A BORSUK–ULAM TYPE THEOREM OVER ITERATED SUSPENSIONS OF REAL PROJECTIVE SPACES

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Abstract. A CW complex $B$ is said to be I-trivial if there does not exist a $\mathbb{Z}_2$-map from $S^{i-1}$ to $S(\alpha)$ for any vector bundle $\alpha$ over $B$ and any integer $i$ with $i > \dim \alpha$. In this paper, we consider the question of determining whether $\Sigma^k \mathbb{R}P^n$ is I-trivial or not, and to this question we give complete answers when $k \neq 1, 3, 8$ and partial answers when $k = 1, 3, 8$. A CW complex $B$ is I-trivial if it is “W-trivial”, that is, if for every vector bundle over $B$, all the Stiefel–Whitney classes vanish. We find, as a result, that $\Sigma^k \mathbb{R}P^n$ is a counterexample to the converse of this statement when $k = 2, 4$ or $8$ and $n \geq 2k$.

1. Introduction and results

For a real vector bundle $\alpha$ over a CW complex $B$, the index of $\alpha$, denoted by $\text{ind} \alpha$, is defined to be the largest integer $i$ for which there exists a $\mathbb{Z}_2$-map from the $(i - 1)$-sphere $S^{i-1}$ to $S(\alpha)$ (see [1, 2, 6]). Here, $S(\alpha)$ is the sphere bundle of $\alpha$ and it is regarded as $\mathbb{Z}_2$-space by the antipodal map on each fiber. The sphere $S^{i-1}$ is also regarded as $\mathbb{Z}_2$-space by the antipodal map. Obviously we have the inequality $\text{ind} \alpha \geq \dim \alpha$ for any $\alpha$. The underlying space $B$ is said to be I-trivial if the equality $\text{ind} \alpha = \dim \alpha$ holds for every vector bundle $\alpha$ over $B$. With this terminology, the classical Borsuk–Ulam theorem can be restated as the point space is I-trivial. Also, the sphere $S^n$ is I-trivial if and only if $n \neq 1, 2, 4, 8$ (see [5, 7]). As for the stunted projective space $FP^n_m$, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with a function $\lambda_F(n)$ suitably defined, it is shown in [9] that $FP^n_m$ is I-trivial if and only if $\lambda_F(n) < m \leq n$. Note that in [9] the symbol $\theta(n)$ is used instead of $\lambda_F(n)$.

In this paper, we investigate whether, for positive integers $k$ and $n$, the $k$-fold suspension $\Sigma^k \mathbb{R}P^n$ of $\mathbb{R}P^n$ is I-trivial or not.

In general, for a vector bundle $\alpha$, we have $\text{ind} \alpha = \dim \alpha$ if the total Stiefel–Whitney class $W(\alpha)$ is equal to 1 (see [6, Proposition 2.2]). The underlying space $B$ is said to be W-trivial if $W(\alpha) = 1$ holds for every vector bundle $\alpha$ over
B. Thus, we immediately have the following lemma, which is of fundamental importance in our study.

**Lemma 1.1.** If $B$ is $W$-trivial, then it is $I$-trivial.

Concerning whether $\Sigma^k \mathbb{R}P^n$ is $W$-trivial or not, we quote the following result from [10].

**Theorem 1.2** ([10, Theorem 1.4]). For positive integers $k$ and $n$, the $k$-fold suspension $\Sigma^k \mathbb{R}P^n$ of $\mathbb{R}P^n$ is not $W$-trivial if and only if $k$ and $n$ satisfy one of the following conditions:

1. $k = 1, 2, 4$ or $8$ and $n \geq k$.
2. $k = 3, 5$ or $7$ and $n + k = 4$ or $8$.
3. $k = 6$ and $n = 2$ or $3$.

Accordingly, our goal is to determine whether or not $\Sigma^k \mathbb{R}P^n$ is $I$-trivial for each pair $(k, n)$ as in the above theorem. The results obtained in this paper are summarized as the theorem below. Note that if $k \geq 9$, $\Sigma^k \mathbb{R}P^n$ is $I$-trivial for all $n \geq 1$ since it is $W$-trivial.

**Theorem 1.3.** Let $n$ be a positive integer.

1. $\Sigma \mathbb{R}P^n$ is not $I$-trivial if $n = 1, 2$ or $3$.
2. $\Sigma^2 \mathbb{R}P^n$ is not $I$-trivial if and only if $n = 2$ or $3$.
3. $\Sigma^3 \mathbb{R}P^n$ is not $I$-trivial only if $n = 1$ or $5$. It is not $I$-trivial if $n = 1$.
4. $\Sigma^4 \mathbb{R}P^n$ is not $I$-trivial if and only if $4 \leq n \leq 7$.
5. $\Sigma^5 \mathbb{R}P^n$ is not $I$-trivial if and only if $n = 3$.
6. $\Sigma^6 \mathbb{R}P^n$ is not $I$-trivial if and only if $n = 2$ or $3$.
7. $\Sigma^7 \mathbb{R}P^n$ is not $I$-trivial if and only if $n = 1$.
8. $\Sigma^8 \mathbb{R}P^n$ is not $I$-trivial only if $8 \leq n \leq 15$. It is not $I$-trivial if $n = 8$.

Thus, the following three cases remain still unsettled: (i) $k = 1$ with $n \geq 4$, (ii) $k = 3$ with $n = 5$, and (iii) $k = 8$ with $8 < n \leq 15$.

The next corollary, which is an immediate consequence of Theorems 1.2 and 1.3, shows that the converse of Lemma 1.1 is not always true.

**Corollary 1.4.** The following spaces are $I$-trivial, although they are not $W$-trivial.

1. $\Sigma^2 \mathbb{R}P^n$ with $n \geq 4$.
2. $\Sigma^4 \mathbb{R}P^n$ with $n \geq 8$.
3. $\Sigma^6 \mathbb{R}P^n$ with $n \geq 16$.

