On the coefficients of certain family of modular equations

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The definition of the modular equation

- $\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$: the complex upper half plane
- $j(z) = q^{-1} + 744 + 196884q + \cdots$: the elliptic modular function on $SL_2(\mathbb{Z})$ with $z \in \mathcal{H}$ and $q = e^{2\pi iz}$.

Consider a function

$$
\Psi_n(X, z) = \prod_{a > 0 \atop ad = n} \prod_{0 \leq b < d \atop (a, b, d) = 1} (X - j \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z)).
$$

The coefficients of $\Psi_n(X, z)$ are holomorphic modular functions and they are polynomials in $j(z)$. So there exists a polynomial $\Phi_n(X, Y) \in \mathbb{C}[X, Y]$ s.t. $\Phi_n(X, j(z)) = \Psi(X, z)$. 
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The coefficients of \( \Psi_n(X, z) \) are holomorphic modular functions and they are polynomials in \( j(z) \). So there exits a polynomial \( \Phi_n(X, Y) \in \mathbb{C}[X, Y] \) s.t. \( \Phi_n(X, j(z)) = \Psi(X, z) \).
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**Definition + Theorem**

\( \Phi_n(X, Y) \) is said to be the *nth modular equation*.

- \( \Phi_n(X, Y) \) is a polynomial with integral coefficients.
- \( \Phi_n(X, Y) \) is irreducible as a polynomial in \( X \) over \( \mathbb{C}(Y) \).
- \( \Phi_p(X, Y) \) satisfies the Kronecker’s congruences.

But \( \Phi_n(X, Y) \) has very large coefficients even for small \( n \).

\[
\Phi_3(X, Y) = X(X + 2^{15} \cdot 3 \cdot 5^3)^3 + Y(Y + 2^{15} \cdot 3 \cdot 5^3)^3 - X^3Y^3 \\
+ 2^3 \cdot 3^2 \cdot 31X^2Y^2(X + Y) - 2^2 \cdot 3^3 \cdot 9907XY(X^2 + Y^2) \\
+ 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367X^2Y^2 \\
+ 2^{16} \cdot 3^5 \cdot 5^3 \cdot 17 \cdot 263XY(X + Y) \\
- 2^{31} \cdot 5^6 \cdot 22973XY.
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Introduction

The modular equation and the main theorem

Proof of theorem 1

The definition of the modular equation

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The growth of coefficients of modular equation

- \( P(X_1, \cdots, X_r) \in \mathbb{C}[X_1, \cdots, X_r] \): a nonzero polynomial
- \( h(P(X_1, \cdots, X_r)) \): the logarithmic height of \( P(X_1, \cdots, X_r) \) defined by the logarithm of the maximum of the absolute values of its coefficients
- \( f, g \): complex valued functions defined on some set \( S \)
- \( h \): a real valued positive function on \( S \)
- \( f = g + O(h) \): there exists an absolute positive constant \( A \) such that \( |f - g| \leq A \cdot h \) on \( S \).
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The growth of coefficients of modular equation

**Theorem (K. Mahler, 1972)**

\[ h(\Phi_{2^n}(X, Y)) \leq 2^n (36n + 57) \log 2. \]

**Theorem (P. Cohen, 1984)**

Let \( \psi(n) = n \prod_{p|n} (1 + \frac{1}{p}) \). Then we have

\[ h(\Phi_n(X, Y)) = 6\psi(n) \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right\}. \]
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The modular equation for $\Gamma(5)$

- $\Gamma(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{5} \right\}$
- $j_5(z) = q^{-1/5}(1 + q - q^3 + q^5 + \cdots)$: a Hauptmodul of $\Gamma(5)$.

Note that

$$\frac{1}{j_5(z)} = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}},$$

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- $\Phi_{n}^{j_5}(X, Y)$: the $n$th modular equation for $j_5(z)$ for $(n, 5) = 1$.

