HYPERSURFACES CUTTING OUT A PROJECTIVE VARIETY

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Abstract. Let \( X \) be a nondegenerate projective variety of degree \( d \) and codimension \( e \) in a projective space \( \mathbb{P}^N \) defined over an algebraically closed field. We study the following two problems: Is the length of the intersection of \( X \) and a line \( L \) in \( \mathbb{P}^N \) at most \( d - e + 1 \) if \( L \not\subseteq X \)? Is the scheme-theoretic intersection of all hypersurfaces of degree at most \( d - e + 1 \) containing \( X \) equal to \( X \)? To study the second problem, we look at the locus of points from which \( X \) is projected non-birationally.

§0 Introduction.

Let \( X \subseteq \mathbb{P}^N \) \((N = n + e)\) be a projective variety of dimension \( n \), degree \( d \), and codimension \( e \) over an algebraically closed field \( k \), and we always assume that \( X \) is nondegenerate in \( \mathbb{P}^N \), i.e., \( X \) is not contained in any hyperplane in \( \mathbb{P}^N \). Let \( m \) be a positive integer. We say that \( X \) is \( m \)-regular if its ideal sheaf \( \mathcal{I}_{X/\mathbb{P}^N} \) satisfies \( H^i(\mathbb{P}^N, \mathcal{I}_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(m-i)) = 0 \) for all \( i > 0 \) (see [16], Lecture 14). By \( E_m(X) \), we denote the scheme-theoretic intersection of all hypersurfaces in \( \mathbb{P}^N \), containing \( X \), of degree at most \( m \). We consider the following conditions on \( X \):

\((A_m)\) \( l(X \cap L) := \text{length}(\mathcal{O}_{X \cap L}) \leq m \) for each line \( L \) in \( \mathbb{P}^N \) with \( L \not\subseteq X \).
\((B_m)\) \( X = E_m(X) \)
\((C_m)\) \( X \) is \( m \)-regular.

The purpose here is to study \((A_m)\) and \((B_m)\) for \( m = d - e + 1 \). To study \((B_m)\), we also look at the structure of the locus, denoted by \( B(X) \) and \( C(X) \) (see (0.1)), of points from which \( X \) is projected non-birationally.

First we briefly look at the three conditions. It is well-known that \((C_m)\) implies \((B_m)\) since the \( m \)-regularity of \( X \) implies that the homogeneous ideal of \( X \) is generated in degree \( \leq m \) ([16], Lec.14.Prop.). Also it is clear that \((B_m)\) implies \((A_m)\). On the other hand, the condition \((C_m)\) for \( m = d - e + 1 \) is the famous conjecture on Castelnuovo-Mumford regularity, and sometimes the implication \((A_m) \Rightarrow (C_m)\) for \( m \) close to \( d - e \), with few trivial exceptions, is also included in the conjecture (see [6]; [9], §4). The conjecture is true for \( n = 1 \) ([9], Theorems 1.1 and 3.1), and the first part of it is true

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for a smooth surface \(X\) of \(\text{char}(k) = 0\) ([15]). For \(n \geq 3\), there are nice approaches to the conjecture but it is still open (see [5] and [13] for information). As evidences of the regularity conjecture, it is natural to expect \((A_m)\) and \((B_m)\) for \(m = d - e + 1\).

One of key ideas to study \((A_m)\) and \((B_m)\) for \(m = d - e + 1\) is based on [17], Theorem 1, where it was shown that \(X = E_d(X)\) as set, and \(X = E_d(X)\) as scheme if \(X\) is smooth. To obtain these results, Mumford considered the image \(\pi(X)\) from which \(X_1\), where it was shown that \(A\) the conjecture but it is still open (see [5] and [13] for information). As evidences of the Theorem 2.

As an application of Theorem 2, we have another proof of Theorem 1 in \(\text{char}(k) = 0\), [2], [3], [14], and [18]. When \(X\) is smooth, we have a sharp bound including another invariant of \(X\) ([19]).

Next we deal with \((B_{d-e+1})\). To state our result, we introduce some notation:

\[
\begin{align*}
B(X) & : = \{v \in \mathbb{P}^N \setminus X \mid l(X \cap \langle v, x \rangle) \geq 2 \text{ for general } x \in X\}, \\
C(X) & : = \{u \in X \setminus \text{Sing } X \mid l(X \cap \langle u, x \rangle) \geq 3 \text{ for general } x \in X\},
\end{align*}
\]

where \(\text{Sing } X\) denotes the singular locus of \(X\). In other words, these are loci of points from which \(X\) is projected not birationally: in the former, points off \(X\), and in the latter, points on \(X \setminus \text{Sing } X\). When \(e = 1\), it is clear that \((B_d)\) holds and \(B(X) = \mathbb{P}^N \setminus X\) and \(C(X) = X \setminus \text{Sing } X\) if \(d \geq 3\). Thus we consider \((B_{d-e+1})\) for \(e \geq 2\).

**Theorem 1.** (\(\text{char}(k) \geq 0\)) For a standard secant line \(L\) to \(X\), we have \(l(X \cap L) \leq d - e + 1\).

For a non-standard secant line, the same result is expected. But at this stage, no proof about it would be known for \(\text{char}(k) \geq 0\). The result here is a generalization of results in \(\text{char}(k) = 0\), [2], [3], [14], and [18]. When \(X\) is smooth, we have a sharp bound including another invariant of \(X\) ([19]).

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As an application of Theorem 2, we have another proof of Theorem 1 in \(\text{char}(k) = 0\), since a standard secant line \(L\) to \(X\) is not contained in \(B(X)\) (see Remark 3.6).

As another application, we will prove \((B_{d-e+1})\) for a special case:

**Theorem 2.** (\(\text{char}(k) = 0\)) Assume \(e \geq 2\).

1. (Calabri and Ciliberto ([4], Corollary 2) and Sommese, Verschelde, and Wampler ([22])) As sets, \(X \subseteq E_{d-e+1}(X) \subseteq X \cup B(X)\).
2. As schemes, \(X\) and \(E_{d-e+1}(X)\) are equal outside \(B(X), C(X)\), and \(\text{Sing } X\).

As an application of Theorem 2, we have another proof of Theorem 1 in \(\text{char}(k) = 0\), since a standard secant line \(L\) to \(X\) is not contained in \(B(X)\) (see Remark 3.6).

As another application, we will prove \((B_{d-e+1})\) for a special case:

**Corollary 3.** (\(\text{char}(k) = 0\)) Suppose that \(X(\subseteq \mathbb{P}^N)\) with \(e \geq 2\) is contained in the image \(v_l(\mathbb{P}^m) \subseteq \mathbb{P}^M\) \((M = (m+l) - 1)\) of an \(l\)th \((l \geq 2)\) Veronese embedding \(v_l\) of a projective space \(\mathbb{P}^m\) in \(\mathbb{P}^M(\supset \mathbb{P}^N)\) for some \(m > 0\). Then \(B(X)\) and \(C(X)\) are empty. Consequently \(X = E_{d-e+1}(X)\) as sets; and \(X = E_{d-e+1}(X)\) as schemes if \(X\) is smooth.
Finally we will study the structure of $B(X)$ and $C(X)$. We show that $B(X)$ is a closed subset of $P^N \setminus X$ and that $C(X)$ is a closed subset of $X \setminus \text{Sing } X$ in (4.1) and (4.2). The set $B(X)$ was firstly studied by Beniamino Segre [20] and [21], and later by Calabri and Ciliberto [4]. Segre [20] proved the closure of $B(X)$ is the union of a finite number of linear subspaces of dimension at most $n-1$ (see Theorem 4.3). Based on Segre’s result, we say more about $B(X)$ and $C(X)$. Conventionally, we mean $\dim \emptyset = -1$.

**Theorem 4.** (char $k = 0$) If $e \geq 2$, then $\dim B(X) \leq \min\{n - 1, \dim \text{Sing } X + 1\}$. In particular, if $X$ is smooth (i.e., $\dim \text{Sing } X = -1$) and $e \geq 2$, then $B(X)$ is a finite set.

**Theorem 5.** (char $k = 0$) Assume $e \geq 2$. Let $Z$ be an irreducible component of $C(X)$. Then the closure of $Z$ is a linear subspace of dimension $l \leq \min\{n - 1, \dim \text{Sing } X + 2\}$.

As a consequence of Theorems 2, 3, 4 and 5, we have the following.

**Corollary 6.** (char $k = 0$) Suppose that $X$ is smooth, $n \geq 2$ and $e \geq 2$. Then $B(X)$ is a finite set and $C(X)$ is the union of a finite number of linear subspaces of dimension $\leq 1$. Consequently the intersection of all hypersurfaces containing $X$, of degree $\leq d - e + 1$ is equal to $X$ as scheme, except a finite union of linear subspaces of dimension $\leq 1$.