Throughout this paper, all cohomology is assumed to have coefficients $\mathbb{Z}_2$ unless otherwise stated. We often consider a homomorphism from $H^*(\Sigma^k \mathbb{R}P^n)$ to $H^*(\mathbb{R}P^n)$. We denote by $s^i a$ the non-zero element of $H^{k+i}(\Sigma^k \mathbb{R}P^n)$ ($1 \leq i \leq n$), where $a$ represents the generator of $H^*(\mathbb{R}P^n)$ and $s^k$ represents $k$-fold suspension. Besides this notation, we use the letter $t$ to denote the generator of
$H^*(\mathbb{R}P^m)$ for $m$ not necessarily equal to $n$. Hom($\tilde{H}^*(\Sigma^k\mathbb{R}P^n), \tilde{H}^*(\mathbb{R}P^m)$) denotes the group consisting of all homomorphisms (of degree 0) from $\tilde{H}^*(\Sigma^k\mathbb{R}P^n)$ to $\tilde{H}^*(\mathbb{R}P^m)$ as graded algebra over the Steenrod algebra mod 2.

Also, it is to be understood that the notation $\xi$ always denotes the canonical line bundle over $\mathbb{R}P^n$ (for various values of $n$).

2. Preliminaries

Let $B$ be a CW complex. The following proposition, which will be used to show that $B$ is I-trivial, is a slight generalization of Proposition 2.4 of [6].

**Proposition 2.1.** Let $\alpha$ be a vector bundle over $B$ with dim $\alpha = m$ and let $k$ be an integer with $m \leq k$.

(1) If $g^*(\omega(\alpha)) = 1$ for any map $g : \mathbb{R}P^k \rightarrow B$, then we have $m \leq \text{ind } \alpha \leq k$.

(2) If $g^*(\omega(\alpha)) = 1$ for any map $g : \mathbb{R}P^m \rightarrow B$, then we have $\text{ind } \alpha = m$.

**Proof.** Part (2) is the special case when $k = m$ in part (1). We prove part (1). Assume that $\text{ind } \alpha > k$. Then there exists a $\mathbb{Z}_2$-map $f : S^k \rightarrow S(\alpha)$. We consider the induced map $\tilde{f} : \mathbb{R}P^k \rightarrow P(\alpha)$, where $P(\alpha)$ denotes the associated projective bundle of $\alpha$. Denoting by $e$ the $\mathbb{Z}_2$-Euler class of the line bundle $\alpha \rightarrow P(\alpha)$, we have $\tilde{f}^*(e) = t$, where $t$ is the generator of $H^*(\mathbb{R}P^k)$. In $H^m(P(\alpha))$, there is a well-known relation as follows:

$$e^m = \sum_{i=0}^{m-1} w_{m-i}(\alpha) e^i.$$

Applying $\tilde{f}^*$ to this relation, we obtain the following relation in $H^m(\mathbb{R}P^k)$:

$$t^m = \sum_{i=0}^{m-1} g^*(w_{m-i}(\alpha)) t^i,$$

where $g$ denotes the composite of $\tilde{f}$ and the projection $p : P(\alpha) \rightarrow B$. Here, if we suppose that $g^*(\omega(\alpha)) = 1$, that is, $g^*(w_i(\alpha)) = 0$ for all $i > 0$, then the above relation is reduced to $t^m = 0$. However, we have $t^m \neq 0$ since $m \leq k$. This is a contradiction, so that the proof of Proposition 2.1 is completed. \qed

**Remark.** In the above proposition, the assumption of (1) holds especially when Hom($\tilde{H}^*(B), \tilde{H}^*(\mathbb{R}P^k)$) = 0. This gives Proposition 2.4 of [6]. Also, the assumption of (2) trivially holds when $W(\alpha) = 1$. This gives Proposition 2.2 of [6].

The following proposition will be used to show that $B$ is not I-trivial.

**Proposition 2.2.** Let $d = 2, 4$ or 8, and let $p_d$ denote the Hopf vector bundle over $S^d$. Assume there exist maps $f : B \rightarrow S^d$ and $g : \mathbb{R}P^d \rightarrow B$ such that

$$(f \circ g)^* : H^d(S^d) \rightarrow H^d(\mathbb{R}P^d)$$
is an isomorphism. Then we have $\text{ind} f^*(\rho_d) > d$. Consequently, $B$ is not $I$-trivial.

**Proof.** Let us consider the sequence of maps

$$\mathbb{R}P^d \xrightarrow{g} B \xrightarrow{f} S^d$$

and the Hopf vector bundle $\rho_d$ over $S^d$. We put $\alpha = f^*(\rho_d)$ and prove that $\text{ind} \alpha > \dim \alpha (= d)$. Let $s$ and $t$ be the generators of $H^d(S^d)$ and $H^1(\mathbb{R}P^d)$ respectively. We assume that $(f \circ g)^*$ is an isomorphism. Then, we have

$$W(g^*(\alpha)) = (f \circ g)^* W(\rho_d)$$

$$= (f \circ g)^*(1 + s)$$

$$= (1 + t)^d$$

$$= (1 + t)^d,$$

since $d$ is a power of 2. Let $\xi$ be the canonical line bundle over $\mathbb{R}P^d$. Since $W(\xi) = 1 + t$, we have $W(g^*(\alpha) \otimes \xi) = (1 + t + t)^d = 1$ by an analogous formula to Formula III of Theorem 4.4.3 in [4]. Thus, we see that $g^*(\alpha) \otimes \xi$ is orientable and the only obstruction to its non-zero cross section lies in $H^d(\mathbb{R}P^d; \pi_d-1(S^{d-1}))$. This obstruction vanishes, since $w_d(g^*(\alpha) \otimes \xi) = 0$ and since the mod 2 reduction $H^d(\mathbb{R}P^d; \mathbb{Z}) \to H^d(\mathbb{R}P^d)$ is an isomorphism. Therefore, we can decompose $g^*(\alpha) \otimes \xi$ into the form $1 \oplus \beta$ for some vector bundle $\beta$ with $\dim \beta = d - 1$. Tensoring with $\xi$, we have $g^*(\alpha) = \xi \oplus (\beta \otimes \xi)$, so that we obtain the bundle monomorphism

$$\xi \xrightarrow{g^*(\alpha)} \xrightarrow{g} \alpha.$$

Restricting this bundle monomorphism to the sphere bundles, we obtain a $\mathbb{Z}_2$-map $S(\xi) \to S(\alpha)$. Since $S(\xi) = S^d$, we have $\text{ind} \alpha \geq d + 1$ and the proposition follows. $\Box$

### 3. $k$-fold suspension for the case $k = 2, 4$ or $8$

In this section, we consider $\Sigma^k \mathbb{R}P^n$ with $k = 2, 4$ or $8$ and prove parts (2), (4) and (8) of Theorem 1.3. We write $d$ instead of $k$. If $n < d$, then $\Sigma^d \mathbb{R}P^n$ is W-trivial by Theorem 1.2, so it is I-trivial. Therefore, we consider only the case $n \geq d$ hereafter. We remark that especially in the case where $n = d = 8$, $\Sigma^d \mathbb{R}P^n$ is seen to be not I-trivial. In fact, denoting by $\xi$ the canonical line bundle over $\mathbb{R}P^7$, we have

$$\Sigma^8 \mathbb{R}P^8 = \Sigma^8 (\mathbb{R}P^7)^{\xi} = (\mathbb{R}P^7)^{\xi+8} = (\mathbb{R}P^7)^{9\xi} = \mathbb{R}P^{16},$$

which is not I-trivial by Theorem 1.2 of [9]. Thus the second half of (8) follows. We prove the following theorem, which immediately leads to parts (2), (4) and the first half of (8).