**Theorem**

- $\Phi_{n}^{j_5}(X, Y)$ satisfies $\Phi_{n}^{j_5}(j_5(z), j_5(nz)) = 0$.
- $\Phi_{n}^{j_5}(X, Y)$ is a polynomial with integral coefficients.
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$$\Phi_{3}^{j_5}(X, Y) = X^4Y^3 + X^3 - 3X^2Y^2 - XY^4 - Y.$$
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### The growth of coefficients of modular equation for $\Gamma(5)$

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<tr>
<td>32</td>
<td>$2^{12}3^{144}5^{144}11^{72}17^{18}23^{36}29^{36}47^{27}53^{18}59^{18}71^{9}83^{18}89^{18}$</td>
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**Theorem (B. Cais and B. Conrad, 2006)**

For a positive integer $n$ with $(n, 5) = 1$, we have

$$h(\Phi_n^{j_5}(X, Y)) = \frac{1}{10}\psi(n) \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right\}.$$ 

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For a positive integer $n$ with $(n, 5) = 1$, we have

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Note that $[\Gamma(1) : \Gamma(5)] = 60$. 
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The main question

Question

- $\Gamma$: a genus zero discrete subgroup of $SL_2(\mathbb{R})$
- $f(z)$: a Hauptmodul of $\Gamma$

When can we define the modular equation $\Phi^f_n(X, Y)$?
If so, can we show

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Congruence subgroups

For a positive integer \( N \), we define congruence subgroups

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}
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The modular equation of classical congruence subgroup

- $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$: a genus zero subgroup
- $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$: a Hauptmodul of $\Gamma$ with $a_n \in \mathbb{R}$. 
The modular equation of classical congruence subgroup

Definition + Theorem

For an integer $n$ with $(n, mN) = 1$ we define the $n$th modular equation as

$$\Phi_n^f(X, f(z)) = f(z)^{rn} \cdot \prod_{a > 0 \atop ad = n} \prod_{0 \leq b < d \atop (a, b, d) = 1} \left( X - f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right)(z).$$

where $\sigma_a \in SL_2(\mathbb{Z})$ s.t $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod mN$.

If $a_n \in \mathbb{Z}$, then $\Phi_n^f(X, Y) \in \mathbb{Z}[X, Y]$ and $\Phi_p^f(X, Y)$ satisfies the Kronecker’s congruences.
Note that $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$.

If $\Gamma' = \Gamma_0(N_1) \cap \Gamma_0^0(N_2) \cap \Gamma_1(N_3) \cap \Gamma_1^1(N_4) \cap \Gamma(N_5)$ is an arbitrary intersection of classical congruence subgroups, then we have

$$\alpha^{-1}\Gamma'\alpha = \Gamma_1(N) \cap \Gamma_0(mN)$$

where $N = \text{lcm}(N_3, N_4, N_5)$ and

$$\alpha = \begin{pmatrix} \text{lcm}(N_2, N_4, N_5) & 0 \\ 0 & 1 \end{pmatrix}, m = \frac{\text{lcm}(N_1, N_3, N_5)\text{lcm}(N_2, N_4, N_5)}{N}.$$
The modular equation of classical congruence subgroup

\( \Gamma' = \Gamma_0(N_1) \cap \cdots \cap \Gamma(N_5) : \) a genus zero subgroup.

\( g(z) = q_h^{-1} + \sum_{n=0}^{\infty} a_n q^n : \) a Hauptmodul of \( \Gamma' \) \( (q_h = e^{2\pi iz/h}) \)

\[ \Rightarrow \Gamma = \Gamma_1(N) \cap \Gamma_0(mN) : \) a genus zero subgroup.

\[ \Rightarrow f(z) := (g \circ \alpha)(z) = g(hz) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n \) is a Hauptmodul of \( \Gamma. \)

The \( n \)th modular equation \( \Phi^g_n(X, Y) \) for \( g(z) \) is irreducible as a polynomial in \( X \) over \( \mathbb{C}(Y) \) satisfying \( \Phi^g_n(g(z), g(nz)) = 0. \)

\[ \Rightarrow \Phi^g_n(g(hz), g(hnz)) = \Phi^g_n(f(z), f(nz)) = 0. \]

\[ \Rightarrow \Phi^f_n(X, Y) = \Phi^g_n(X, Y). \]
The modular equation of classical congruence subgroup

\begin{itemize}
  \item $\Gamma' = \Gamma_0(N_1) \cap \cdots \cap \Gamma(N_5)$: a genus zero subgroup.
  \item $g(z) = q_h^{-1} + \sum_{n=0}^{\infty} a_n q^n_h$: a Hauptmodul of $\Gamma'$ ($q_h = e^{2\pi iz/h}$)
\end{itemize}

$\Rightarrow \Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$: a genus zero subgroup.