Moreover we will show that the inequality in Theorem 5 is sharp by giving an example in (4.10). Also, in Theorem 4.11, we study the singular locus of $X$ contained in the boundary of $C(X)$.

We organize this paper as follows. In §1, we summarize some results of inner projections which we will use later. In §2, we prove Theorem 1. In §3, we prove Theorem 2 and Corollary 3. In §4, we look at the structure of $B(X)$ and $C(X)$ and prove Theorems 4.4 which is a strong form of Theorem 4, and Theorem 5.

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**Notation.** We use standard terminology from algebraic geometry, e.g, [11]. By a point, we always mean a closed point. A general point means a point off a finite union of suitable proper closed subvarieties. For a point $x$ of $X$, by $T_x(X)$, we denote the embedded tangent space to $X$ at $x$ in $P^N$. By $\langle Y, Z \rangle$, we denote the linear span of subschemes $Y$ and $Z$ of $P^N$, the smallest linear subspace containing both $Y$ and $Z$. By $\text{Sing } X$ (resp. $\text{Sm } X$), we denote the singular locus (resp. smooth locus) of $X$.

§1 Inner projection.

In this section, we summarize some results of inner projections.

(1.1) Let $X \subseteq P^N$ be as in §0. Let $\Lambda \subseteq P^N$ be a linear subspace of dimension $l$ with $\Lambda \cap X \neq \emptyset$ but $\Lambda \not\supseteq X$. The linear projection $\pi_{\Lambda} : P^N \setminus \Lambda \to P^{N-l-1}$ from $\Lambda$ induces a morphism $\pi_{\Lambda,X} : X \setminus \Lambda \to P^{N-l-1}$. Let $V$ be the linear space of linear forms on $P^N$, and let $W \subseteq V$ be the subspace of linear forms vanishing on $\Lambda$, i.e., $P^N = P(V)$ and
Proof of Theorem 1.

First we claim that if \( \mathbb{P}^N \) is a general \( e \)-dimensional linear subspace containing \( L \), then \( M \cap X \) is finite and containing at least \( e - 1 \) distinct
points off $L$. To prove this, consider the linear projection $\pi_L: \mathbb{P}^n \setminus L \rightarrow \mathbb{P}^{n-2}$ from $L$. The closure $\bar{X}$ of $\pi_L(X \setminus L)$ has dimension $n$ by assumption. Since $X$ is nondegenerate, since so is $\bar{X} \subseteq \mathbb{P}^{n-2}$. Thus $\bar{d} := \deg \bar{X} \geq e - 1$, (see for example [10], (18.12); [8], (I.4.2)). Since $\bar{M} := \pi_L(M \setminus L)$ is a general $(e - 2)$-dimensional linear subspace of $\mathbb{P}^{n-2}$, by Bézout’s and Bertini’s Theorems, $\bar{X} \cap \bar{M}$ is $\bar{d}$ distinct points, contained in a nonempty open subset $U$ of $\bar{X}$ over which the induced morphism $\pi_{L,X}$ is a finite morphism (see (1.1.2) for $U$). This implies the claim.

Let $M \subseteq \mathbb{P}^n$ be a general $e$-dimensional linear subspace containing $L$. By the first part, as sets, $(M \setminus L) \cap X = \{x_1, \ldots, x_m\}$ and $L \cap X = \{y_1, \ldots, y_t\}$ for some distinct points $x_j$ and $y_i$ with

\[(2.1.1) \quad m \geq \bar{d} \geq e - 1.\]

Take a general $(e - 2)$-dimensional linear subspace $\Lambda \subseteq M$. We may assume that $\Lambda$ is disjoint from $X$, $L$, and lines $\langle y_i, x_j \rangle$ and $\langle x_j, x_k \rangle$ ($i = 1, \ldots, t$; $j \neq k = 1, \ldots, m$). Consequently, $\langle \Lambda, x_j \rangle \cap L$ is a point, say $z_j$, and $\{z_j\}$ are $m$ distinct points off $X \cap L$. Now consider the linear projection $\pi_\Lambda: \mathbb{P}^n \setminus \Lambda \rightarrow \mathbb{P}^n$ from $\Lambda$. The image $X' := \pi_\Lambda(X) \subseteq \mathbb{P}^n$ is a hypersurface of degree $d'(\leq d)$, since the projection $\pi_{\Lambda,X}: X \rightarrow \mathbb{P}^n$ is finite. Let $F$ be the hypersurface obtained by pulling-back of $X' \subseteq \mathbb{P}^n$ by $\pi_\Lambda$. In other words, $F$ is the cone over $X'$ with vertex $\Lambda$, if we consider $\mathbb{P}^n$ to be a subspace of $\mathbb{P}^n$, disjoint from $\Lambda$. Thus $\deg F = d' \leq d$, $F \supseteq X$ and $F \cap L \supseteq (X \cap L) \cup \{z_1, \ldots, z_m\}$. Moreover $F \not\supseteq L$, since the closure of $\pi_\Lambda^{-1}(\pi_\Lambda(L))$ is $\langle L, \Lambda \rangle = M$ and $M \cap X$ is finite. Consequently $d' = l(F \cap L) \geq l(X \cap L) + m$, and hence

\[(2.1.2) \quad l(X \cap L) \leq d' - m \leq d - m \leq d - e + 1. \quad \square\]

**Remark 2.2.** Under the same notation and assumptions as in Theorem 1, if $l(X \cap L) = d - e + 1$, then the closure $\bar{X}$ of $\pi_L(X \setminus L)$ is a variety of minimal degree, i.e., $\bar{d} = \bar{e} + 1$ (see [10], (19.9); [8], (I.2.2), (I.5.10)). Furthermore, if char($k$) = 0, then $\pi: X \setminus L \rightarrow \bar{X}$ is separable and hence $\bar{X}$ is birational to $X$: Indeed, if $l(X \cap L) = d - e + 1$, then it follows from (2.1.2) that $m = e - 1$ and $d' = d$. Hence $m = \bar{d} = e - 1$ by (2.1.1), as required.

**Remark 2.3.** If a secant line $L$ meets $X$ only at $SmX$ and $e \geq 2$, by Bertini-type Theorem, $X \cap H$ is irreducible and reduced for a general hyperplane $H \supseteq L$, and hence $L$ is standard (see [19], Lemma for the proof. see also [14] and [18], Lemma 2.1). On the other hand, if $X$ has bad singularity at $X \cap L$, for a general hyperplane $H \supseteq L$ of $\mathbb{P}^n$, $X \cap H$ is not necessary irreducible (see [18], Example 2.2(2)). Moreover, if $X$ is integral but not normal at $x \in X \cap L$, then $X$ has Serre’s condition $S_1$ but not $S_2$ at $x$ (see [1], VII (2.2) and (2.13)), and hence $X \cap H$ is not $S_1$ at $x$, i.e., $X \cap H$ is not reduced. Thus to obtain Theorem 1, the method of taking hyperplane section used in [2], [14], and [19] does not work in case $X \cap L \not\subseteq SmX$. 
§3 Hypersurfaces containing projective variety: Proof of Theorem 2.

Let $X \subseteq \mathbb{P}^N$ be as in §0. In this section, we always assume $\text{char}(\mathbb{k}) = 0$ and will prove Theorem 2. This assumption is necessary only to apply the trisecant lemma, which asserts for general points $x, y$ of $X$, $l(X \cap \langle x, y \rangle) = 2$ if $e \geq 2$ and $\text{char}(\mathbb{k}) = 0$: In fact, the set of points $x \neq y \in X$ with $l(X \cap \langle x, y \rangle) \geq 3$ is a closed subset of $X \times X \setminus \Delta_X$, and also a proper subset by the trisecant lemma (see for example [11], IV 3.8) for curves which are obtained by hyperplane sections of $X$.

Theorem 2 is reduced to proving Propositions 3.1 and 3.2. For scheme-theoretic part (2), see [17], p.34, Lemma.

**Proposition 3.1.** (char $\mathbb{k} = 0$) Let $v$ be a point of $\mathbb{P}^N \setminus X$. Suppose $v \notin B(X)$ if $e \geq 2$. Then there exists a hypersurface $F$ of degree $\leq d - e + 1$ such that $F \supseteq X$ but $v \notin F$.

