In this paper, we present arc-length estimations for quadratic rational Bézier curves using the length of polygon and distance between both end points. Our arc-length estimations coincide with the arc-length of the quadratic rational Bézier curve exactly when the weight \( w \) is 0, 1 and \( 1 \). We show that for all \( w > 0 \) our estimations are strictly increasing with respect to \( w \). Moreover, we find the parameter \( \mu^* \) which makes our estimation coincide with the arc-length of the quadratic rational Bézier curve when it is a circular arc too. We also show that \( \mu^* \) has a special limit, which is used for optimal estimation. We present some numerical examples, and the numerical results illustrates that the estimation with the limit value of \( \mu^* \) is an optimal estimation.

1. Introduction

Quadratic rational Bézier curve has been widely used in CAD/CAM and Solid Modeling [8, 16, 19]. It is also called by conic section. Its arc-length can not be obtained algebraically, while the arc-length of quadratic Bézier curve can be obtained[6, 17]. To find the arc-length of quadratic rational Bézier curve is an important problem in CAGD(Computer Aided Geometric Design), Geometric Modeling[20, 21, 22, 23, 26] and Geodesy on spheroid[7, 18, 24, 25].

Gravesen[14] presented a good estimation for the arc-length of Bézier curve of degree \( n \) as 

\[
L_G = \frac{2}{n+1}L_b + \frac{n-1}{n+1}L_p
\]

where \( L_p \) is the length of control polygon and \( L_b \) is the distance between both end control points of Bézier curve. He showed that his approximate has the approximation order five and extended his result to the rational case[15] and triangular surface case. Roulier and Piper[21, 22, 23] presented theorems and algorithms which produce polynomial/rational Bézier or parametric curves having specified arc-length subject to certain constraints. Floater et al.[11, 12, 13] presented estimations of arc-length for a parametric curve having high approximation order using only samples of points.
The estimation method for arc-length of the quadratic rational Bézier curves is well-known, but no previous method reproduces the arc-length of special shapes of quadratic rational Bézier curves, e.g. quadratic Bézier curves or circular arcs.

We present the arc-length estimations for the quadratic rational Bézier curves reproducing the arc-length of special shapes. Our estimations are greatly inspired by Gravesen’s method. The arc-length of quadratic rational Bézier curve for given control polygon is strictly increasing as the weight $w > 0$ is increasing. We show that our arc-length estimations have the same property. Also, the estimations have a parameter $\mu$ which is used to reproduce the arc-length of special shape of the quadratic rational Bézier curve, for example, line segment($w = 0$), parabola($w = 1$), or control polygon($w = \infty$). Moreover, we present the parameter $\mu^*$ which makes our estimation coincides with the arc-length of the quadratic rational Bézier curve when it is a circular arc too. We find the limit of $\mu^*$ under symmetry of quadratic rational Bézier curve. Using some numerical examples, we analyze the quality of our arc-length estimations with respect to the parameter $\mu$. It illustrates the limit of $\mu^*$ is the optimal estimation of our estimations.

The remainder of this paper is structured as follows. In Section 2, we present our arc-length estimations which coincide with the arc-length of quadratic rational Bézier curve for $w = 0, 1, \infty$. In section 3, The arc-length estimations coinciding with the length of circular arc simultaneously are investigated. We give some numerical examples and state optimal estimation of our estimations in Section 4.

2. ARC-LENGTH ESTIMATIONS FOR QUADRATIC RATIONAL BÉZIER CURVE

In this section we present arc-length estimation for quadratic rational Bézier curve using the polygonal length, the distance between both end points, and weight. The Bézier curve of
degree $n$ with control points $b_i$, $i = 0, \ldots, n$ is defined by

$$b(t) = \sum_{i=0}^{n} b_i B_i^n(t), \quad t \in [0, 1]$$

where $B_i^n(t)$, $(i = 0, \ldots, n)$ is the Bernstein polynomial $B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$. The arc-length estimation of Gravsen[14] for the Bézier curve of degree $n$ is

$$L_p = \sum_{i=0}^{n-1} |\Delta b_i| \quad \text{and} \quad L_b = |\Delta b_n|.$$ 

where $\Delta b_i = b_{i+1} - b_i$, $(i = 0, \ldots, n-1)$, $\Delta b_n = b_0 - b_n$, and $|\mathbf{x}|$ is the length of the vector $\mathbf{x}$. Quadratic rational Bézier curve with the control points $b_i$, $(i = 0, 1, 2)$ can be defined in standard form[1, 3, 9, 10]