$\Rightarrow f(z) := (g \circ \alpha)(z) = g(hz) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ is a Hauptmodul of $\Gamma$.

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$\Rightarrow \Phi_n^f(X, Y) = \Phi_n^g(X, Y)$.
The modular equation of classical congruence subgroup

For an example, since

\[
\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(5) \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_1(5) \cap \Gamma_0(25)
\]

and \( f(z) := j_5(5z) \) is a Hauptmodul of \( \Gamma_1(5) \cap \Gamma_0(25) \), we have the same \( n \)th modular equations \( \Phi_{j_5}^n(X, Y) = \Phi^n_f(X, Y) \) when \((n, 5) = 1\).
The modular equation for noncongruence subgroup

- $N > 1$: an integer
- $e$: a Hall divisor of $N$ (a positive divisor of $N$ s.t $(e, N/e) = 1$).

**Definition**

For a Hall divisor $e$ of $N$, an Atkin-Lehner involution of $\Gamma_0(N)$ is a matrix with determinant $1$ of the form

\[
\begin{pmatrix}
    a\sqrt{e} & b/\sqrt{e} \\
    cN/\sqrt{e} & d\sqrt{e}
\end{pmatrix}
\]

where $a, b, c, d \in \mathbb{Z}$. 

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where $a, b, c, d \in \mathbb{Z}$. 
The modular equation for noncongruence subgroup

- $W_e$ : the set of all Atkin-Lehner involutions with a fixed Hall divisor $e$ of $N$.
- These sets satisfy the multiplication rule:

$$W_e W_f = W_f W_e = W_k \text{ where } k = \frac{e}{(e,f)} \cdot \frac{f}{(e,f)}.$$ 

- $S$ : a subset of the Hall divisors of $N$ closed under the above multiplication rule.
- $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ : the subgroup of $SL_2(\mathbb{R})$ generated by all elements of $\Gamma_0(N)$ and $W_e$ for all $e \in S$. 

The modular equation for noncongruence subgroup

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The modular equation for noncongruence subgroup

Definition + Theorem (I. Chen and N. Yui, 1996)

Suppose $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ has genus zero.

- $\exists!$ a Hautmodul $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ s.t. $a_n \in \mathbb{Z}$.

- For a positive integer $n$ prime to $N$, the $n$-th modular equation $\Phi_n^f(X, Y) = 0$ can be defined as

$$
\Phi_n^f(X, f(z)) = \prod_{a>0} \prod_{0 \leq b < d} \prod_{ad=n, (a,b,d)=1} \left( X - f \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z) \right).
$$

- $\Phi_n^f(X, Y)$ has integral coefficients and is irreducible as a polynomial in $X$ over $\mathbb{C}(Y)$. 
The main theorem 1 (Cho, Kim and Park)

Let \( f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n \) be a Hauptmodul of \( \Gamma = \Gamma_1(N) \cap \Gamma_0(mN) \) with \( a_n \in \mathbb{R} \). For a positive integer \( n \) with \( (n, mN) = 1 \), we have

\[
h(\Phi^f_n(X, Y)) = \frac{6\psi(n)}{[\Gamma(1) : \Gamma]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right\}
\]

and

\[
\lim_{n \to \infty, (n, mN) = 1} \frac{h(\Phi^f_n(X, Y))}{h(\Phi^j_n(X, Y))} = \frac{1}{[\Gamma(1) : \Gamma]}.
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The main theorem

Let $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ be a Hauptmodul of $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ with $a_n \in \mathbb{R}$. For a positive integer $n$ with $(n, N) = 1$, we have

$$h(\Phi_n^f(X, Y)) = \sum_{e \in S} \frac{6\psi(n)}{[\Gamma(1) : \Gamma_0(N/e)]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right\}$$

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\]
Lemma

Γ (and $< \Gamma_0(N), W_e >_{e \in S}$) have no elliptic points on $i\mathbb{R}_{>1}$.

We have assumed that $f(z) = q^{-1} + \cdots$ has real Fourier coefficients.

$\Rightarrow$ $f(it)$ is real and $f(it) \to \infty$ as $t \to \infty$

$\Rightarrow$ $f'(z)$ is nonvanishing on $i\mathbb{R}_{>1}$ by Lemma.