**Proof.** When $e = 1$, this is clear. By induction on $e$, suppose $e \geq 2$. Let $x$ be a general point of $X$, so that $x \in Sm X$ and $l(X \cap \langle v, x \rangle) = 1$ by $v \notin B(X)$. Consider the projection $\pi_x : \mathbb{P}^N \setminus \{x\} \to \mathbb{P}^{N-1}$ from $x$ and let $\bar{X}$ be the closure of $\pi_x(X \setminus \{x\})$. By (1.1.1) and (1.1.3), $\bar{X}$ is a nondegenerate projective variety of degree $\bar{d} \leq d - 1$, and codimension $\bar{e} = e - 1$, and $\bar{X} = \pi_x(X \setminus \{x\}) \cup \pi_x(T_x(X) \setminus \{x\})$. Thus $\bar{v} := \pi_x(v) \notin \bar{X}$, since $l(X \cap \langle v, x \rangle) = 1$. We assume, for a moment, that $\bar{v} \notin B(\bar{X})$ for $\bar{e} = e - 1 \geq 2$, and will complete the induction. By the induction, we have a hypersurface $\bar{F} \subseteq \mathbb{P}^{N-1}$, of degree $\leq \bar{d} - \bar{e} + 1(\leq d - e + 1)$ such that $\bar{F} \supseteq \bar{X}$ but $\bar{v} \notin \bar{F}$. Let $F$ be the hypersurface obtained by pulling-back of $\bar{F}$ by $\pi_x$. Then $\deg F = \deg \bar{F} \leq d - e + 1, F \supseteq X$ and $v \notin F$, since $F$ is the cone over $\bar{F}$ with vertex $x$, as required.

To conclude the proof, we will show $\bar{v} \notin B(\bar{X})$ for $e \geq 3$. By contradiction, we assume that for a general point $y \in X$ with $\bar{y} := \pi_x(y) \in Sm \bar{X}$,

$$l(\langle \bar{v}, \bar{y} \rangle \cap \bar{X}) \geq 2. \tag{3.1.1}$$

By the generality of $x$ and $y$ and trisecant lemma, the projection $\pi_{x,X} : X \setminus \{x\} \to \mathbb{P}^{N-1}$ is embedding at $y$ (see (1.1.3)(4)), and consequently $T_{\bar{x}}(\bar{X}) = \pi_x(T_x(X))$ (see (1.2)). Moreover, $v \notin T_x(X), v \notin T_{\bar{x}}(\bar{X}), x \notin \langle T_x(X), v \rangle, y \notin \langle T_x(X), v \rangle$ since $v \notin B(X)$ and $x$ and $y$ are general in nondegenerate $X$, and consequently

$$\bar{v} \notin T_{\bar{x}}(\bar{X}) = \pi_x(T_x(X)) \quad \text{and} \quad \langle \bar{y}, \bar{v} \rangle \cap \pi_x(T_x(X) \setminus \{x\}) = \emptyset. \tag{3.1.2}$$

By (3.1.1) and (3.1.2), there is a point $\bar{z} \in \langle \bar{v}, \bar{y} \rangle \cap \bar{X}$ distinct from $\bar{y}$ which is an image $\pi_x(z)$ of some $z \in X$. This means $\langle v, x, y \rangle$ contains $z \in X$ off $\langle v, x \rangle$ and $\langle v, y \rangle$, since $x$ and $y$ are general and $v \notin B(X)$. Thus for the linear projection $\pi_v : \mathbb{P}^N \setminus \{v\} \to \mathbb{P}^{N-1}$ from $v$, the image $\pi_v(X)$ has a secant line meeting at the three distinct points $\pi_v(x)$, $\pi_v(y)$ and $\pi_v(z)$. This contradicts the trisecant lemma, since $\pi_v(x)$ and $\pi_v(y)$ are general points of $\pi_v(X)$ because of the generality of $x$ and $y$ in $X$. \(\square\)
Proposition 3.2. (char $k = 0$) Let $u$ be a smooth point of $X$ with embedded tangent space $T_u(X) \subseteq \mathbb{P}^N$ and let $w$ be a point of $\mathbb{P}^N \setminus T_u(X)$. Suppose $u \not\in C(X)$ if $e \geq 2$. Then there exists a hypersurface $F$ containing $X$, of degree $\leq d - e + 1$, such that $F$ is smooth at $u$ with $w \not\in T_u(F) \subseteq \mathbb{P}^N$.

Proof. When $e = 1$, this is clear. By induction, suppose $e \geq 2$. Let $x$ be a general point of $X$. Then $x \in \text{Sm} X$ with $l(X \cap \langle x, u \rangle) = 2$ and $x \not\in \langle T_u(X), w \rangle$. Consider the projection $\pi_x: \mathbb{P}^N \setminus \{x\} \to \mathbb{P}^{N-1}$ from $x$, and let $X$ be the closure of $\pi_x(X \setminus \{x\})$. The projection $\pi_{x,u}: X \setminus \{x\} \to \mathbb{P}^{N-1}$ is an embedding at $u$ (see (1.1.3)), and hence $\bar{u} := \pi_x(x) \in \text{Sm} \bar{X}$, and $\bar{w} := \pi_x(w) \not\in T_u(\bar{X}) = \pi_x(T_u(X))$ (see (1.2)). Moreover $\bar{X}$ is a nondegenerate projective variety of degree $\bar{d} \leq d - 1$, dimension $\bar{n} = n$, and codimension $\bar{e} = e - 1$, with $\bar{X} = \pi_x(X \setminus \{x\}) \cup \pi_x(T_x(X) \setminus \{x\})$ (see (1.1.1) and (1.1.3)). We assume, for a moment, that $\bar{u} \not\in C(\bar{X})$ for $e \geq 3$, and will complete the induction. By induction, we have a hypersurface $\bar{F} \subseteq \mathbb{P}^{n+e-1}$ containing $\bar{X}$, of degree $\leq \bar{d} - \bar{e} + 1(\leq d - e + 1)$, smooth at $\bar{u}$ with $\bar{w} \not\in T_{\bar{u}}(\bar{F})$. The hypersurfaces $F$ obtained by pulling-back of $\bar{F}$ by $\pi_x$ is of degree $\leq \bar{d} - \bar{e} + 1$, smooth at $u$ with $w \not\in T_u(F)$, as required.

To conclude our proof, we will show $\bar{u} \not\in C(\bar{X})$ for $e \geq 3$. By contradiction, assume that for general $y \in X$ with $\bar{y} := \pi_x(y) \in \text{Sm} \bar{X}$, $l(\langle \bar{y}, \bar{u} \rangle \cap \bar{X}) \geq 3$. By the generality of $x$ and $y$ and trisecant lemma, $\pi_{x,u}$ is embedding at $y$, and hence $T_{\bar{y}}(\bar{X}) = \pi_x(T_y(X))$. Moreover, $y \not\in \langle T_u(X), x \rangle$, $y \not\in \langle T_x(X), u \rangle$, and $x \not\in \langle T_y(X), u \rangle$. Hence $\bar{y} \not\in T_{\bar{u}}(\bar{X})$, $\bar{y} \not\in \langle \bar{u}, \pi_x(T_x(X) \setminus \{x\}) \rangle$, and $\bar{u} \not\in T_{\bar{y}}(\bar{X})$. Consequently there is a point $\bar{z} \in \bar{X} \setminus \langle \bar{y}, \bar{u} \rangle$ distinct from $\bar{y}$ and $\bar{u}$ with $\bar{z} = \pi_x(z)$ for some $z \in X$. This means that $z$ lies on $\langle u, x, y \rangle$, but off $\langle u, x \rangle$ and $\langle u, y \rangle$ by $u \not\in C(X)$. Thus for the projection $\pi_{u,u}: \mathbb{P}^N \setminus \{u\} \to \mathbb{P}^{N-1}$ from $u$, the closure $X'$ of $\pi_u(X \setminus \{u\})$ has a general secant line $\langle \pi_u(x), \pi_u(y) \rangle$ meeting at the three distinct points, which contradicts the trisecant lemma. \hfill $\square$

Remark 3.3. If $v \in B(X)$, then $\pi_x(v) \in \pi_x(X \setminus \{x\})$ for any $x \in X$. Thus the points of $B(X)$ cannot separate from $X$ by the hypersurfaces obtained in (3.1). On the other hand, if $u \in C(X)$, then $\pi_x(u) \in \text{Sing} \pi_x(X \setminus \{x\})$ for any $x \in X$. Thus at the points of $C(X)$, the hypersurfaces in (3.2) cannot separate tangent space of $X$ from $\mathbb{P}^N$.

Example 3.4. Let $X \subseteq \mathbb{P}^N (N = n + e)$ be a projective variety of dimension $n$, degree $d$, and codimension $e$ over an algebraically closed field $k$ of char $k = 0$. Assume $X$ is a variety of minimal degree, or of delta genus $\Delta(X) = 0$, i.e., $d = e + 1$ (see [10], (19.9); [8], (1.2.2), (1.5.10)). Assume $e \geq 2$, or equivalently $d \geq 3$. Then $B(X) = \emptyset$. Moreover, $X$ has no 3-secant line and hence $C(X) = \emptyset$.