$$r(t) = \frac{b_0 B_0^2(t) + w b_1 B_1^2(t) + b_2 B_2^2(t)}{B_0^2(t) + w B_1^2(t) + B_2^2(t)},$$

where $w > 0$ is the weight associated with $b_1$. We assume that the control points are not collinear in this paper. Let $L(w)$ be the arc-length of quadratic rational Bézier curve of the weight $w$ for given control polygon $b_i$, $(i = 0, 1, 2)$. The closer $w$ is to zero, the closer $r(t)$ is to the line segment $b_0b_2$, and the closer $w$ is to infinity, the closer $r(t)$ is to the control polygon $b_0, b_1, b_2$, as shown in Figure 1. Thus by the limits of $L(w)$ as $w \to 0$ and $w \to \infty$, we have $L(0) = |\Delta b_2| = L_b$ and $L(\infty) = |\Delta b_0| + |\Delta b_1| = L_p$.

Now, we present the arc-length estimation $\lambda_\mu$ for the quadratic Bézier curve $r(t)$, $t \in [0, 1]$ with the parameter $\mu \in \mathbb{R}$ as follows, which is an extension of the Gravsen’s estimation for quadratic Bézier curve ($n = 2$):

$$\lambda_\mu(r) = \frac{\mu}{\mu + w} L_b + \frac{w}{\mu + w} L_p. \quad (2.2)$$

Since the quadratic rational Bézier curve $r$ depends on the control polygon $b_i$, $(i = 0, 1, 2)$ and the weight $w$, we can write $\lambda_\mu(r) = \lambda_\mu(b_i, w)$. The estimation $\lambda_2(b_i, 1)$ is equal to that of Gravsen for quadratic case, $n = 2$, in Equation (2.1).

**Proposition 2.1.** The arc-length estimation $\lambda_\mu(b_i, w)$ coincides with the arc-length of the quadratic rational Bézier curve $r(t)$ for $w = 0, \infty$, and it is strictly increasing with respect to $w$ for any control polygon $b_0, b_1, b_2$ and for all positive $\mu > 0$.

**Proof.** Equation (2.2) yields $\lambda_\mu(b_i, 0) = L_b$ and $\lambda_\mu(b_i, \infty) = L_p$, so $\lambda_\mu$ coincides with the arc-length $L(w)$ for $w = 0, \infty$. By the derivative

$$\frac{\partial \lambda_\mu(b_i, w)}{\partial w} = \frac{\mu(L_p - L_b)}{(\mu + w)^2}$$
and by the inequality \( L_p > L_b \), \( \lambda_\mu(b_i, w) \) is strictly increasing with respect to \( w \). \( \square \)

The arc-length of quadratic Bézier curve with control points \( b_i, i = 0, 1, 2 \) is well-known \([6, 17]\) as
\[
L_q = \frac{\beta_1 |\Delta b_1| + \beta_3 |\Delta b_0|}{2\beta_2} + \frac{|\Delta b_0 \times \Delta b_1|^2}{8(\beta_2)^{3/2}} \cdot \ln \left( \frac{|\Delta b_1|\sqrt{\beta_2} + \beta_1}{|\Delta b_0|\sqrt{\beta_2} - \beta_3} \right)
\]
where
\[
\beta_i = \frac{1}{4} i |\Delta b_0|^2 + (4 - i) |\Delta b_1|^2 - |\Delta b_2|^2
\]
for \( i = 1, 2, 3 \). If we take \( \mu = \hat{\mu} \) where
\[
\hat{\mu} = \frac{L_p - L_q}{L_q - L_b},
\]
then the estimation \( \lambda_\mu(b_i, w) \) coincide with the arc-length \( L(w) \) for \( w = 1 \), and Equation (2.2) yields
\[
\lambda_\mu(b_i, w) = \frac{w L_p(L_q - L_b) + L_b(L_p - L_q)}{w(L_q - L_b) + L_p - L_q}.
\]
(2.4)

It follows from \( L_b < L_q < L_p \) that \( \hat{\mu} > 0 \), for all \( w > 0 \). For the symmetric quadratic rational Bézier curve \( r(t) \), i.e., \( |\Delta b_0| = |\Delta b_1| \), the value \( \hat{\mu} \) has the special limit as follows.

**Proposition 2.2.** If the quadratic rational Bézier curve \( r(t) \) is symmetric, then
\[
\lim_{b_1 \to (b_0 + b_2)/2} \hat{\mu} = 2 \quad \text{and} \quad \lim_{b_1 \to \infty} \hat{\mu} = 1
\]
independently of the length \( |\Delta b_2| \).