$\Rightarrow$ $f(it)$ is strictly increasing for $t \geq 1$.

$\Rightarrow$ We can choose real numbers $s > 1$ and $1 \leq t_0 \leq t_1$ such that $f(it_0) = s$, $f(it_1) = 2s$. 
Proof of the main theorem 1

Lemma

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We have assumed that \(f(z) = q^{-1} + \cdots\) has real Fourier coefficients.

⇒ \(f(it)\) is real and \(f(it) → ∞\) as \(t → ∞\)

⇒ \(f'(z)\) is nonvanishing on \(i\mathbb{R}_{>1}\) by Lemma.

⇒ \(f(it)\) is strictly increasing for \(t ≥ 1\).

⇒ We can choose real numbers \(s > 1\) and \(1 ≤ t_0 ≤ t_1\) such that \(f(it_0) = s, f(it_1) = 2s\).
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\[ \Rightarrow f(it) \text{ is real and } f(it) \to \infty \text{ as } t \to \infty \]
\[ \Rightarrow f'(z) \text{ is nonvanishing on } i\mathbb{R}_{>1} \text{ by Lemma.} \]
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Lemma 1

Let $1 \leq t_0 \leq t \leq t_1$. We have

$$h(\Phi_n(X,f(it))) = \frac{6\psi(n)}{[\Gamma(1) : \Gamma]} \left( \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right),$$

Lemma (P. Cohen)

Let $P(X) \in \mathbb{C}[X]$ be any nonzero polynomial of degree $\leq D$. Then for any $s > 0$, there exists an absolute constant $c_s > 0$, depending only on $s$, such that

$$\left| (h(P(X))) - \log \sup_{s \leq x \leq 2s} |P(x)| \right| \leq c_s D.$$
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Proof of the main theorem 1. Let $D = \psi(n)$. If we write
\[ \Phi^f_n(X, Y) = P_0(Y)X^D + P_1(Y)X^{D-1} + \cdots + P_D(Y), \]
then \( h(\Phi^f_n(X, Y)) = \max_{0 \leq j \leq D} h(P_j(Y)) \). Since \( \deg P_j(Y) \leq D \),

\[
h(\Phi^f_n(X, Y)) = \max_{0 \leq j \leq D} \log \sup_{s \leq y \leq 2s} |P_j(y)| + \mathcal{O}(D) = \sup_{s \leq y \leq 2s} \max_{0 \leq j \leq D} \log |P_j(y)| + \mathcal{O}(D) = \sup_{s \leq y \leq 2s} h(\Phi^f_n(X, y)) + \mathcal{O}(D) = \sup_{t_0 \leq t \leq t_1} h(\Phi^f_n(X, f(it))) + \mathcal{O}(D),
\]

because the interval \([t_0, t_1]\) corresponds bijectively to the interval \([s, 2s]\) \(\square\)
Proof of the main theorem 1. Let $D = \psi(n)$. If we write
\[ \Phi^f_n(X, Y) = P_0(Y)X^D + P_1(Y)X^{D-1} + \cdots + P_D(Y), \]
then
\[ h(\Phi^f_n(X, Y)) = \max_{0 \leq j \leq D} h(P_j(Y)). \]
Since $\deg P_j(Y) \leq D$,
\[
\begin{align*}
    h(\Phi^f_n(X, Y)) & = \max_{0 \leq j \leq D} \log \sup_{0 \leq j \leq D} |P_j(y)| + O(D) \\
                    & = \sup_{s \leq y \leq 2s} \max_{0 \leq j \leq D} \log |P_j(y)| + O(D) \\
                    & = \sup_{s \leq y \leq 2s} h(\Phi^f_n(X, Y)) + O(D) \\
                    & = \sup_{t_0 \leq t \leq t_1} h(\Phi^f_n(X, f(it))) + O(D),
\end{align*}
\]
because the interval $[t_0, t_1]$ corresponds bijectively to the interval $[s, 2s] \square$
Thank you!
Proof of lemma 1

For \( t_0 \leq t \leq t_1 \), we have

\[
 h\left( \Phi_f^n(X, f(it)) \right) = \sum_{a>0} S_d(t) + \mathcal{O}(\psi(n)),
\]

where

\[
 S_d(t) = \sum_{0 \leq b < d, (a,b,d)=1} \log \max\{1, |f \circ \sigma_a \left( \frac{ait+b}{d} \right)|\}
\]