Proof. Suppose $B(X) \neq \emptyset$, and we will show that $e = 1$. Consider the projection $\pi_v: \mathbb{P}^N \setminus \{v\} \to \mathbb{P}^{N-1}$ from a point $v \in B(X)$. Set $\bar{X} = \pi_v(X)$ and $\bar{d} = \deg \bar{X}$. Then $\bar{d} \leq d/2(= (e + 1)/2)$. Since $\bar{X}$ is nondegenerate in $\mathbb{P}^{N-1}$, we have $\bar{d} \geq N - n = e$ by [10], (18.12) or [8],(I.4.2). From these two inequalities, we obtain $e \leq 1$, and hence $e = 1$, as required. The second part follows from the first part and Theorem 2, noting that
\(d - e + 1 = 2\). (Or directly, if there is a 3-secant line \(L\), consider the projection from \(L\), and follow the same argument as above. Then we have a contradiction \(N = n\).) \(\square\)

**3.5.** Proof of Corollary 3. We have only to show that \(B(X)\) and \(C(X)\) are empty. First we will prove \(B(X) = \emptyset\). By contradiction, assume there is a point \(v \in B(X)\). For general \(x_1 \neq x_2 \in X\), consider the lines in \(\mathbb{P}^m\) joining \(x_i\) and \(v\), and observe on the lines, there exist points \(y_i \in X\) different from \(x_i\). Let \(L_i (i = 1, 2)\) be the line in \(\mathbb{P}^m\) joining the preimages of \(x_i\) and \(y_i\) in \(\mathbb{P}^m\). Note that \(v \in \langle v_i(L_i) \rangle\). If \(L_1 = L_2\), then \(X = v_i(L_1)\) with \(B(X) \neq \emptyset\), and hence \(X\) is conic by Example 3.3, contradiction. Thus \(L_1 \neq L_2\). On the other hand, it is easy to see for two distinct lines \(\ell_1, \ell_2\) in \(\mathbb{P}^m\), that

1. \(\langle v_i(\ell_1) \rangle \cap \langle v_i(\ell_2) \rangle = \emptyset\) if \(\ell_1\) and \(\ell_2\) are disjoint, and that
2. \(\langle v_i(\ell_1) \rangle \cap \langle v_i(\ell_2) \rangle = v_i(\ell_1 \cap \ell_2)\) if \(\ell_1 \cap \ell_2 \neq \emptyset\).

Therefore \(v \in v_i(L_1 \cap L_2)\). This implies \(v_i(L_1)\) has a 3-secant line \(\langle v, x_i \rangle = \langle x_i, y_i \rangle\), which contradicts Example 3.3, as required. The second part \(C(X) = \emptyset\) follows from the same argument above for \(u \in C(X)\) instead of \(v \in B(X)\). \(\square\)

**Remark 3.6.** Proposition 3.1 implies Theorem 1 in case \(\text{char} \mathbb{k} = 0\): Indeed, since \(L\) is standard, \(L \not\subseteq B(X)\) by the argument as in the proof of (4.4). By (3.1), we obtain a hypersurface \(F\) containing \(X\) but not \(L\). Consequently \(l(X \cap L) \leq l(F \cap L) \leq d - e + 1\).

§4 Study of \(B(X)\) and \(C(X)\): Proof of Theorem 4 and 5.

Let \(X \subseteq \mathbb{P}^N\) be as in §0. In this section, we assume \(\text{char} \mathbb{k} = 0\). We will study the structure of \(B(X)\) and \(C(X)\) in (0.1), and prove Theorems 4 and 5. As a consequence, we will obtain Corollary 6.

First we will show \(B(X)\) and \(C(X)\) are algebraic sets.

**Lemma 4.1.** \(B(X)\) is a closed subset of \(\mathbb{P}^N \setminus X\).

**Proof.** For a hyperplane \(H \subseteq \mathbb{P}^N\), we set \(U := \mathbb{P}^N \setminus (X \cup H)\). We have only to show that \(B(X) \cap U\) is closed in \(U\). To this purpose, consider the morphism \(\varpi : U \times X \to U \times \mathbb{P}^N\) defined by \((u, x) \mapsto (u, \langle u, x \rangle \cap H)\), that is the family of the projections \(\pi_{u,X} : X \to \mathbb{P}^N\) from \(u \in U\) to \(H \cong \mathbb{P}^N\). Note that \(\varpi\) is projective, since \(U \times X \to U\) is projective and \(U \times \mathbb{P}^N \to U\) is separated (see [11], Ex.II.4.9). Moreover \(\varpi\) is finite, since \(\varpi\) is quasi-finite (see [11], Ex.III.11.2). Hence \(W := \{(u, \bar{x}) \in U \times \mathbb{P}^N\mid \dim_k(u, \bar{x}) \omega_{u,X} O_U \otimes k(u, \bar{x}) \geq 2\}\) is the set of points \((u, \bar{x}) \in U \times \mathbb{P}^N\) whose fibre \(\varpi^{-1}(u, \bar{x}) \cong \pi^{-1}_u(\bar{x})\) is of length at least 2. Moreover \(W\) is closed in \(U \times \mathbb{P}^N\) by [11], Ex.II.5.8, and hence the first projection \(p_1 : W \to U\) is projective. Therefore a point \(u \in U\) is contained in \(B(X) \cap U\) if and only if the fibre \(W_u := p_1^{-1}(u)\) is dense in \(\pi_u(X)\), i.e., \(W_u = \pi_u(X)\). Then \(W_u = \pi_u(X)\) if and only if \(\dim W_u = n\), since \(X\) is irreducible and \(\pi_u\) is finite. Thus \(B(X) \cap U = \{u \in U \mid \dim W_u \geq n\}\), and hence \(B(X) \cap U\) is closed in \(U\), by [11], Ex.II.3.22 (d) and the properness of \(p_1\), as required. \(\square\)
Lemma 4.2. \( C(X) \) is a closed subset of the smooth locus \( \text{Sm} X \) of \( X \).

Proof. For a hyperplane \( H \subseteq \mathbb{P}^N \), we have only to show that \( C(X) \setminus H \) is closed in \( X_0 := \text{Sm} X \setminus H \). Recall that the linear projection \( \pi_{z,X} : X \setminus \{z\} \to \mathbb{P}^{N-1} \) from \( z \in X_0 \) to \( H \cong \mathbb{P}^{N-1} \subseteq \mathbb{P}^N \) is extendable to the morphism \( \hat{\pi}_{z,X} : \hat{X}_z \to \mathbb{P}^{N-1} \) by taking the blowing-up \( \sigma_z : \hat{X}_z \to X \) of \( X \) at \( z \) (see (1.1)). To construct the family of \( \hat{\pi}_{z,X} \) over \( z \in X_0 \), consider the family of linear projections \( (\mathbb{P}^N \setminus H) \times \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1} \cong H \subseteq \mathbb{P}^N \), \( (z,x) \mapsto (z,x) \cap H \) whose base locus is the diagonal of \( \mathbb{P}^N \setminus H \). Its restriction \( \varpi_2 : \mathcal{X} := X_0 \times X \dashrightarrow \mathbb{P}^N \) is a rational map whose base locus is the diagonal \( \Delta_{X_0} \) of \( X_0 \). By taking the blowing up \( \sigma : \hat{\mathcal{X}} \to \mathcal{X} \) by the ideal sheaf of \( \Delta_{X_0} \) with reduced structure, we have an extension \( \hat{\varpi}_2 : \hat{\mathcal{X}} \to \mathbb{P}^{N-1} \) of \( \varpi_2 \). Since \( J_{\Delta_{X_0}/\mathcal{X}} \{z\} \times X \cong J\{z\}/X \) by a local computation, \( \sigma \) is the family of blowing-ups \( \sigma_z \) for \( z \in X_0 \). Thus \( \hat{\varpi} := (p_1 \circ \sigma) \times \hat{\varpi}_2 : \hat{\mathcal{X}} \to X_0 \times \mathbb{P}^{N-1} \) is the required family such that \( \hat{\varpi}^{-1}(z,\bar{x}) = \hat{\pi}_{z,X}^{-1}(\bar{x}) \) for \( (z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1} \). Note that \( \hat{\varpi} \) is projective, since \( \hat{\mathcal{X}} \to X_0 \) is projective and \( X_0 \times \mathbb{P}^{N-1} \to X_0 \) is separated.