**Proof.** Let \( h = |b_1 - (b_0 + b_2)/2| \). For sufficiently small \( h > 0 \), the following series expansions hold.
\[
L_b = |\Delta b_2|
\]
\[
L_p = 2|\Delta b_0| = 2\sqrt{h^2 + (|\Delta b_2|/2)^2} \approx |\Delta b_2| + \frac{2h^2}{|\Delta b_2|} + O(h^4)
\]
\[
L_q \approx |\Delta b_2| + \frac{2h^2}{3|\Delta b_2|} + O(h^4)
\]
(2.5)
\[
\frac{L_p - L_q}{L_q - L_b} \approx 2 - \frac{6}{5} \frac{h^2}{|\Delta b_2|^2} + O(h^4).
\]
Thus we have the limit
\[
\lim_{b_1 \to (b_0 + b_2)/2} \hat{\mu} = 2.
\]
Figure 2. Graph of $\hat{\mu}$ for the symmetric quadratic Bézier curve, i.e. $|\Delta b_0| = |\Delta b_1|$: The horizontal axis is the distance between $b_1$ and $(b_0 + b_2)/2$.

Also, for sufficiently large $h > 0$, we have following series expansions.

\[ L_p = 2|\Delta b_0| = 2h + \frac{|\Delta b_2|^2}{4h} + \mathcal{O} \left( \frac{1}{h^3} \right) \]

\[ L_q \approx h + \frac{1}{8}|\Delta b_2|^2 \left(1 + 4 \ln 2 - 2 \ln |\Delta b_2| - 2 \ln h \right) \frac{1}{h} + \mathcal{O} \left( \frac{1}{h^3} \right) \]  \quad (2.6)

\[ \frac{L_p - L_q}{L_q - L_b} \approx 1 + \frac{|\Delta b_2|}{h} + \mathcal{O} \left( \frac{1}{h^2} \right) \]

which yield

\[ \lim_{b_1 \to \infty} \hat{\mu} = 1. \]

Figure 2 illustrates Proposition 2.2.

**Remark 2.3.** In general the finer the quadratic rational Bézier curve is subdivide, the closer the middle control point $b_1$ is to the point $(b_0 + b_2)/2$ and the closer the weight is to one[3, 10]. Thus Proposition 2.2 means that $\mu = 2$ is a good candidate for optimal estimation of $\lambda_\mu$. In Section 4, Figure 6 illustrates this assertion.

3. Arc-length estimations for quadratic rational Bézier curve coinciding with length of circular arc

In this section we present another arc-length estimation $L_\mu$ with parameter $\mu$ which coincides with the arc-length of the quadratic rational Bézier curve $r(t)$ for $w = 0, 1, \infty$, simultaneously. The arc-length estimation $L_\mu$ is a natural extension of $\lambda_\mu(b_i, w)$ in Equation (2.4) as.
follows:

\[ L_\mu(r) = \frac{u^\mu L_p(L_q - L_b) + L_b(L_p - L_q)}{u^\mu (L_q - L_b) + L_p - L_q}, \]  

(3.1)

which satisfies \( L_1(b_i, w) = \lambda_1(b_i, w). \)

**Proposition 3.1.** The estimation \( L_\mu(b_i, w) \) coincides with the arc-length of \( r(t) \) for \( w = 0, 1, \infty \), and it is strictly increasing with respect to \( w \) for any \( b_0, b_1, b_2 \) and for all \( \mu > 0 \).

**Proof.** Since Equation (3.1) yields \( L_\mu(b_i, 0) = L_b, L_\mu(b_i, 1) = L_q \) and \( L_\mu(b_i, \infty) = L_p, \) it satisfies \( L_\mu(b_i, w) = L(w) \) for \( w = 0, 1, \infty \). By the derivative

\[ \frac{\partial L_\mu(b_i, w)}{\partial w} = \frac{\mu w^{\mu-1}(L_q - L_b)(L_p - L_q)(L_p - L_b)}{(w^\mu (L_q - L_b) + L_p - L_q)^2} \]

and by the inequalities \( L_b < L_q < L_p, \) we have \( \partial L(b_i, w)/\partial w > 0 \) for all positive \( \mu > 0 \), and the assertion follows. \( \square \)

Figure 3 shows the monotonicity of the estimations \( \lambda_\mu(b_i, w) \) and \( L_\mu(b_i, w) \) with respect to \( w. \)

