7-spheres using Pontryagin class

Kervaire, Milnor: Differentiable structures on $S^n (n \neq 4)$

Donaldson, Freedman: infinitely many $\mathbb{R}^4$

Floer: uncountably many $\mathbb{R}^4$
Thank You !!!