To prove \( C(X) \setminus H \) is closed in \( X_0 \), we consider

\[ W := \{(z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1} | \dim k(z,\bar{x}) \hat{\varpi}_* \mathcal{O}_{\hat{\mathcal{X}}} \otimes k(z,\bar{x}) \geq 2 \text{ or } \dim \hat{\pi}_{z,X}^{-1}(\bar{x}) \geq 1 \}. \]

By \([11], \text{Ex. II.5.8, and Ex. II.3.22 (d)}\), \( W \) is closed in \( X_0 \times \mathbb{P}^{N-1} \), and the first projection \( p_1 : W \to X_0 \) is projective. Moreover \( W \) is the set of points \( (z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1} \) whose fibre \( \hat{\varpi}^{-1}(z,\bar{x}) \cong \hat{\pi}_{z,X}^{-1}(\bar{x}) \) is of length at least 2, since \( \hat{\varpi} \) is finite around \( (z,\bar{x}) \) if \( \dim \hat{\varpi}^{-1}(z,\bar{x}) = 0 \). A point \( (z,\bar{x}) \in X_0 \times \mathbb{P}^{N-1} \) belongs to \( W \) if and only if \( l(X \cap \{z,\bar{x}\}) \geq 3 \), since \( l(\hat{\pi}_{z,X}^{-1}(\bar{x})) \geq 2 \) if and only if \( l(X \cap \{z,\bar{x}\}) \geq 3 \) (see (1.1.3)). Thus \( z \in X_0 \) belongs to \( C(X) \setminus H \) if and only if the fibre \( W_z := p_1^{-1}(z) \) is equal to \( \hat{X} \). The last condition is equivalent to \( \dim W_z = n \). Therefore \( C(X) \setminus H \) is closed in \( X_0 \), as required.

Next we study the structure of \( \text{B}(X) \). Recall the following result.

Theorem 4.3. (Beniamino Segre [20], see also [4]) Let \( X \subseteq \mathbb{P}^N \) \((N = e + n)\) be a nondegenerate, projective variety of dimension \( n \) and codimension \( e \). If \( e \geq 2 \) and \( \text{B}(X) \neq \emptyset \), then every irreducible component of the closure of \( \text{B}(X) \) is a linear subspace of dimension at most \( n - 1 \). Moreover \( \dim \text{B}(X) = n \) if and only if \( e = 1 \).

Based on this result and the idea of the proof, we will prove Theorem 4.4 and Theorem 5. Theorem 4 is an immediate consequence of Theorem 4.4.

Theorem 4.4. Let \( X \) be as in §10. Assume \( n \geq 2 \) and \( e \geq 2 \). Let \( \Lambda \) be an irreducible component of the closure of \( \text{B}(X) \), of dimension \( l \geq 0 \). Then

1. \( \dim X \cap \Lambda = l - 1 \), and
2. \( X \cap \Lambda \subseteq \text{Sing} X \).

Consequently \( \dim \Lambda \leq \dim \text{Sing} X + 1 \), and in particular, \( \dim \text{B}(X) \leq \dim \text{Sing} X + 1 \).

Proof. When \( l = 0 \), the assertion is trivial, so we assume \( l \geq 1 \). By (4.3), \( \Lambda \) is linear. First we observe the linear projection \( \pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1} \) from \( \Lambda \), in particular,
its fibre and image. For a general point $x \in X \setminus \Lambda$, let $X_x$ be the closure of $\pi^{-1}_{\Lambda,X}(\bar{x})$ over $\bar{x} := \pi_{\Lambda}(x)$. Then $\dim X_x = l$: Indeed, $\dim X_x \geq l$, since a line joining $x$ and general $v \in \Lambda \cap B(X)$ contains a point of $X$ different from $x$. Hence $\dim X_x = l$, since $\pi^{-1}_{\Lambda,X}(\bar{x}) \subseteq \langle \Lambda, x \rangle$ and $\Lambda \not\subseteq X$. Consequently the closure $\bar{X}$ of $\pi_{\Lambda,X}(X \setminus \Lambda)$ has dimension $n - l$. Moreover $d := \deg \bar{X} \geq 2$: If not, $\bar{X}$ is linear and nondegenerate in $\mathbb{P}^{n-1}$, and hence $\bar{X} = \mathbb{P}^{n-1}$, which implies $e = 1$, contradiction.

Now (1) is clear, since $\dim X_x = l$ and $X_x \subseteq \langle \Lambda, x \rangle$ with $\Lambda \not\subseteq X$.

Next we will show (2) in case $n = 2$. Then $l = 1$ by assumptions $l \geq 1$ and (4.3). Hence $\dim X_x = 1$ and $\bar{X} \subseteq \mathbb{P}^{n-2}$ is a nondegenerate curve of degree $d \geq 2$. Let $H$ be a general hyperplane containing $\Lambda$. The first step is to show

\[ (4.4.1) \quad \Sing(X \cap H) \supseteq X \cap \Lambda. \]

Since $\bar{H} := \pi_{\Lambda}(H \setminus \Lambda)$ is a general hyperplane in $\mathbb{P}^{n-2}$, by Bézout’s Theorem, $\bar{X} \cap \bar{H}$ is $\bar{d}$ distinct points, say $\bar{x}_1, \ldots, \bar{x}_{\bar{d}}$, which lie on $\pi_{\Lambda,X}(X \setminus \Lambda)$ (see (1.1.1)). Consequently $X \cap H$ contains $\bar{d} (\geq 2)$ distinct curves $X_{\bar{x}_i}$, each of which contains $X \cap \Lambda$ by Lemma 4.5 below. This implies (4.4.1). Now to obtain (2), we assume, to the contrary, that there is a point $z \in \Lambda \cap \Sm X$. Then $H \not\supseteq T_z(X)$ by the generality of $H \supseteq \Lambda$ and $\Lambda \neq T_z(X)$. Hence $X \cap H$ is smooth at $z$. This contradicts (4.4.1).

We will show (2) in case $n > 2$. By contradiction, assume that there is a point $z \in \Lambda \cap \Sm X$. Take a general line $L$ through $z$, contained in $\Lambda$, not contained in $X$, so that $L \cap B(X)$ is dense open in $L$ by (4.1). Let $H'$ be a general hyperplane containing $L$. We claim that the reduced induced structure $X' := (X \cap H')_{\red}$ is irreducible and nondegenerate in $H' \cong \mathbb{P}^{n-1}$ such that $z \in \Sm X'$ and the closure of $B(X')$ contains $L$. If the claim is proved, by induction on $n$, we have a projective surface $X'$ such that the closure of $B(X')$ contains $L$ with $L \cap \Sm X' \neq \emptyset$, which contradicts the case $n = 2$. Thus we have only to show the claim. By the same argument as in the first paragraph, a general fibre of the linear projection $\pi_{L,X} : X \setminus L \to \mathbb{P}^{n-2}$ has dimension 1. Hence the image of $\pi_{L,X}$ has dimension $n - 1 (\geq 2)$. By Bertini’s theorem [24], (1.6.3), $X \cap H'$ is irreducible and generically reduced. (To be precise, apply Bertini’s Theorem for the normalization $\bar{X}$ of $X$ and the pull-back $\bar{H}'$ of $H'$, and take push-forward of $\bar{X} \cap \bar{H}'$.) Moreover, since $X$ is integral, $(X \setminus L) \cap H'$ satisfies Serre condition $S_1$ by [7], (3.4.6), and hence $(X \setminus L) \cap H'$ is reduced (see [1], (VII.2.2)). On the other hand, since $X \subseteq \mathbb{P}^N$ is nondegenerate, so is $\pi_{L,X}(X \setminus L) \subseteq \mathbb{P}^{n-2}$, and also so is its hyperplane section (see [7], p.116). Consequently $(X \setminus L) \cap H'$ is nondegenerate. Moreover, $X \cap H'$ is smooth at $z \in \Lambda \cap \Sm X$, since $H' \not\supseteq T_z(X)$ by the generality of $H' \supseteq L$ and $T_z(X) \neq L$. Finally it is clear that $L \not\subseteq X'$ and $L \setminus X' \subseteq B(X')$. In sum, $X'$ satisfies the property we have claimed. This complete the proof. \hfill \Box

**Lemma 4.5.** Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension $n$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension $l$ $(1 \leq l < n)$, not contained in $X$. Assume the image of the linear projection $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-l-1}$ has
dimension $n - 1$. Then for each $x \in X \setminus \Lambda$, the closure $X_x$ of the fibre of $\pi_{\Lambda, X}$ over $\bar{x} := \pi_{\Lambda, X}(x)$ is a hypersurfaces of $\langle \Lambda, \bar{x} \rangle$, containing $X \cap \Lambda$.