**Proof.** The coefficients of polynomial \( P(x) = (x - w_1) \cdots (x - w_d) \) lie between \( 2^{-d}M \) and \( 2^dM \) where \( M = \prod_{j=1}^d \max\{1, |w_j|\} \).

\[
 \Rightarrow h(P) = \sum_{j=1}^d \log \max\{1, |w_j|\} + \mathcal{O}(d).
\]

\[
 \therefore \Phi_f^n(X, f(it)) = f(z)^r_n \prod_{a>0} \prod_{ad=n} \prod_{0 \leq b < d, (a,b,d)=1} \left( X - (f \circ \sigma_a) \left( \frac{ait+b}{d} \right) \right)
\]

\[
 \Rightarrow h\left( \Phi_f^n(X, f(it)) \right) = r_n \log f(it)
\]

\[
 + \sum_{a>0} \sum_{ad=n} \sum_{0 \leq b < d} \log \max \{1, |f \circ \sigma_a \left( \frac{ait+b}{d} \right)|\} + \mathcal{O}(\psi(n)),
\]

where \( r_n \log f(it) = \mathcal{O}(\psi(n)) \). \( \square \)
Proof of lemma 1

Next goal is to calculate each term in the summation

\[ S_d(t) = \sum_{0 \leq b < d \atop (a,b,d)=1} \log \max \left\{ 1, |f \circ \sigma_a \left( \frac{ait + b}{d} \right)| \right\}. \]

Lemma

For \( z = \xi + i\eta \in \mathcal{H} \), let \( g(z) = a_{-1}q_h^{-1} + \sum_{n=0}^{\infty} a_n q_h^n \) with \( q_h = e^{2\pi i z/h} \) for a positive integer \( h \). We assume that if \( a_{-1} = 0 \) (respectively, \( a_{-1} \neq 0 \)), then \( g(z) \) (respectively, \( q_h g(z) \)) is absolutely convergent for \( \eta > 0 \). Then for \( \eta \geq 1/2 \), we have

\[ \log \max\{1, |g(z)|\} = \begin{cases} \mathcal{O}(1) & \text{if } a_{-1} = 0, \\ 2\pi i\eta/h + \mathcal{O}(1) & \text{if } a_{-1} \neq 0. \end{cases} \]
Proof of lemma 1

It is necessary to study the behavior of Hauptmodul at each cusp of \( \Gamma = \Gamma_1(N) \cap \Gamma_0(mN) \) (or \( < \Gamma_0(N), W_e >_{e \in S} \)).

**Lemma**

Let \( \Gamma = \Gamma_1(N) \cap \Gamma_0(mN) \) and

\[
\Delta = \{ \pm (1 + Nk) \in (\mathbb{Z}/mN\mathbb{Z})^\times \mid k = 0, \cdots, m - 1 \}.
\]

We assume that \( a, c, a', \) and \( c' \) are integers such that \((a, c) = (a', c') = 1\). By \( \pm \frac{1}{0} \) we mean \( \infty \). Then the cusp \( \frac{a}{c} \) is equivalent to \( \frac{a'}{c'} \) under \( \Gamma \) if and only if there exist \( x \in \Delta \) and \( n \in \mathbb{Z} \) such that

\[
\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \begin{pmatrix} x^{-1}a + nc \\ xc \end{pmatrix} \mod mN.
\]
Proof of lemma 1

Let $M$ be a positive integer. P. Cohen proved that

$$I_M = \left[ \frac{1}{M+1}, \frac{M+2}{M+1} \right) = \bigcup_{k=1}^{M} \bigcup_{\substack{h=1 \\ (h,k)=1}}^{k} I_M(h/k),$$

which is a disjoint union of sets $I_M(h/k)$. Here each $I_M(h/k)$ is an interval of the form $[\rho_1^{(h/k)}, \rho_2^{(h/k)})$ containing $h/k$ and

$$\frac{1}{2Mk} \leq \frac{1}{(M+1)k} < \frac{1}{(M+1)k} \leq h/k < \frac{1}{2Mk} \leq \rho_2^{(h/k)} - h/k \leq \frac{1}{(M+1)k}.$$
Proof of lemma 1

Note that we may reindex the sum in $S_d(t)$ via

$$b \rightarrow \begin{cases} 
    b & \text{if } \frac{b}{d} \in \left[ \frac{1}{N+1}, 1 \right), \\
    b + d & \text{if } \frac{b}{d} \in \left[ 0, \frac{1}{N+1} \right).
\end{cases}$$

For real numbers $h, k$ and $x$, we put

$$g_{h,k}(x) = \frac{2\pi nt / d^2 k^2}{(\frac{at}{d})^2 + (x - \frac{h}{k})^2}.$$
Lemma

Let $T(t, z, b, d) = \log \max \{1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \}$.