**Theorem 3.1.** Let $d = 2, 4$ or $8$.

1. If $n \geq 2d$, then $\Sigma^d \mathbb{R}P^n$ is I-trivial.
(2) If \( d \leq n \leq 2d - 1 \), then we have the following:
   (a) For any vector bundle \( \alpha \) over \( \Sigma^d \mathbb{R}P^n \) with \( \dim \alpha \neq 2d \), we have \( \text{ind} \alpha = \dim \alpha \).
   (b) For a vector bundle \( \alpha \) over \( \Sigma^d \mathbb{R}P^n \) with \( \dim \alpha = 2d \), we have \( \text{ind} \alpha = 2d \) or \( 2d + 1 \).
   (c) In the case where \( d = 2 \) or \( 4 \), there exists a vector bundle \( \alpha \) over \( \Sigma^d \mathbb{R}P^n \) with \( \dim \alpha = 2d \) such that \( \text{ind} \alpha = 2d + 1 \); consequently, \( \Sigma^d \mathbb{R}P^n \) is not I-trivial.

The above theorem can be equivalently rewritten as the following two propositions.

**Proposition 3.2.** Let \( d = 2, 4 \) or \( 8 \) (and \( n \geq d \)). Let \( \alpha \) be a vector bundle over \( \Sigma^d \mathbb{R}P^n \) with \( \dim \alpha = m \).

1. If \( m \neq 2d \), then we have \( \text{ind} \alpha = m \).
2. If \( m = 2d \), then we have \( \text{ind} \alpha = m \) or \( m + 1 \), and in particular, we have \( \text{ind} \alpha = m \) in the case \( n \geq 2d \).

**Proposition 3.3.** Let \( d = 2 \) or \( 4 \), and assume that \( d \leq n \leq 2d - 1 \). Then there exists a vector bundle \( \alpha \) over \( \Sigma^d \mathbb{R}P^n \) with \( \dim \alpha = 2d \) such that \( \text{ind} \alpha = 2d + 1 \).

To prove Proposition 3.2, we prepare the following lemma, in which we merely suppose that \( d \) is even and we do not necessarily require that \( d = 2, 4 \) or \( 8 \).

**Lemma 3.4.** Let \( d, n, \) and \( m \) be positive integers with \( d \) even. Assume that \( m = 2d + 1 \).

1. If \( m - d + 1 \) is not a power of \( 2 \), we have
   \[
   \text{Hom}(\tilde{H}^*(\Sigma^d \mathbb{R}P^n), \tilde{H}^*(\mathbb{R}P^m)) = 0.
   \]
2. If \( m - d + 1 \) is a power of \( 2 \), we have
   \[
   \text{Hom}(\tilde{H}^*(\Sigma^d \mathbb{R}P^n), \tilde{H}^*(\mathbb{R}P^m)) \cong \begin{cases} 
   0 & (m > n + d), \\
   \mathbb{Z}_2 & (m \leq n + d),
   \end{cases}
   \]

   and in the latter case, the non-zero homomorphism is non-zero only in dimension \( m \).

**Proof.** Let \( \varphi : \tilde{H}^*(\Sigma^d \mathbb{R}P^n) \to \tilde{H}^*(\mathbb{R}P^m) \) be a homomorphism. For a non-negative integer \( r \), we claim that the following three statements are true:

   (i) \( \varphi(s^da^{2^r}) = 0 \).
   (ii) If \( 2^r - 1 + d \neq m \) and \( r \geq 2 \), then \( \varphi(s^da^{2^r-1}) = 0 \).
   (iii) If \( \varphi(s^da^{2^r-1}) = 0 \) and \( r \geq 2 \), then \( \varphi(s^da^i) = 0 \) for \( 2^r + 1 \leq i \leq 2^{r+1} - 2 \).

   We first show statement (i). To show \( \varphi(s^da^{2^r}) = 0 \) for \( r = 0 \), we use the Steenrod square \( Sq^d \). We clearly have \( Sq^d(s^da) = s^dSq^da = 0 \). On the other hand, we have \( Sq^d(a^{i+1}) = (\frac{i+1}{d})a^{i+1} \neq 0 \) since \( d \) is even and \( 2d + 1 \leq m \).
Therefore we obtain $\varphi(s^d a) \neq t^{d+1}$, that is, $\varphi(s^d a) = 0$. For $r \geq 1$, we also have $\varphi(s^d a^{2^r}) = 0$ since $a^{2^r} = S \varphi^{2^{r-1}} S \varphi^{2^{r-2}} \cdots S \varphi a$.

Next we show statement (ii). If $2^r - 1 + d > m$, we clearly have $\varphi(s^d a^{2^r-1}) = 0$ in $\tilde{H}^*(\mathbb{R}P^m)$ for dimensional reasons. So let $2^r - 1 + d < m$. We use $S \varphi^1$ and the result of (i). First, we have $S \varphi^1(s^d a^{2^r-1}) = (2^r - 1) s^d a^{2^r-1} = s^d a^{2^r}$ since $r \geq 1$. Secondly, we have $S \varphi^1(t^{2^r-1+d}) = (2^r - 1 + d)t^{2^r+d} = t^{2^r+d} \neq 0$ since $d$ is even, $r \geq 1$ and $2^r + d \leq m$. Thirdly, we have $\varphi(s^d a^{2^r}) = 0$ by statement (i).

From these, we obtain $\varphi(s^d a^{2^r-1}) \neq t^{2^r-1+d}$, that is, $\varphi(s^d a^{2^r-1}) = 0$.