The length of circular arc is well-known. We make our estimation \( L_\mu(r) \) coincides with the length of circular arc by special choice of \( \mu. \) The quadratic rational Bézier curve is a circular arc if and only if it is symmetric and the weight satisfies \( w = L_b/L_p \) [2, 4, 5, 8, 16]. Moreover, the angle of the circular arc is \( 2\alpha, \) where \( \alpha = \arccos w = \arccos(L_b/L_p) \) and the radius is \( L_b/2\sin \alpha, \) so its length is

\[ L_c = \frac{\alpha \cdot L_b}{\sin \alpha} = \frac{\arccos(L_b/L_p) \cdot L_b \cdot L_p}{\sqrt{L_p^2 - L_b^2}}. \]

Solving the equation \( L_\mu(b_i, w) = L_c \) we have \( \mu = \mu^*, \) where

\[ \mu^* = \frac{1}{\ln(L_b/L_p)} \ln \left( \frac{(L_p - L_q)(L_c - L_b)}{(L_p - L_c)(L_q - L_b)} \right). \]  

(3.2)

Since \( L_b < L_q < L_p \) and \( L_b < L_c < L_p, \) the definition of \( \mu^* \) in Equation (3.2) is well defined.

The value \( \mu^* \) is equal to zero if and only if \( (L_p - L_q)(L_c - L_b) = (L_p - L_c)(L_q - L_b) \) which is equivalent to \( L_q = L_c. \) Let the plane curve \( \gamma \) be set of all points \( b_1 = [x_1, y_1] \) satisfying \( L_q = L_c \) for fixed \( b_0 \) and \( b_2. \) Figure 4 shows that the curve \( \gamma \) is passing through the mid-point of \( b_0 \) and \( b_2, \) and the slope of \( \gamma \) with respect to the line \( \overline{b_0b_2} \) at the point is \( \pm \pi/4. \) Inside of the curve \( \gamma, \mu^* < 0, \) and Outside \( \mu^* > 0. \)

In the following proposition we confirm that \( L_{\mu^*}(b_i, w) \) reproduces the arc-length of \( r(t) \) when it is a circular arc, i.e., \( w = L_b/L_p \) and \( |\Delta b_0| = |\Delta b_1|. \)

**Proposition 3.2.** If \( b_1 \) does not lie on the curve \( \gamma, \) then the estimation \( L_{\mu^*}(b_i, w) \) coincides with the arc-length of the quadratic Bézier curve \( r(t) \) for \( w = 0, 1, \infty \) and for it to be a circular arc.
Figure 3. Left column: The conic section having the control points (magenta) \( b_0 = [-1,0] \), \( b_2 = [1,0] \) and \( b_1 = [0,2], [0.5,3], \) or \([1.5,1.5]\) (from top to bottom) with the weights \( w = 3 \) (green), 1 (red), or 1/3 (blue).

Right column: The real arc-length \( L(w) \) of the conic (black), its estimations \( \lambda_2(b_i, w) \) (red), \( \lambda_{\hat{\mu}}(b_i, w) = \mathcal{L}_1(b_i, w) \) (green) and \( \mathcal{L}_{\mu^*}(b_i, w) \) (blue), where \( (\hat{\mu}, \mu^*) = (1.581, 1.213), (1.448, 1.208) \), or \((1.812, 1.043)\)(from top to bottom).

The estimation \( \mathcal{L}_\mu(b_i, w) \) always coincides with \( L(w) \) at \( L_b, L_q \) and \( L_p \).
Proof. Proposition 3.1 shows that $L_{\mu^*}(b, w)$ coincides with the arc-length $L(w)$ for $w = 0, 1, \infty$ if $b \notin \gamma$ or equivalently $\mu^* \neq 0$. Thus it suffices to show that the estimation reproduces the arc-length for $r(t)$ to be a circular arc. If $r(t)$ is a circular arc, then it is symmetric, i.e., $|\Delta b_0| = |\Delta b_1|$, and the weight is $w = L_b/L_p$. The symmetry of $r(t)$ implies $L_q > L_c$ and so $\mu^* > 0$ or equivalently $|\mu^*| = \mu^*$. Also the weight $w = L_b/L_p$ yields

$$w^{\mu^*} = \left( \frac{L_b}{L_p} \right)^{\frac{1}{\ln(L_b/L_p)}} \ln \left( \frac{(L_p-L_q)(L_c-L_b)}{(L_p-L_c)(L_q-L_b)} \right) = \frac{(L_p-L_q)(L_c-L_b)}{(L_p-L_c)(L_q-L_b)},$$