(1) If $at/d \geq 1/2$, then

$$T(t, z, b, d) = \begin{cases} 
2\pi nt/d^2 + O(1) & \text{if } \bar{a} \in \Delta, \\
O(1) & \text{otherwise.}
\end{cases}$$

(2) Put $M = \lfloor d/\sqrt{nt} \rfloor$. If $at/d \leq 1$, then $M \geq 1$ and, for $b/d \in I_M(h/k)$, then

$$T(t, z, b, d) = \begin{cases} 
g_{h,k}(b/d) + O(1) & \text{if } k \equiv 0 \mod mN \text{ and } \bar{h} \in \bar{a}\Delta, \\
O(1) & \text{otherwise.}
\end{cases}$$
Now, we calculate $S_d(t)$ more precisely. To this end we need the following lemma.

Lemma

Let $k, j$ and $a$ be positive integers satisfying $j|k$ and $(j, a) = 1$. We further let $ζ$ be a primitive $k$th root of unity and let

$$c'_k(l) = \sum_{\substack{h \in (\mathbb{Z}/k\mathbb{Z}) \times \\ h \equiv a \mod j}} ζ^{hl} \text{ for } l \in \mathbb{Z}.$$ 

Then

$$|c'_k(l)| \leq j \cdot (k, l) \text{ for any } l \in \mathbb{Z}.$$ (1)
Lemma 2

(1) If $d < \sqrt{nt}$, then $S_d(t) = O(n/d)$.
(2) If $d \geq \sqrt{nt}$, then

$$S_d(t) = \frac{1}{[\Gamma(1) : \Gamma]} \cdot \frac{6d}{(a, d)} \phi((a, d)) \log(d^2/n)$$

$$+ O \left( \sigma_1 \left( \frac{d}{(a, d)} \right) \right) + O \left( \frac{d\sigma_1((a, d))}{(a, d)} \right),$$

where $\phi(x)$ is the Euler function and $\sigma_1(x)$ is the sum of positive divisors of $x$. 

Proof of lemma 1
Sketch of proof of lemma 2

Proof of lemma 2.(2) Note that the assumption $d \geq \sqrt{nt}$ implies $\frac{at}{d} \leq 1$. Put $M = \left\lfloor \frac{d}{\sqrt{nt}} \right\rfloor \geq 1$. Then we have

$$S_d(t) = \sum_{k=1}^{M} \sum_{\substack{h=1 \atop (h,k)=1}}^{k} \sum_{\substack{b/d \in I_M(h/k) \atop 0 \leq b < d \atop (a,b,d)=1}} \log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait + b}{d} \right) \right| \right\}$$

$$= \sum_{1 \leq h \leq k \leq M \atop (h,k)=1} \sum_{\substack{b/d \in I_M(h/k) \atop 0 \leq b < d \atop (a,b,d)=1}} g_{h,k}(b/d) + O(d).$$
Sketch of proof of lemma 2

Take

\[ \sum_{b/d \in IM(h/k)} g_{h,k}(b/d) = k^{-2} \sum_{f|(a,d)} \mu(f)F_f \left( \frac{dh}{fk} \right) + O \left( \frac{\sqrt{n\sigma_1((a,d))}}{k(a,d)} \right), \]

where \( F_f(\theta) = \frac{2\pi^2 d}{f} \sum_{v \in \mathbb{Z}} e^{-2\pi |v|nt/df} e^{2\pi iv\theta} \) and \( \mu(x) \) is the Möbius function.