**Proof.** Let $\tilde{X}$ be the closure of $\pi_{\Lambda, X}(X \setminus \Lambda)$ in $\mathbb{P}^{N-1}$, and let $\nu: Y \to \tilde{X}$ be a desingularization of $\tilde{X}$. Considering the target $\mathbb{P}^{N-1}$ of $\pi_{\Lambda, X}$ to be a subspace of $\mathbb{P}^N$ disjoint from $\Lambda$, we have a morphism $\tilde{X} \to \mathbb{G} := \text{Grass}(l + 1, N)$ from $\tilde{X}$ to the Grassmann of $(l + 1)$-plane in $\mathbb{P}^N$, defined by $\tilde{x} \to \langle \Lambda, \tilde{x} \rangle$. Let $\mathcal{Q}_Y$ be the pull-back of the universal quotient bundle $\mathcal{Q}$ on $\mathbb{G}$ by $Y \to \tilde{X} \to \mathbb{G}$. So that $\mathcal{Q}_Y \cong \mathcal{O}_{\mathbb{P}^l} \oplus \mathcal{Q}_Y(1)$, where $\mathcal{O}_Y(1) = \nu^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)|_{\tilde{X}})$. The projective bundle $\mathbb{P} := \mathbb{P}(\mathcal{Q}_Y)$ with projection $\tau: \mathbb{P} \to Y$ has a natural morphism $\phi: \mathbb{P} \to \mathbb{P}^N$ defined by the tautological line bundle $\mathcal{O}_\mathbb{P}(1)$, which is embedding except on $\mathbb{P}(\mathcal{Q}_Y)$ and the fibres of $\tau$ over the points of $Y$ at which $Y \to \mathbb{P}^{N-1}$ is not embedding. Thus $\phi(\mathbb{P})$ contains $X$, and $X$ meets an open subset of $\phi(\mathbb{P})$ where $\phi$ is an embedding. Hence we can consider a prime divisor $\tilde{X}$ on smooth variety $\mathbb{P}$, with birational, surjective, induced morphism $\tilde{X} \to X$. Then $\tilde{X}$ is a member of a linear system $|\mathcal{O}_\mathbb{P}(\mu) \otimes \tau^*\mathcal{M}|$ for some positive integer $\mu$ and a line bundle $\mathcal{M}$ on $Y$ (see [11], (II.6.11), (II.6.11A), (II.6.13), and (Ex.III.12.5)). We write $\mathcal{Q}_Y = \mathcal{O}_Y z_0 + \cdots + \mathcal{O}_Y z_l + \mathcal{O}_Y(1) z_{l+1}$ with formal basis $z_i$ (or homogeneous coordinates of fibres). Then $\tilde{X}$ is zero of

$$F = \sum_{\mu_0, \ldots, \mu_{l+1} \geq 0, \mu_0 + \cdots + \mu_{l+1} = \mu} f_{\mu_0 \cdots \mu_{l+1}} z_0^{\mu_0} \cdots z_{l+1}^{\mu_{l+1}} \in H^0(\mathcal{O}_\mathbb{P}(\mu) \otimes \tau^*\mathcal{M})$$

for some $f_{\mu_0 \cdots \mu_{l+1}} \in H^0(Y, \mathcal{M} \otimes \mathcal{O}_Y(\mu_{l+1}))$. By $\mathbb{P}_y^l$, we denote the fiber of $\mathbb{P}_y(\mathcal{O}_Y z_0 + \cdots + \mathcal{O}_Y z_l) \to Y$ over $y \in Y$. Then $\tilde{X} \cap \mathbb{P}_y^l$ is the subset of $\mathbb{P}_y^l(\subset \mathbb{P})$ defined by

$$F|_{\mathbb{P}_y^l} = \sum_{\mu_0, \ldots, \mu_l \geq 0, \mu_0 + \cdots + \mu_l = \mu} f_{\mu_0 \cdots \mu_l}(y) z_0^{\mu_0} \cdots z_l^{\mu_l}$$

and $\tilde{X} \cap \mathbb{P}_y^l$ is mapped into $X \cap \Lambda$. If we consider a rational map, defined by $\{f_{\mu_0 \cdots \mu_l} \subseteq H^0(Y, \mathcal{M})$, from $Y$ to the set $\mathbb{P}^M (M = (l+1) - 1)$ of linear forms of degree $\mu$ on $\mathbb{P}^l$ with homogeneous coordinates $z_0, \ldots, z_l$, then this map is a morphism from $Y$ whose image is a one point, since $\phi(\tilde{X} \cap \mathbb{P}_y^l) \subseteq X \cap \Lambda$ for each $y \in Y$ and since the points of $\mathbb{P}^M$ whose zeros, as $\mu$-form of $\mathbb{P}^l$, are mapped into $X \cap \Lambda$ are finite. Consequently $\mathcal{M} \cong \mathcal{O}_Y$ and $f_{\mu_0 \cdots \mu_l} \in H^0(\mathcal{O}_Y) \cong \mathbb{k}$, and moreover $f_{\mu_0 \cdots \mu_l} \neq 0$ for some $\mu_0, \ldots, \mu_l$. Therefore set-theoretically, $\phi(\tilde{X} \cap \mathbb{P}_y^l) = X \cap \Lambda$ for each $y \in Y$. This implies that the image $\phi(\tilde{X} \cap \mathbb{P}(\mathcal{Q}_Y \otimes \mathbb{k}(y)))$ contains $X \cap \Lambda$. For $x \in X \setminus \Lambda$, $X_x = \bigcup_{y \in \nu^{-1}(x)} \phi(\tilde{X} \cap \mathbb{P}(\mathcal{Q}_Y \otimes \mathbb{k}(y)))$ as sets. Hence $X_x$ is a hypersurface, containing $X \cap \Lambda$, as required. □

**Example 4.6.** The induced morphism $\tilde{X} \cap \mathbb{P}_y^l \to X \cap \Lambda$ in the proof of Lemma 4.5, is bijective but not necessary isomorphic: Let $\mathbb{P}$ be the the projective bundle $\mathbb{P}_1(\mathcal{Q})$ over $Y := \mathbb{P}^1$, associated with vector bundle $\mathcal{Q} = \mathcal{O}_\mathbb{P}_1 z_0 + \mathcal{O}_\mathbb{P}_1 z_1 + \mathcal{O}_\mathbb{P}_1(4) z_2$ on $\mathbb{P}_1$, with projection $\tau: \mathbb{P} \to \mathbb{P}^1$ and the tautological bundle $\mathcal{O}_\mathbb{P}(1)$. Here $z_i$ are formal bases. Let
s, t be the homogeneous coordinates of \( \mathbb{P}^1 \). Let \( \phi: \mathbb{P} \to \mathbb{P}^5 \) be the morphism defined by \( z_0, z_1, s^4 z_2, s^3 t z_2, s^3 t^2 z_2, t^4 z_2 \in H^0(O_\mathbb{P}(1)) \). Let \( X_1 \) (\( i = 1, 2 \)) be divisors defined by \( F_1 = z_1^2 - s^2 t^2 z_2 \) and \( F_2 = z_1^2 - s^3 t z_2 \) in \( H^0(O_\mathbb{P}(2)) \). Then the image \( X_i = \phi(X_1) \) are nondegenerate projective surfaces of degree 8. The line \( L \) (or \( \phi(\mathcal{O}_{\mathbb{P}}(z_0 \oplus \mathcal{O}_{\mathbb{P}}(z_1))) \)) is contained in the closure of \( B(X_1) \). Moreover \( X_1 \cap L = X_2 \cap L \) as set, and \( l(X_1 \cap L) = 4 \) and \( l(X_2 \cap L) = 2 \). On the other hand, \( l(\tilde{X}_1 \cap \mathbb{P}^1) = l(\tilde{X}_2 \cap \mathbb{P}^1) = 2 \) for each \( y \in \mathbb{P}^1 \).

**Remark 4.7.** In Theorem 4.4, to obtain \( \dim B(X) \leq \dim \text{Sing } X + 1 \) for \( e \geq 2 \), there is an easier argument using Bertini’s Theorem as follows. Assume to the contrary \( l := \dim B(X) \geq \dim \text{Sing } X + 2 \). By (4.3), \( l \leq n - 1 \). For \( m = N - l + 1 \), let \( M \subseteq \mathbb{P}^N \) be a general \( m \)-dimensional linear subspace. By Bertini’s Theorem, \( X' := X \cap M \) is a smooth projective variety of dimension \( n - l + 1 \), nondegenerate in \( M \) (see [10], (18.10) or [7], (3.5.8)). Moreover the closure of \( B(X') \) contains a line \( L \) with \( L \not\subseteq X' \), since \( B(X') \supseteq B(X) \cap M \). Let \( M' \subseteq M \) be a general \( (e + 1) \)-dimensional linear subspace containing \( L \). Since \( L \cap X' \subseteq \text{Sm } X' \), by Bertini-type-Theorem (see [14], (2.1); [18], (2.1)), \( X'' := X' \cap M' \) is smooth nondegenerate projective curve in \( M' \) with \( \dim B(X'') \geq 1 \). Since \( L \cap B(X'') \) is dense in \( L \) (or apply Theorem 4.3), \( X'' \) lies on the 2-plane spanned by \( L \) and a general point \( x \in X'' \), and consequently \( e = 1 \), contradiction.