Finally we show statement (iii). Let us put $i = 2^r + 1$. We note that $2 \leq j \leq 2r - 1$. Then we have $s^d a^i = s^d a^{2^r-1+j} = S \varphi^i(s^d a^{2^r-1})$ since $(2^r-1)$ is odd. From this, we see that $\varphi(s^d a^i) = 0$ if $\varphi(s^d a^{2^r-1}) = 0$.

Now, we are ready to prove Lemma 3.4. By statement (i), we have $\varphi(s^d a^i) = 0$ for $i = 1, 2$ and also for $i = 2^r$ with $r \geq 2$. First we assume that $m - d + 1$ is not a power of 2. Then, for any integer $r$ with $r \geq 2$, we have $2^r - 1 + d \neq m$, so that we have $\varphi(s^d a^i) = 0$ for $i = 2^r - 1$ by (ii). Hence we have $\varphi(s^d a^i) = 0$ for $i$ with $2^r + 1 \leq i \leq 2^r + 2$ by (iii). Thus we obtain $\varphi(s^d a^i) = 0$ for $i$ with $2^r - 1 \leq i \leq 2^r + 1 - 2$ for all $r \geq 2$, and we conclude that $\varphi = 0$. This proves part (1) of the lemma.

Next we prove part (2). Assume that $m - d + 1$ is a power of 2, say, $2^t$. Then we have $2^t - 1 + d = m$. Using (i), (ii) and (iii) just as in the previous argument, we obtain $\varphi(s^d a^i) = 0$ unless $i = 2^t - 1$ or $2^t + 1 \leq i \leq 2^{t+1} - 2$. For dimensional reasons, we also have $\varphi(s^d a^i) = 0$ in $\tilde{H}^*(\mathbb{R}P^m)$ for $2^t + 1 \leq i \leq 2^{t+1} - 2$ since $m < i + d$. Therefore, we obtain $\varphi(s^d a^i) = 0$ unless $i = 2^t - 1$. In the case where $m > n + d$, we have $s^d a^{2^t-1} = 0$ in $\tilde{H}^*(\Sigma^d \mathbb{R}P^n)$ for dimensional reasons, so that we conclude that $\varphi = 0$. In the case where $m \leq n + d$, the formula $\varphi(s^d a^{2^t-1}) = t^m$ actually defines the only non-zero homomorphism from $\tilde{H}^*(\Sigma^d \mathbb{R}P^n)$ to $\tilde{H}^*(\mathbb{R}P^m)$. Indeed, there is no integer $i$ with $0 < i < 2^t - 1$ such that $s^d a^{2^t-1} = S \varphi^i(s^d a^{2^t-1-i})$ since $(2^t-1-i)$ is even. This proves part (2) and the proof of Lemma 3.4 is thus completed.

Now we are ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** Let $\alpha$ be an $m$-dimensional vector bundle over $\Sigma^d \mathbb{R}P^n$, where $d = 2, 4$ or 8.

First, we consider the case where $m < 2d$. Recall that the smallest integer $i$ so that $w_i(\alpha)$ is non-zero must be a power of 2. Since dim $\alpha < 2d$ and $\Sigma^d \mathbb{R}P^n$ is $d$-connected, it follows that $w_i(\alpha) = 0$ for all $i > 0$, that is, $W(\alpha) = 1$. Hence, by Proposition 2.1 (and the remark after it), we obtain $\text{ind} \alpha = m$ as required.

Next, we consider the case where $m > 2d$. If $m - d + 1$ is not a power of 2, we have $\text{Hom}(\tilde{H}^*(\Sigma^d \mathbb{R}P^n), \tilde{H}^*(\mathbb{R}P^m)) = 0$ by Lemma 3.4, so that we obtain $\text{ind} \alpha = m$ as required, again by Proposition 2.1 (and the remark after it). Assume that $m - d + 1$ is a power of 2, say, $2^t$. Then we claim that $w_m(\alpha) = 0$. 

□
To show this, we calculate $Sq^{d-1} w_{2d}(\alpha)$ using the Wu formula [11]. We have

$$
Sq^{d-1} w_{2d} = \sum_{i=0}^{d-1} \binom{2^d - d + i}{i} w_{d-1-i} w_{2^i + i}
$$

$$
= \binom{2^d - 1}{d - 1} w_{2d} + w_{d-1} w_{2d - d - 1}
$$

$$
= w_m,
$$

where we have abbreviated $w_i(\alpha)$ as $w_i$. Note that $w_j = 0$ for $0 < j \leq d - 1$ since $\Sigma^d \mathbb{R} P^n$ is $d$-connected, and also note that $\binom{2^d - 1}{d - 1}$ is odd since $d - 1 < 2^d - 2$ by the assumption $m > 2d$. On the other hand, $w_{2d}$ is either zero or $s^d a^{d-d}$, and we have

$$
Sq^{d-1} (s^d a^{d-d}) = \binom{2^d - d}{d - 1} s^d a^{d-1}
$$

$$
= 0,
$$

since $\binom{2^d - d}{d - 1}$ is even for $d = 2, 4$ or $8$. Therefore, whether $w_{2d}$ is zero or not, we conclude that $w_m = 0$. Now, let $g : \mathbb{R} P^n \to \Sigma^d \mathbb{R} P^n$ be an arbitrary map. By Lemma 3.4, the homomorphism $g^* : \tilde{H}^*(\Sigma^d \mathbb{R} P^n) \to \tilde{H}^*(\mathbb{R} P^n)$ is zero except possibly in dimension $m$. Since $w_m(\alpha) = 0$ by the previous argument, we see that $g^*(W(\alpha)) = 1$. Therefore, by Proposition 2.1, we obtain $\text{ind } \alpha = m$. By this, we have completed the proof of part (1) of Proposition 3.2.