\[ S_d(t) = \sum_{f|(a,d)} \mu(f) \sum_{1 \leq h \leq k \leq M} \sum_{\substack{h \equiv 0 \mod mN \\ h \in \Delta
\frac{n\sigma_1((a,d))}{(a,d)}}.} \]
We now consider the sum

$$\sum_{1 \leq h \leq k \leq M \atop (h,k)=1 \atop k \equiv 0 \mod mN \atop \overline{h} \in \overline{a} \Delta} k^{-2} F_f(dh/fk) = \frac{2\pi^2 d}{f} \sum_{v \in \mathbb{Z}} C_M(dv/f) e^{-2\pi |v|nt/df}, \quad (2)$$

where

$$C_M(l) = \sum_{1 \leq k \leq M \atop k \equiv 0 \mod mN} k^{-2} c_k(l)$$

and

$$c_k(l) = \sum_{1 \leq h \leq k \atop (h,k)=1 \atop \overline{h} \in \overline{a} \Delta} e^{2\pi i hl/k} \text{ for any } l \in \mathbb{Z}.$$
We have to calculate $C_M(l)$ and $c_k(l)$ to know the upper bound of the sum of (2). By lemma, $|c_k(l)| \leq |\Delta|mN(k, l)$ for $l \in \mathbb{Z} - \{0\}$. So when $l \neq 0$, we have

$$|C_M(l)| \leq |\Delta|mN \sum_{k=1}^{\infty} k^{-2}(k, l) \leq |\Delta|mN \sum_{d|l} d \sum_{j=1}^{\infty} \frac{1}{j^2 d^2}$$

$$= |\Delta|mN \frac{\pi^2}{6} \sum_{d|l} \frac{|l|}{d} = |\Delta|mN \frac{\pi^2}{6} \sigma_1(|l|).$$

So $|C_M(l)| = O\left(\frac{\sigma_1(|l|)}{|l|}\right)$ for $l \neq 0$. 
For \( l = 0 \), \( \pi : (\mathbb{Z}/k\mathbb{Z})^\times \to (\mathbb{Z}/mN\mathbb{Z})^\times \) gives us

\[
c_k(0) = |\pi^{-1}(\Delta)| = |\Delta||\ker \pi| = |\Delta| \frac{\phi(k)}{\phi(mN)}.
\]

\[
C_M(0) = \sum_{\substack{1 \leq k \leq M \\ k \equiv 0 \mod mN}} k^{-2} \frac{|\Delta|}{\phi(mN)} \phi(k)
\]

\[
= \frac{6}{\pi^2} \frac{|\Delta|}{\phi(mN)[\Gamma(1) : \Gamma_0(mN)]} \log M + \mathcal{O}(1)
\]

\[
= \frac{6}{\pi^2[\Gamma(1) : \Gamma]} \log M + \mathcal{O}(1),
\]

because

\[
[\Gamma(1) : \Gamma] = [\Gamma(1) : \Gamma_0(mN)][\Gamma_0(mN) : \Gamma] = \frac{[\Gamma(1) : \Gamma_0(mN)]\phi(mN)}{|\Delta|}.
\]
Therefore we get

\[
\sum_{1 \leq h \leq k \leq M \atop (h,k)=1} k^{-2} F_f(\frac{dh}{fk}) = \frac{12d}{f[\Gamma(1) : \Gamma]} \log M + \mathcal{O}(d/f)
\]

So we obtain

\[
S_d(t) = \sum_{f|\gcd(a,d)} \mu(f) \left( \frac{6d}{f[\Gamma(1) : \Gamma]} \log(d^2/n) + \mathcal{O}(d/f) \right) + \mathcal{O} \left( \sigma_1(\frac{d}{f}) e^{-2\pi n/df} \right) + \mathcal{O} \left( \frac{d\sigma_1((a,d))}{(a,d)} \right).
\]

□
Proof of lemma 1. We have shown that

$$h(\Phi^f_n(X, f(it))) = \sum_{a > 0, ad = n} S_d(t) + O(\psi(n)) = H_1 + H_2 + O(\psi(n)),$$

where

$$H_1 = \sum_{a > 0, ad = n, \frac{d}{n} < \sqrt{n}} S_d(t) = O(\psi(n))$$

and

$$H_2 = \sum_{a > 0, ad = n, d \geq \sqrt{n}} S_d(t) = \frac{6\psi(n)}{[\Gamma(1) : \Gamma]} \left( \log n - 2 \sum_{p|n} \frac{\log p}{p} + O(1) \right).$$