and thus we have

$$L_{\mu^*}(b, L_b/L_p) = \frac{(L_c-L_b)}{(L_c-L_b) + 1} = L_c.$$

Although the estimation $L_{|\mu^*|}(r)$ represents the arc-length of the quadratic rational Bézier curve $r(t)$ when it is circular arc or its weight is 0, 1, $\infty$, simultaneously, the estimation doesn’t work well when $b_1$ lie inside of the curve $\gamma$. Thus we present the limit of $\mu^*$ for symmetric quadratic rational Bézier curve to use it as an optimal estimation of $L_{\mu}$.

**Proposition 3.3.** If the quadratic Bézier curve is symmetric, i.e., $|\Delta b_0| = |\Delta b_1|$, then $\mu^*$ has the limit values

$$\lim_{b_1 \rightarrow (b_0+b_2)/2} \mu^* = \frac{5}{2} \quad \text{and} \quad \lim_{b_1 \rightarrow \infty} \mu^* = 1,$$

independently of scale of $L_b = |\Delta b_2|$. 

**Figure 4.** The curve $\gamma$ (blue) is the set of all points $b_1 = [x_1, y_1]$ satisfying $\mu^* = 0$ when $b_0$ and $b_2$ are fixed.
Proof. Let \( h = |b_1 - (b_0 + b_2)/2| \). For sufficiently small \( h > 0 \), we have Equation (2.5) and the series expansions

\[
L_c \approx |\Delta b_2| + \frac{2h^2}{3|\Delta b_2|} + O(h^4)
\]

\[
\frac{(L_c - L_b)}{(L_p - L_c)} \approx \frac{1}{2} - \frac{9}{10} \frac{h^2}{|\Delta b_2|^2} + O(h^4)
\]

Thus it follows from Equation (3.2) and

\[
\ln \left( \frac{L_p - L_q}{L_q - L_b}(L_c - L_b) \right) \approx -\frac{12}{5} \frac{h^2}{|\Delta b_2|^2} + O(h^4)
\]

\[
\ln \frac{L_b}{L_p} \approx -2 \frac{h^2}{|\Delta b_2|^2} + O(h^4)
\]

that

\[
\lim_{b_1 \to (b_0 + b_2)/2} \mu^* = \frac{6}{5}.
\]

For sufficiently large \( h > 0 \), we have Equation (2.6) and the following series expansions.

\[
L_c \approx \frac{1}{2}|\Delta b_2|\pi - \frac{1}{2}|\Delta b_2|\frac{1}{h} + O \left( \frac{1}{h^2} \right)
\]

\[
\frac{(L_c - L_b)}{(L_p - L_c)} \approx \left( \frac{1}{4}|\Delta b_2|\pi - \frac{1}{2}|\Delta b_2| \right) \frac{1}{h} + O \left( \frac{1}{h^2} \right)
\]
Equation (3.2) and
\[
\ln \left( \frac{L_p - L_q}{L_q - L_b} \right) \left( \frac{L_c - L_b}{L_q - L_b} \right) \approx -\ln(h) + \ln\left( \frac{1}{4} |\Delta b_2| \pi - \frac{1}{2} |\Delta b_2| \right) + \mathcal{O}\left( \frac{1}{h^2} \right)
\]
\[
\ln \frac{L_b}{L_p} \approx -\ln(h) + \ln\left( \frac{1}{2} |\Delta b_2| \right) + \mathcal{O}\left( \frac{1}{h^2} \right)
\]
yield
\[
\lim_{b_1 \to \infty} \mu^* = 1.
\]

Figure 5 illustrates Proposition 3.3.

**Remark 3.4.** By the same reason of Remark 2.3, Proposition 3.3 means that \( \mu = \frac{6}{5} \) is a good candidate for optimal estimation of \( L_\mu \).

### 4. Examples and Comments

In this section we apply our estimation method to quadratic rational Bézier curves.

The examples are the unit circle \( x^2 + y^2 = 1 \) and the ellipse \( (x/2)^2 + y^2 = 1 \). These can be expressed by four segments of the quadratic rational Bézier curve, \( k = 4 \). By Remarks 2.3 and 3.4, we investigate the arc-length estimations \( \lambda_\mu(r) \) and \( L_\mu(r) \) for \( \mu = 2, \frac{6}{5}, \mu^* \) and \( \frac{6}{5} \). The second rows in Tables 1 and 2 show the errors \( |\lambda_\mu(r) - L| \) and \( |L_\mu(r) - L| \). By subdivision of the quadratic rational Bézier curve at each shoulder point\([1, 8, 9]\), the unit circle and the ellipse can be expressed by \( k \) segments of quadratic rational Bézier curves, \( k = 8, \cdots, 64 \). Tables 1 and 2 list the errors \( |\lambda_\mu(r) - L| \) and \( |L_\mu(r) - L| \), and show \( |L_{\frac{6}{5}}(r) - L| \) has the smallest error.