Finally, we consider the case where $m = 2d$. Let $g : \mathbb{R} P^{2d+1} \to \Sigma^d \mathbb{R} P^n$ be an arbitrary map and let us consider the homomorphism $g^* : \tilde{H}^*(\Sigma^d \mathbb{R} P^n) \to \tilde{H}^*(\mathbb{R} P^{2d+1})$. When $d = 4$ or $8$, we have

$$
\text{Hom}(\tilde{H}^*(\Sigma^d \mathbb{R} P^n), \tilde{H}^*(\mathbb{R} P^{2d+1})) = 0
$$

by Lemma 3.4 since $(2d + 1) - d + 1$ is not a power of 2. Hence $g^*$ is the zero homomorphism for $d = 4, 8$. When $d = 2$, $g^*$ is zero except possibly in dimension $2d + 1$, by Lemma 3.4. Since $\dim \alpha = 2d$, we obviously have $w_{2d+1}(\alpha) = 0$. Therefore we have $g^*(W(\alpha)) = 1$ for $d = 2$. Thus we have $g^*(W(\alpha)) = 1$ for $d = 2, 4$ or $8$, so that we can conclude that $\text{ind } \alpha \leq 2d + 1$ by Proposition 2.1. The first half of (2) of Proposition 3.2 is thus obtained.

Now we assume that $n \geq 2d$ in addition to $m = 2d$. Since $\Sigma^d \mathbb{R} P^n$ is $d$-connected and the smallest integer $i$ so that $w_i(\alpha) \neq 0$ is a power of 2, the only Whitney class which is possibly non-zero is $w_{2d}(\alpha)$. Thus we can write as $W(\alpha) = 1 + w_{2d}(\alpha)$. Here we claim that the following is true:

**Assertion 3.5.** Let $d = 2, 4$ or $8$, and assume that $n \not\equiv 0, 6, 7 \pmod{8}$ when $d = 2$. Then we have either $W(\beta) = 1$ or $W(\beta) = 1 + \sum_{i=1}^{n} s^d a_{id}$ for any vector bundle $\beta$ over $\Sigma^d \mathbb{R} P^n$. 

Clearly we have $s^d a^d \neq 0$ and $s^d a^{2d} \neq 0$ since $n \geq 2d$. Hence, admitting that Assertion 3.5 is true, we obtain $W(\alpha) \equiv 1 + w_2(\alpha) = 1$ when $d = 4$ or 8. When $d = 2$, we also have $W(\alpha) \equiv 1 + w_4(\alpha) = 1$ at least for $n = 4$. For $n > 4$, let us consider the inclusion map $i : \Sigma^d \mathbb{RP}^d \hookrightarrow \Sigma^d \mathbb{RP}^n$. Since $i^* : H^j(\Sigma^d \mathbb{RP}^n) \to H^j(\Sigma^d \mathbb{RP}^d)$ is injective, we have $W(\alpha) \equiv 1 + w_4(\alpha) = 1$ also for $n > 4$ from the result for $n = 4$. Thus we have $W(\alpha) = 1$ for $d = 2, 4$ or 8, so that we can conclude that $\text{ind} \alpha = \dim \alpha$ by Proposition 2.1 (and the remark after it). Therefore the proof of Proposition 3.2 is complete if we prove Assertion 3.5.

**Proof of Assertion 3.5.** Since the cup product is trivial in $\tilde{H}^*(\Sigma^d \mathbb{RP}^n)$, we have $W(2\beta) = 1$ for any vector bundle $\beta$ over $\Sigma^d \mathbb{RP}^n$, using the Whitney sum formula. According to the results of Fujii [3], we know that $KO(\Sigma^d \mathbb{RP}^n)$ is isomorphic to $\mathbb{Z}_2$ if $n \not\equiv 0, 6, 7 \pmod{8}$, and also that $KO(\Sigma^d \mathbb{RP}^n)$ and $KO(\Sigma^n \mathbb{RP}^n)$ are cyclic groups of order a power of 2. Therefore, in these cases, there are at most two types of $W(\beta)$, one of which is $W(\beta) = 1$. As is shown in the proof of [10, Proposition 3.1], there exists $\beta$ such that $W(\beta) = 1 + \sum_{i=1}^n s^i a^i$. Thus the assertion follows and the proof of Proposition 3.2 is completed. \hfill \Box

Now we proceed to prove Proposition 3.3.

**Proof of Proposition 3.3.** In view of Proposition 2.2 and part (2) of Proposition 3.2, it suffices to prove the following lemma. \hfill \Box

**Lemma 3.6.** Let $d = 2$ or 4, and assume that $d < n \leq 2d - 1$. Then there exist maps $f : \Sigma^d \mathbb{RP}^n \to S^{2d}$ and $g : \mathbb{RP}^{2d} \to \Sigma^d \mathbb{RP}^n$ such that

$$(f \circ g)^* : H^{2d}(S^{2d}) \longrightarrow H^{2d}(\mathbb{RP}^{2d})$$

is an isomorphism.

**Proof.** First we take, as $f : \Sigma^d \mathbb{RP}^n \to S^{2d}$, the following composition:

$$\Sigma^d \mathbb{RP}^n \hookrightarrow \Sigma^d \mathbb{RP}^{2d-1} \xrightarrow{j} \Sigma^d \mathbb{RP}_d^{2d-1} = \Sigma^d((\mathbb{RP}^{d-1})^d) \xrightarrow{h} \Sigma^d((\mathbb{RP}^{d-1})^d)^d \xrightarrow{q} \Sigma^d(pt)^d = S^{2d},$$

where $j$ is the map collapsing the $(2d-1)$-skeleton, and $q$ is the map induced from the constant map $\mathbb{RP}^{d-1} \to pt$, while $h$ is the map obtained from the fact that $d$ is isomorphic to the $d$-dimensional trivial bundle over $\mathbb{RP}^{d-1}$. Then, it is easy to see that $f^* : H^{2d}(S^{2d}) \to H^{2d}(\Sigma^d \mathbb{RP}^n)$ is an isomorphism.

Next, as $g : \mathbb{RP}^{2d} \to \Sigma^d \mathbb{RP}^n$, we take the following composition:

$$\mathbb{RP}^{2d} \xrightarrow{j'} \mathbb{RP}_{d+1}^{2d} \xrightarrow{h'} (\mathbb{RP}^{d-1})^{(d+1)} \xrightarrow{h''} (\mathbb{RP}^{d-1})^{d+1}$$

$$= \Sigma^d(\mathbb{RP}^{d-1}) \xrightarrow{h} \Sigma^d \mathbb{RP}^{d-1} \xrightarrow{h'} \Sigma^d \mathbb{RP}^n,$$
where \( j' \) is the map collapsing the \( d \)-skeleton, and \( h' \) is the map obtained from the fact that \( \partial \xi \) is isomorphic to the \( d \)-dimensional trivial bundle over \( \mathbb{R}P^{d-1} \). Then, it is immediate that \( g^* : H^{2d}(\Sigma^d \mathbb{R}P^n) \to H^{2d}(\mathbb{R}P^{2d}) \) is an isomorphism. This completes the proof of Lemma 3.6. \( \square \)

Remark. Since \( 8\xi \) is isomorphic to the 8-dimensional trivial bundle over \( \mathbb{R}P^7 \), the above lemma is also true for \( d = 8 \). However, Proposition 2.2 is not true for \( d = 16 \), which is the reason why we cannot apply the same argument as in Proposition 3.3 to the case \( d = 8 \).