(4.8). Proof of Theorem 5. For a general (smooth) point \( x \) of \( X \), let \( \Lambda_x \) be the linear span \( \langle Z, T_x(X) \rangle \). Now, according to Segre [20], we will show that

\[(4.8.1) \quad \dim \Lambda_x = n + 1.\]

To this purpose, consider the linear projection \( \pi_{z,X}: X \setminus \{z\} \to \mathbb{P}^{N-1} \) from a point \( z \in Z \). Since \( z \in \text{Sm } X \), \( \pi_{z,X} \) is generically quasi-finite (see (1.1.3)). By the generic smoothness of \( \pi_{z,X} \), the line \( \langle x, z \rangle \) meets \( X \) at a point \( y \in \text{Sm } X \) distinct from \( x \) and \( z \). Moreover \( T_y(X) \subseteq \langle T_x(X), z \rangle \) (see (1.2.2)). Let \( Y \) be an irreducible component of the closure of the set of the points \( y \in \langle x, z \rangle \cap X \) for moving \( z \in Z \) and fixed \( x \in X \). If \( y \in Y \) is general, the corresponding point \( z \) is also general in \( Z \), and hence,

\[T_y(Y) \subseteq T_y(X) \subseteq \langle T_x(X), z \rangle \supseteq T_x(X).\]

From this, by considering the projection of \( Y \) from \( T_x(X) \), we observe that \( \langle T_x(X), y \rangle \) does not depend on \( y \in Y \) (see (1.2.1)). Consequently \( \langle T_x(X), y \rangle = \langle T_x(X), Y \rangle = \Lambda_x \), which implies (4.8.1).

Let \( \Lambda \) be the intersection of \( \Lambda_x \) for general points \( x \in X \), and set \( l := \dim \Lambda \). If \( l = n + 1 \), then \( \Lambda = \Lambda_x \) and hence \( \Lambda \) contains \( X \), which contradicts \( e \geq 2 \). Thus \( l \leq n \).

Let \( \pi_{\Lambda,X}: X \setminus \Lambda \to \mathbb{P}^{N-l-1} \) be the linear projection from \( \Lambda \) to a subspace \( \mathbb{P}^{N-l-1} \subseteq \mathbb{P}^N \) disjoint from \( \Lambda \), and let \( \tilde{X} \) be the closure of \( \pi_{\Lambda,X}(X \setminus \Lambda) \). For general \( x \in X \), let \( \tilde{X}_x \) be the closure of \( \pi^{-1}_{\Lambda,X}(\tilde{x}) \) over \( \bar{x} := \pi_{\Lambda}(x) \). We claim that \( l < n \), \( \dim \tilde{X}_x = l \), and

\[(4.8.2) \quad l((X \setminus \Lambda) \cap \langle v, x \rangle) \geq 2.\]
for general $v \in \Lambda$. If $l = n$, i.e., $\text{codim}(\Lambda, \Lambda_\nu) = 1$, then $T_x(X) \cap \Lambda = n - 1$, and hence $X \subseteq \langle \Lambda, x \rangle$, which contradicts $e \geq 2$. Thus $l < n$. Then $T_x(X) \cap \Lambda = l$ or $l - 1$, since $\text{codim}(T_x(X), \Lambda_\nu) = 1$. If $\dim T_x(X) \cap \Lambda = l$, by (1.2.1), $\dim \bar{X} = n - l - 1$, and hence $X$ is the cone over $\bar{X}$ with vertex $\Lambda$, which means $Z \subseteq \Lambda \subseteq \text{Sing} X$, contradiction. Therefore $\dim T_x(X) \cap \Lambda = l - 1$. By (1.2.1), $\dim \bar{X} = n - l$, and $\dim X_\nu = l$. The fact $l((X \setminus \{z\}) \cap (z, x^1)) \geq 2$ for $z \in Z$ implies that the $l$-dimensional part of $X_\nu$ is a hypersurface of $\langle \Lambda, x \rangle$, of degree $\geq 2$, not containing $\Lambda$. Thus we have (4.8.2).

Next we will show $\Lambda$ is the closure of $Z$. If $\Lambda \not\subseteq X$, by (4.8.2), a general point of $\Lambda$ lies on $B(X)$, and hence $Z \subseteq \Lambda \cap X \subseteq \text{Sing} X$ by (4.4), contradiction. Thus $\Lambda \subseteq X$.

Since $Z$ is an irreducible component of $C(X)$, if $\dim Z < \dim \Lambda$, then $\Lambda \setminus Z \subseteq \text{Sing} X$ by (4.8.2), and hence $Z \subseteq \Lambda \subseteq \text{Sing} X$, contradiction. Thus $\dim Z = \dim \Lambda$, and hence $\Lambda$ is the closure of $Z$.

Finally we look at $\dim Z$. Note that $\dim Z = \dim \Lambda = \dim X_\nu = l$. By (1.2.2), $T_{x'}(X) \subseteq (T_x(\bar{X}), \Lambda)(= \langle T_x(X), \Lambda \rangle = \Lambda_\nu) \subseteq \mathbb{P}^N$ for each $x' \in X_\nu \cap \text{Sm} X$. Consequently, by Theorem of tangencies ([23], (I.1.7)),

$$l = \dim X_\nu \leq \dim \langle T_x(\bar{X}), \Lambda \rangle = \dim X + \dim (X_\nu \cap \text{Sing} X) + 1 = 2 + \dim (X_\nu \cap \text{Sing} X) \leq 2 + \dim \text{Sing} X.$$

This completes the proof of Theorem 5. \(\square\)

(4.9). Proof of Corollary 6. If $X$ is smooth, then $B(X)$ is a finite set and $C(X)$ is a finite union of linear subspaces of dimension $\leq 1$. The rest follows from Theorem 2. \(\square\)

Here we will show the inequality of Theorem 5 is sharp, by giving an example of $X$ whose $C(X)$ contains a linear subspace of dimension $\dim \text{Sing} X + 2$.

Example 4.10. For integers $l \geq 0$, $n \geq l + 2$, and $a_n \geq \cdots \geq a_{l+1} > a_l = \cdots = a_0 = 0$, let $\mathbb{P}$ be the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ over $\mathbb{P}^1$, associated with vector bundle $\mathcal{E} = \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ on $\mathbb{P}^1$, with projection $\tau: \mathbb{P} \to \mathbb{P}^1$ and the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. Assume $a_n \geq 2$ if $n = l + 2$. For a general member $\tilde{X} \in |\mathcal{O}_{\mathbb{P}^1}(\mu) \otimes \tau^* \mathcal{O}_{\mathbb{P}^1}(1)|$ with an integer $\mu \geq 2$, let $X$ be the image of $\tilde{X}$ by the morphism $\phi: \mathbb{P} \to \mathbb{P}^N (N = n + 1 + \sum_{i=0}^n a_i)$ defined by $|\mathcal{O}_{\mathbb{P}^1}(1)|$. Let $L \subseteq \mathbb{P}^N$ be the $l$-dimensional linear subspace which is the image $\phi(L)$ of the subbundle $\tilde{L} = \mathbb{P}_{\mathbb{P}^1}(\oplus_{i=0}^l \mathcal{O}_{\mathbb{P}^1}(a_i)) \subseteq \mathbb{P}$. Then

\begin{enumerate}
  
  \item $L \subseteq X$;
  
  \item $\text{Sing} X$ is a subset of $L$, of codimension $2$ if $l \geq 1$, and $\text{Sing} X = \emptyset$ if $l = 0$;
  
  \item $C(X)$ contains $L \setminus \text{Sing} X$.
\end{enumerate}

Hence $\dim C(X) \geq \dim L = l \geq \dim \text{Sing} X + 2$ if $l \geq 1$ and, $\dim C(X) \geq 0$ and $\dim \text{Sing} X = -1$ if $l = 0$. Consequently $\dim C(X) = \dim \text{Sing} X + 2$ for $l \geq 1$.