As we expected in Remarks 2.3 and 3.4, Figure 6 illustrates that \( \lambda_2(r) \) is optimal estimation of \( \lambda_\mu(r) \) and \( L_{\frac{6}{5}}(r) \) is optimal estimation of \( L_\mu(r) \). Also we can see the decreasing ratio of the errors for each one subdivision, as shown in Tables 1 and 2. The decreasing ratios of \( |L_{\frac{6}{5}}(r) - L| \) is closer to \( \frac{1}{64} \) as \( k \) increases, while those of others are closer to \( \frac{1}{16} \). This means \( |L_{\frac{6}{5}}(r) - L| \) converges to zero most rapidly in our estimations.

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**References**

Figure 6. (a) The error function $|\lambda_\mu - L|$ (green) for all $\mu \in (0, \infty)$. (b) $|\lambda_\mu - L|$ (green) near $\mu = 2$. (c) The error function $|L_\mu - L|$ (blue) for all $\mu \in (0, \infty)$. (d) $|L_\mu - L|$ (blue) near $\mu = \frac{6}{5}$. Here the quadratic rational Bézier curve is the ellipse $(x/2)^2 + y^2 = 1$ and the number of segment is $k = 64$.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{no. segments} & |\lambda_2 - 2\pi| & |\lambda_\mu - 2\pi| = |\mathcal{L}_1 - 2\pi| & |\mathcal{L}_\mu - 2\pi| \\hline
k = 4 & 1.43 \times 10^{-2} & 3.38 \times 10^{-2} & 0 & 1.47 \times 10^{-3} \\hline
k = 8 & 8.46 \times 10^{-4} & 1.76 \times 10^{-3} & 0 & 2.23 \times 10^{-5} \\hline
k = 16 & 5.21 \times 10^{-5} & 1.05 \times 10^{-4} & 0 & 3.44 \times 10^{-7} \\hline
k = 32 & 3.25 \times 10^{-6} & 6.51 \times 10^{-6} & 0 & 5.36 \times 10^{-8} \\hline
k = 64 & 2.03 \times 10^{-7} & 4.06 \times 10^{-7} & 0 & 8.37 \times 10^{-11} \\hline
\end{array}
\]

Table 1. Errors $|\lambda_\mu - 2\pi|$ and $|\mathcal{L}_\mu - 2\pi|$, after subdivision of the unit circle into $k$ segments. The fractional number in bracket is decreasing ratio after each one subdivision. The value of $(\hat{\mu}, \mu^*)$ is $(1.8026,1.2093), (1.9530,1.2061), (1.9884,1.2071), (1.9971,1.2002), (1.9993,1.2000)$ for $k = 4, \ldots, 64$, respectively.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{no. segments} & |\lambda_2 - L| & |\lambda_\mu - L| = |\mathcal{L}_1 - 2\pi| & |\mathcal{L}_\mu - L| \\hline
k = 4 & 5.40 \times 10^{-2} & 4.69 \times 10^{-2} & 7.42 \times 10^{-2} & 6.90 \times 10^{-3} \\hline
k = 8 & 1.57 \times 10^{-3} & 2.68 \times 10^{-3} & 3.40 \times 10^{-3} & 7.51 \times 10^{-5} \\hline
k = 16 & 6.84 \times 10^{-5} & 1.62 \times 10^{-4} & 2.10 \times 10^{-4} & 5.87 \times 10^{-7} \\hline
k = 32 & 5.01 \times 10^{-6} & 1.00 \times 10^{-5} & 1.34 \times 10^{-5} & 8.36 \times 10^{-9} \\hline
k = 64 & 3.13 \times 10^{-7} & 6.26 \times 10^{-7} & 8.45 \times 10^{-7} & 1.29 \times 10^{-10} \\hline
\end{array}
\]

Table 2. Errors $|\lambda_\mu - L|$ and $|\mathcal{L}_\mu - L|$, after subdivision of the ellipse \((x/2)^2 + y^2 = 1\) into $k$ segments.