4. \( k \)-fold suspension for the case \( k = 3, 5, 6 \) or 7

In this section, we consider \( \Sigma^k \mathbb{R}P^n \) with \( k = 3, 5, 6 \) or 7 and prove parts (3), (5), (6), and (7) of Theorem 1.3.

First, let us consider the case \( k = 7 \). By Theorem 1.2, \( \Sigma^7 \mathbb{R}P^n \) is not I-trivial only if \( n = 1 \). Since \( \Sigma^7\mathbb{R}P^1 (= S^8) \) is not I-trivial, part (7) of Theorem 1.3 is obvious.

Next, we consider the cases \( k = 5 \) and \( k = 6 \). By Theorem 1.2, \( \Sigma^5 \mathbb{R}P^n \) is not I-trivial only if \( n = 3 \), and \( \Sigma^6 \mathbb{R}P^n \) is not I-trivial only if \( n = 2, 3 \). Thus, parts (5) and (6) of Theorem 1.3 follow from the following proposition.

Proposition 4.1.

1. \( \Sigma^5 \mathbb{R}P^3 \) is not I-trivial.
2. \( \Sigma^6 \mathbb{R}P^n \) is not I-trivial for \( n = 2 \) or 3.

Proof. Since \( \mathbb{R}P^3 \) is S-reducible and since the suspension map \( \Sigma^\infty : [S^5, \Sigma^2 \mathbb{R}P^3] \to \{S^5, \mathbb{R}P^3\} \) is surjective, we see that \( \Sigma^5 \mathbb{R}P^3 \) is reducible. Hence \( \Sigma^5 \mathbb{R}P^3 \) is also reducible, so that there exists a map \( g : S^8 \to \Sigma^5 \mathbb{R}P^3 \) such that the composite

\[
S^8 \xrightarrow{\varphi} \Sigma^5 \mathbb{R}P^3 \xrightarrow{f} S^8
\]

is homotopic to the identity map. Here \( f \) is the map which collapses the 7-skeleton. As in the proof of Proposition 2.2, we consider the vector bundle \( \alpha = f^* (\rho) \), where \( \rho \) is the Hopf vector bundle over \( S^9 \). Since \( g^* (\alpha) = id^* (\rho) = \rho \), we obtain a \( \mathbb{Z}_2 \)-map \( S^{15} = S(\rho) \to S(\alpha) \) by restricting the bundle monomorphism \( g : \rho \to \alpha \) to the sphere bundles. Therefore we have \( \text{ind} \alpha \geq 16 \). Since \( \dim \alpha = 8 \), it follows that \( \Sigma^5 \mathbb{R}P^3 \) is not I-trivial and part (1) of the proposition is obtained.

Next let \( n = 2 \) or 3, and consider \( \Sigma^6 \mathbb{R}P^n \). We use Proposition 2.2. As a map \( f : \Sigma^6 \mathbb{R}P^n \to S^8 \), when \( n = 2 \), we take the quotient map which collapses the 7-skeleton. When \( n = 3 \), we take the composite map

\[
\Sigma^6 \mathbb{R}P^3 \to S^8 \cup e^9 \simeq S^8 \vee e^9 \to S^8,
\]

where the first map collapses the 7-skeleton and the last map is the projection. As a map \( g : \mathbb{R}P^8 \to \Sigma^6 \mathbb{R}P^n \), when \( n = 2 \), we take the composite map

\[
\mathbb{R}P^8 \to S^7 \cup 2e^8 \simeq \Sigma^6 \mathbb{R}P^2,
\]
where the first map collapses the 6-skeleton. When \( n = 3 \), we take the composite of this map with the inclusion \( i : \Sigma^6 \mathbb{R}P^2 \to \Sigma^6 \mathbb{R}P^3 \). Taking \( f \) and \( g \) as above, \( f' : H^8(S^8) \to H^8(\Sigma^6 \mathbb{R}P^m) \) and \( g^* : H^8(\Sigma^6 \mathbb{R}P^m) \to H^8(\mathbb{R}P^8) \) are both isomorphisms. Therefore, part (2) of the proposition is obtained from Proposition 2.2.

Finally, in this section, we consider \( \Sigma^3 \mathbb{R}P^n \), the case \( k = 3 \). By Theorem 1.2, \( \Sigma^3 \mathbb{R}P^n \) is not I-trivial only if \( n = 1, 5 \). Clearly \( \Sigma^3 \mathbb{R}P^1(= S^4) \) is not I-trivial, and thus part (3) of Theorem 1.3 is obvious. It is still open whether \( \Sigma^3 \mathbb{R}P^5 \) is I-trivial or not. However, we have a partial result as follows.

**Proposition 4.2.**

1. Let \( \alpha \) be a vector bundle over \( \Sigma^3 \mathbb{R}P^5 \) with \( \dim \alpha = m \).
   
   (a) If \( m \neq 8, 9 \), then we have \( \text{ind} \alpha = m \).
   
   (b) If \( m = 8 \) or \( 9 \), then we have \( m \leq \text{ind} \alpha \leq 10 \).
2. \( \Sigma^3 \mathbb{R}P^5 \) is not I-trivial if and only if there exists a map \( g : \mathbb{R}P^8 \to \Sigma^3 \mathbb{R}P^5 \) such that \( g^* : H^8(\Sigma^3 \mathbb{R}P^5) \to H^8(\mathbb{R}P^8) \) is an isomorphism.

**Proof.** By Proposition 2.1 (and the remark after it), part (1) of the proposition immediately follows from the following two assertions:

(i) If \( m < 8 \), we have \( \text{W}(\alpha) = 1 \).

(ii) If \( m \geq 10 \), we have \( \text{Hom}(\tilde{H}^*(\Sigma^3 \mathbb{R}P^5), \tilde{H}^*(\mathbb{R}P^m)) = 0 \).