Proof. Note that $X \setminus L$ is smooth, since $\phi$ gives an embedding of $\mathbb{P} \setminus \tilde{L}$ into $\mathbb{P}^N$ and since $\tilde{X}$ is smooth by the generality of $\tilde{X}$. To see (1) and (2), let us look at $\tilde{X} \cap \tilde{L}$ and the
induced morphism $\tilde{X} \cap \tilde{L} \to L$. Let $s, t$ be the homogeneous coordinates of $\mathbb{P}^1$ and let $z_i$ be the formal basis of $O_{\mathbb{P}^1}(a_i)$ in $E$. Then $\tilde{X}$ is defined as a subscheme of $\mathbb{P}$ by

$$F = \sum_{\mu_0, \ldots, \mu_n \geq 0, \mu_0 + \cdots + \mu_n = \mu} f_{\mu_0, \ldots, \mu_n} z_0^{\mu_0} \cdots z_n^{\mu_n} \in H^0(O_{\mathbb{P}}(\mu) \otimes \tau^*O_{\mathbb{P}^1}(1)), \quad (\mu_i \geq 0,)$$

for some homogeneous polynomials $f_{\mu_0, \ldots, \mu_n} \in \mathbb{k}[s, t]$ of degree $1 + \sum_{i=1}^n a_i \mu_i$. Moreover $\tilde{X} \cap \tilde{L}$ is defined by $F|\tilde{L}$ in $\tilde{L} \cong \mathbb{P}^1 \times L$. Since $\tilde{L}$ is defined by $z_{i+1} = \cdots = z_n = 0$ in $\mathbb{P}$, the degree of $F|\tilde{L}$ with respect to $s$ and $t$ is one, and we may write $F|\tilde{L} = h_1 s + h_2 t$ for some $h_1$ and $h_2 \in \mathbb{k}[z_0, \ldots, z_l]$. Let $W$ be the subscheme of $L$ defined by $h_1 = h_2 = 0$. Then $\text{codim}(W, L) = 2$ if $l \geq 1$, and $W = \emptyset$ if $l = 0$, since $\tilde{X}$ is general and hence $h_1$ and $h_2$ are general. Thus together with $\phi^{-1}(L) = \tilde{L}$ implies that $\tilde{X} \to X$ is one-to-one and unramified at every point of $L \setminus W$ and also $X$ contains $L$. Thus $X \setminus W \subseteq \text{Sm} X$.

To show (2), we will prove $W \subseteq \text{Sing} X$. For a point $x \in W$, take general points $\tilde{x}_1 \neq \tilde{x}_2$ of $\phi^{-1}(x) \cong \mathbb{P}^1$ so that $\tau|\tilde{X}: \tilde{X} \to \mathbb{P}^1$ is unramified at $\tilde{x}_i$. Set $\mathbb{P}^n_i = \tau^{-1}(\tau(\tilde{x}_i))$ and $\mathbb{P}^n = \phi(\mathbb{P}^n_i) \cong \mathbb{P}^n_i$. To look at the dimension of the Zariski tangent space $\Theta_{\tilde{x}_i, X}$ to $X$ at $x$, consider $\Theta_{\tilde{x}_i, X}$ as a subspace of $\Theta_{x, \mathbb{P}^N}$. The space $\Theta_{\tilde{x}_i, X}$ contains the image of $\Theta_{x, X}$ by $\tau$. Then $\dim \Theta_{\tilde{x}_i, X} = \dim \Theta_{x, \mathbb{P}^N} - 2$ and $\Theta_{x, \mathbb{P}^N} \supseteq \Theta_{\tilde{x}_i, X}$. Hence $\Theta_{x, \mathbb{P}^N} \cong \Theta_{x, \mathbb{P}^N}$ and $\Theta_{\tilde{x}_i, X} \cap \Theta_{x, \mathbb{P}^N} = \Theta_{x, \mathbb{P}^N}$ as subspaces of $\Theta_{x, \mathbb{P}^N}$, we have $\dim \Theta_{x, \mathbb{P}^N} = 2(n - 1) - l = n + (n - l - 2)$. If $n \geq l + 3$, this means $x \in \text{Sing} X$. When $n = l + 2$, we take another general point $\tilde{x}_3 \in \phi^{-1}(x)$, and the same argument implies $x \in \text{Sing} X$.

Now (3) is easy: In fact, for $p \in \mathbb{P}^1$, $\tilde{X} \cap \tau^{-1}(p)$ is a hypersurface of degree $\mu \geq 2$ in $\tau^{-1}(p)(\cong \mathbb{P}^n)$, and hence $l(x \cap \langle x, y \rangle) \geq 3$ for every $x \in L \setminus W$ and general $y \in X$. □

Finally we look at the relation between $\text{Sing} X$ and the boundary of $C(X)$.

**Theorem 4.11.** Let $X \subseteq \mathbb{P}^N$ be a nondegenerate, projective variety of dimension $n$ and codimension $e \geq 2$. Let $\Lambda$ be an irreducible component of the closure of $C(X)$, of dimension $l$, which is necessarily linear by Theorem 5.

1. Assume $l \geq 2$. (Hence necessarily $n \geq l + 1 > l \geq 2$.) Then $\dim \text{Sing} X \geq n - 2$, or $\dim \Lambda \cap \text{Sing} X \geq l - 2$.

2. Assume $l \geq 3$. (Hence necessarily $n \geq l + 1 > l \geq 3$.) Then $\dim \Lambda \cap \text{Sing} X \geq l - 3$.

**Proof.** (1). Assume $\dim \Lambda \cap \text{Sing} X \leq l - 3$, and we will show $\dim \text{Sing} X \geq n - 2$. By the assumption, there exists a 2-dimensional subspace $M$ of $\Lambda$ with $M \subseteq \text{Sm} X$, and consequently $M \subseteq C(X)$ by (4.2). Consider the linear projection $\pi_{M, X}: X \setminus M \to \mathbb{P}^{N-3}$ from $M$ and let $\bar{X}$ be the closure of $\pi_{M, X}(X \setminus M)$. Let $x$ be a general point of $X$ and set $\bar{x} := \pi_{M, X}(x)$. Let $X_{\bar{x}}$ be the closure of $\tau_{\bar{x}}^{-1}(\langle \bar{x}, x \rangle \cap (X \setminus M)$ over $\bar{x}$. Since $M \subseteq C(X)$, we have $\dim X_{\bar{x}} = 2$ or 3, and hence $\dim \bar{X} = n - 2$ or $n - 3$. In the latter, $X$ is the cone over $\bar{X}$ with vertex $\Lambda$ and hence $\Lambda \subseteq \text{Sing} X$, contradiction. Thus $\dim \bar{X} = n - 2$. Hence $\dim (T_x(\bar{X}), M) = n + 1$ and $T_{\bar{x}}(X) \subseteq \langle T_x(\bar{X}), M \rangle$ for each
\[ x' \in X_\bar{x} \cap \text{Sm } X \text{ (see (1.2.2))}. \] By Theorem of tangencies ([23], (I.1.7)), \( \dim X_\bar{x} \cap \text{Sing } X \geq \dim X_\bar{x} - \dim (T_\bar{x}(\bar{X}), M) + n - 1 = 0. \) This implies that \( \langle M, x \rangle \cap (X \setminus M) \cap \text{Sing } X \neq \emptyset, \) since \( M \subseteq \text{Sm } X \) and \( X_\bar{x} \setminus M = \langle M, x \rangle \cap (X \setminus M). \) By the generality of \( x \in X, \) Sing \( X \) dominates \( \bar{X}, \) and consequently \( \dim \text{Sing } X \geq n - 2, \) as required.

(2). Assume \( \dim \Lambda \cap \text{Sing } X \leq l - 4 \) to get contradiction. There exists a 3-dimensional subspace \( M' \) of \( \Lambda \) with \( M' \subseteq \text{Sm } X, \) and consequently \( M' \subseteq C(X). \) Consider the linear projection \( \pi_{M',X}: X \setminus M' \rightarrow \mathbb{P}^{N-4} \) from \( M' \) and the closure \( \bar{X} \) of \( \pi_{M',X}(X \setminus M'). \) For a general point \( x \in X, \) let \( X_\bar{x} \) be the closure of \( \pi_{M',X}^{-1}(\bar{x}) \) over \( \bar{x} := \pi_{M',X}(x). \) By the same argument as in (1), \( \dim X_\bar{x} = 3, \) \( \dim \bar{X} = n - 3, \) and \( \dim X_\bar{x} \cap \text{Sing } X \geq 1. \) Since \( X_\bar{x} \subseteq \langle M', x \rangle, \) we have \( X_\bar{x} \cap \text{Sing } X \cap M' \neq \emptyset, \) which contradicts \( M' \subseteq \text{Sm } X. \)

**Remark 4.12.** In Example 4.10, by (4.11), we can relax the assumption \( n \geq l + 2 \) to \( n \geq l + 1 \) if \( l \geq 2. \) Indeed, \( W \supseteq \text{Sing } X \) without the assumption. Since \( \dim W = l - 2, \) we have \( \dim \Lambda \cap \text{Sing } X \geq l - 2 \) by (4.11)(1). Since \( W \) is irreducible by the generality of \( \bar{X} \) and hence by the generality of \( h_1 \) and \( h_2, \) we have \( W = \text{Sing } X = \Lambda \cap \text{Sing } X. \)

**References**


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