We show that these assertions are true. Since \( \Sigma^3 \mathbb{R}P^5 \) is 3-connected, we have \( w_i(\alpha) = 0 \) for \( 0 < i < 4 \). Hence, by Lemma 3.3 of [10], we have \( \text{Sq}^1 w_4(\alpha) = 0 \). On the other hand, we have \( \text{Sq}^1(s^3 \alpha) \neq 0 \). Therefore we obtain \( w_{i}(\alpha) \neq s^3 \alpha \), that is, \( w_4(\alpha) = 0 \). Since the smallest integer \( k \) so that \( w_4(\alpha) \neq 0 \) is a power of 2, it follows that \( w_i(\alpha) = 0 \) for \( i < 8 \) and assertion (i) follows. Assume that \( m \geq 10 \) and let \( \varphi : \tilde{H}^*(\Sigma^3 \mathbb{R}P^5) \to \tilde{H}^*(\mathbb{R}P^m) \) be an arbitrary homomorphism. Since \( \text{Sq}^i(s^3 \alpha) = 0 \) and \( \text{Sq}^i t^4 = t^8 \) \( \neq 0 \), we obtain \( \varphi(s^3 \alpha) \neq t^4 \), that is, \( \varphi(s^3 \alpha) = 0 \). Similarly, since \( \text{Sq}^1(s^3 \alpha) = 0 \) and \( \text{Sq}^1 t^6 = t^{10} \neq 0 \), we obtain \( \varphi(s^3 \alpha) \neq 0 \). Since \( s^3 \alpha^2 = \text{Sq}^1(s^3 \alpha) \), \( s^3 \alpha^3 = \text{Sq}^1(s^3 \alpha^2) \) and \( s^3 \alpha^5 = \text{Sq}^2(s^3 \alpha^3) \), we see that \( \varphi(s^3 \alpha^j) = 0 \) for all \( j \) \((1 \leq j \leq 5)\), so that assertion (ii) follows.

Next we prove part (2) of the proposition. The “if” part is immediate from Proposition 2.2, taking as \( f : \Sigma^3 \mathbb{R}P^5 \to S^8 \) the map which collapses the 7-skeleton. Suppose that \( \Sigma^3 \mathbb{R}P^5 \) is not I-trivial, and let \( \alpha \) be a vector bundle over \( \Sigma^3 \mathbb{R}P^5 \) such that \( \text{ind} \alpha > \dim \alpha \). We put \( m = \dim \alpha \). Then \( m \) must be either 8 or 9 by part (1). From Proposition 2.1, there exists a map \( g : \mathbb{R}P^m \to \Sigma^3 \mathbb{R}P^5 \) such that \( g^*(W(\alpha)) \neq 1 \). Here, we recall that we have \( w_i(\alpha) = 0 \) for \( i < 8 \). Also, we have \( w_9(\alpha) = 0 \) for dimensional reasons. Thus, we see that \( g^*(w_9(\alpha)) = t^8 \). Therefore, when \( m = 8 \), \( g \) is the desired map. When \( m = 9 \), the composite of \( g \) and the inclusion \( \mathbb{R}P^8 \to \mathbb{R}P^9 \) gives the desired map. This proves the “only if” part, and the proof is complete. □

**Remark.** It is easy to see that

\[
\text{Hom}(\tilde{H}^*(\Sigma^3 \mathbb{R}P^5), \tilde{H}^*(\mathbb{R}P^8)) \cong \mathbb{Z}_2
\]
and its only non-zero homomorphism $\varphi$ is given by
\[
\varphi(s^ja^j) = \begin{cases} 
\ell^{j+3} & (j = 3, 5), \\
0 & (j \neq 3, 5).
\end{cases}
\]
Proposition 4.2 implies that this homomorphism is realizable by a map if and only if $\Sigma \mathbb{R}P^5$ is not I-trivial.

5. $k$-fold suspension for $k = 1$

In this section, we consider $\Sigma \mathbb{R}P^n$ and prove part (1) of Theorem 1.3, that is, the following proposition.

**Proposition 5.1.** $\Sigma \mathbb{R}P^n$ is not I-trivial for $n = 1, 2$ or $3$.

**Proof.** The proposition is clearly true when $n = 1$ since $\Sigma \mathbb{R}P^1 = S^2$. By Theorem 1.2 of [8], it is also true when $n = 2$. Furthermore, there is a 3-dimensional bundle $\alpha_3$ over $\Sigma \mathbb{R}P^2$ such that $\text{ind} \alpha_3 = 4$ (see [8, Theorem 4.1]).

Let $S^3 \xrightarrow{p} \Sigma \mathbb{R}P^2 \xrightarrow{i} \Sigma \mathbb{R}P^3$
be the standard cofibration and let us consider the exact sequence

$$[S^3, BO(3)] \xrightarrow{F} [\Sigma \mathbb{R}P^2, BO(3)] \xrightarrow{i_*} [\Sigma \mathbb{R}P^3, BO(3)].$$

Since $[S^3, BO(3)] \cong \pi_2(O(3)) = 0$, $i_*$ is surjective. Hence, we can take $\beta \in [\Sigma \mathbb{R}P^3, BO(3)]$ so that $i_* \beta = \alpha_3$. Then we have $\text{ind} \beta \geq \text{ind} i_* \beta = 4$, which implies that $\Sigma \mathbb{R}P^3$ is not I-trivial.

It is still open for $n \geq 4$ whether $\Sigma \mathbb{R}P^n$ is I-trivial or not. For a positive integer $m$, let $\lambda(m)$ denote the largest integer $r$ such that $2^r \leq m$. We have the following proposition.

**Proposition 5.2.** Let $n, m$ be positive integers and let $\alpha$ be a vector bundle over $\Sigma \mathbb{R}P^n$ with $\text{dim} \alpha = m$. Then we have $\text{ind} \alpha = m$ unless $n \leq m < 2^{\lambda(n+1)+1}$. When $n \neq 3, 7$, we also have $\text{ind} \alpha = m$ for $n = m$.

The following lemma, which is an analogue of Lemma 3.4, shows that $\text{ind} \alpha = m$ if $m \geq 2^{\lambda(n+1)+1}$.

**Lemma 5.3.** For positive integers $n$ and $m$, we have

$$\text{Hom}(\tilde{H}^*(\Sigma \mathbb{R}P^n), \tilde{H}^*(\mathbb{R}P^m)) \cong \begin{cases} 
0 & (m \geq 2^{\lambda(n+1)+1}), \\
\mathbb{Z}_2 & (m < 2^{\lambda(n+1)+1}).
\end{cases}$$

In the latter case, the non-zero homomorphism is non-zero only in dimension $2^{\lambda(m)}$.

**Proof.** Let $\varphi : \tilde{H}^*(\Sigma \mathbb{R}P^n) \rightarrow \tilde{H}^*(\mathbb{R}P^m)$ be a homomorphism. We put $\lambda(n+1) = r$ and $\lambda(m) = \ell$, that is, $2^r \leq n + 1 < 2^{r+1}$ and $2^\ell \leq m < 2^{\ell+1}$. The lemma follows from the following two assertions:

(i) If $i + 1$ is not a power of 2, then $\varphi(sa^i) = 0$. 

(ii) If \( i + 1 = 2^j \) and \( j \neq \ell \), then \( \varphi(sa^i) = 0 \).

In fact, from (i) and (ii) we have \( \varphi(sa^i) = 0 \) unless \( i = 2^\ell - 1 \). In the case where \( m \geq 2r + 1 \), we have \( n + 1 < 2r + 1 \leq 2^r \), so that we have \( sa^{2^r - 1} = 0 \) in \( \tilde{H}^*(\Sigma \mathbb{R}P^n) \) for dimensional reasons. Therefore we obtain \( \varphi = 0 \) in this case. In the case where \( m < 2r + 1 \), we have \( 2^\ell \leq 2r \leq n + 1 \), and the formula \( \varphi(sa^{2^r - 1}) = \ell 2^r \) actually defines a homomorphism from \( \tilde{H}^*(\Sigma \mathbb{R}P^n) \) to \( \tilde{H}^*(\mathbb{R}P^n) \), since there is no integer \( i \) with \( 0 < i < 2^\ell - 1 \) such that \( sa^{2^r - 1} = Sq^i(sa^{2^r - 1}) \). Thus the lemma follows from assertions (i) and (ii).

Now we prove assertions (i) and (ii). Let us write \( i + 1 = 2^j + k \) with \( 0 < k < 2^j \). Then we have \( \varphi(sa^i) = \varphi(Sq^k(sa^{2^j - 1})) = Sq^k(\varphi(sa^{2^j - 1})) \), since \( (2^j - 1) \) is odd. Here, if \( \varphi(sa^{2^j - 1}) = 0 \), we clearly have \( \varphi(sa^i) = 0 \), while if \( \varphi(sa^{2^j - 1}) = \ell 2^j \), we have \( \varphi(sa^i) = Sq^k(\ell 2^j) = 0 \), since \( (2^j - 1) \) is even. Therefore, whether \( \varphi(sa^{2^j - 1}) \) is zero or not, we obtain \( \varphi(sa^i) = 0 \). Thus assertion (i) follows.

Next, let \( i + 1 = 2^j \) and \( j \neq \ell \). In the case where \( j \geq \ell + 1 \), we have \( \varphi(sa^{2^j - 1}) = 0 \) in \( \tilde{H}^*(\mathbb{R}P^n) \) for dimensional reasons. In the case where \( j \leq \ell - 1 \), we have \( Sq^j(\ell 2^j) = \ell 2^{j+1} \neq 0 \), since \( j + 1 \leq \ell \). On the other hand, we obviously have \( Sq^j(sa^{2^j - 1}) = 0 \). From these, we obtain \( \varphi(sa^{2^j - 1}) \neq \ell 2^j \), that is, \( \varphi(sa^{2^j - 1}) \neq 0 \) also in this case. Thus assertion (ii) follows and the proof of the lemma is complete.

\( \square \)

**Proof of Proposition 5.2.** When \( m \geq 2^{\lambda(n+1)+1} \), we have \( \text{ind} \alpha = m \) by the above lemma, using Proposition 2.1 (and the remark after it). To show that \( \text{ind} \alpha = m \) when \( m < n \), we investigate possibilities of types of Stiefel–Whitney classes just like Assertion 3.5. By [3, Theorem 1], we have

\[
\tilde{KO}(\Sigma \mathbb{R}P^n) \cong \begin{cases} 
\mathbb{Z} + \mathbb{Z}_2 & \text{if } n \equiv 3 \pmod{4}, \\
\mathbb{Z}_2 & \text{if } n \not\equiv 3 \pmod{4}.
\end{cases}
\]

Moreover, when \( n \equiv 3 \pmod{4} \), we have the following exact sequence:

\[
0 \rightarrow \tilde{KO}(\Sigma \mathbb{R}P^n) \xrightarrow{\ell} \tilde{KO}(\Sigma \mathbb{R}P^n) \xrightarrow{J_n} \tilde{KO}(S^{n+1}) = \mathbb{Z} \rightarrow 0.
\]

Recall that \( S^{n+1} \) is \( W \)-trivial if \( n \neq 1, 3, 7 \) (see [5]). Since the cup product is trivial in \( \tilde{H}^*(\Sigma \mathbb{R}P^n) \), we see that there are at most three types of non-trivial Stiefel–Whitney classes if \( n = 3 \) or \( 7 \), while there is at most one type if \( n \neq 3, 7 \). On the other hand, by [10, Proposition 3.1], there is a vector bundle over \( \Sigma \mathbb{R}P^n \) whose Stiefel–Whitney class is \( 1 + \sum_{i=1}^n sa^i \). It follows that the possibilities are \( 1 + sa^n, 1 + \sum_{i=1}^n sa^i \) and \( 1 + \sum_{i=1}^{n-1} sa^i \) when \( n = 3 \) or \( 7 \), while the only possibility is \( 1 + \sum_{i=1}^{n-1} sa^i \) when \( n \neq 3, 7 \).

Now, we show that \( \text{ind} \alpha = m \) when \( m < n \). Since \( m < n \), we have \( w_i(\alpha) = 0 \) for \( i = n \) and \( i = n + 1 \), for dimensional reasons. From the above possibilities of Stiefel–Whitney classes, we must have \( W(\alpha) = 1 \), so that we obtain \( \text{ind} \alpha = m \).
Likewise, we obtain \( \text{ind} \alpha = m \) when \( m = n \) and \( n \neq 3, 7 \). This completes the proof. \( \square \)

References


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