ON CONFORMAL AND QUASI-CONFORMAL CURVATURE TENSORS OF AN $N(k)$-QUASI EINSTEIN MANIFOLD

ALIAKBAR HOSSEINZADEH AND ABOLFAZ TALESHIAN

Abstract. We consider $N(k)$-quasi Einstein manifolds satisfying the conditions $C(\xi, X).S = 0$, $\tilde{C}(\xi, X).S = 0$, $P(\xi, X).C = 0$, $P(\xi, X).\tilde{C} = 0$ and $\tilde{P}(\xi, X).\tilde{C} = 0$ where $C$, $\tilde{C}$, $P$ and $\tilde{P}$ denote the conformal curvature tensor, the quasi-conformal curvature tensor, the projective curvature tensor and the pseudo projective curvature tensor, respectively.

1. Introduction

The notion of a quasi Einstein manifold was introduced by M. C. Chaki in [3]. A non flat $n$-dimensional Riemannian manifold $(M, g)$ is said to be a quasi Einstein manifold if its Ricci tensor $S$ satisfies

\begin{equation}
S(X, Y) = ag(X, Y) + bg(X)\eta(Y), \quad \forall X, Y \in TM
\end{equation}

for some smooth functions $a$ and $b \neq 0$, where $\eta$ is a non zero 1-forms such that

\begin{equation}
g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1
\end{equation}

for the associated vector field $\xi$. The 1-form $\eta$ is called the associated 1-form and the unit vector field $\xi$ is called the generator of the manifold. If $b = 0$, then the manifold reduced to an Einstein manifold.

The Ricci operator $Q$ of a Riemannian manifold $(M, g)$ is defined by

\begin{equation}
S(X, Y) = g(QX, Y).
\end{equation}

For a quasi Einstein manifold [3], the Ricci operator satisfies

\begin{equation}
Q = aI + b\eta \otimes \xi.
\end{equation}

From (1.1) and (1.2) we obtain

\begin{equation}
S(X, \xi) = (a + b)\eta(X),
\end{equation}

\begin{equation}
r = na + b,
\end{equation}

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where \( r \) is the scalar curvature of \( M^n \).

If the generator \( \xi \) belongs to \( k \)-nullity distribution \( N(k) \), then the quasi Einstein manifold is called as an \( N(k) \)-quasi Einstein manifold [12]. In [12], it was shown that a conformally flat quasi Einstein manifold is an \( N(k) \)-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an \( N(k) \)-quasi Einstein manifold. The derivation conditions \( R(\xi, X).R = 0 \) and \( R(\xi, X).S = 0 \) were also studied in [12], where \( R \) and \( S \) denote the curvature and Ricci tensor, respectively. In [9], it was proved that in an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold \( k = \frac{a+b}{n-1} \). In [7], derivation conditions \( R(\xi, X),\rho = 0, \rho(\xi, X).S = 0 \) and \( \rho(\xi, X),\rho = 0 \) were studied where \( \rho \) is the projective curvature tensor, also physical examples of \( N(k) \)-quasi Einstein manifolds were given. The derivation conditions \( R(\xi, X).C = 0, R(\xi, X).\hat{C} = 0 \) were studied in [8], where \( C \) and \( \hat{C} \) denote the conformal curvature tensor and quasi conformal curvature tensor, respectively. The theory of \( N(k) \)-quasi Einstein manifolds deals with subjects such as nullity of curvature like tensors and especially it concerns with the notion of \( k \)-nullity distribution which has been in the center of many works such as [1], [4] and [6] and the recent non-Riemannian analogue [2]. In this paper, we consider \( N(k) \)-quasi Einstein manifolds satisfying the conditions \( C(\xi, X).S = 0, \hat{C}(\xi, X).S = 0, \hat{P}(\xi, X).C = 0, \hat{P}(\xi, X).\hat{C} = 0 \) and \( P(\xi, X).C = 0, P(\xi, X).\hat{C} = 0 \) where \( C, \hat{C}, P \) and \( \hat{P} \) denote the conformal curvature tensor, the quasi-conformal curvature tensor, the projective curvature tensor and the pseudo-projective curvature tensor, respectively.

2. \( N(k) \)-quasi Einstein manifolds

Let \( R \) denote the Riemannian curvature tensor of a Riemannian manifold \( M \). The \( k \)-nullity distribution \( N(k) \) [11], of a Riemannian manifold defined by

\[
N(k): p \rightarrow N_p(k) = \{ Z \in T_p M \mid R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}\}
\]

for all \( X, Y \in TM^n \), where \( k \) is some smooth function. In a quasi Einstein manifold \( M \), if the generator \( \xi \) belongs to some \( k \)-nullity distribution \( N(k) \), then it is said to be an \( N(k) \)-quasi Einstein manifold [12].

Lemma 2.1 ([9]). In an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold it follows that

\[
(2.1) \quad k = \frac{a + b}{n - 1}.
\]

Let \( (M^n, g) \) be an \( N(k) \)-quasi Einstein manifold. Then, we have [9]

\[
(2.2) \quad R(Y, Z)\xi = \frac{a + b}{n - 1} \{ \eta(Z)Y - \eta(Y)Z \}.
\]

The equation (2.2) is equivalent to

\[
(2.3) \quad R(\xi, Y)Z = \frac{a + b}{n - 1} \{ g(Y, Z)\xi - \eta(Z)Y \} = -R(\xi, Y)Z.
\]
Theorem 2.2 ([12]). An $n$-dimensional conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold.

In [7], we view the following physical examples of $N(k)$-quasi Einstein manifolds.

Example 2.3 ([7]). A conformally flat perfect fluid spacetime $(M^4, g)$ satisfying Einstein’s equation without cosmological constant is an $N(\frac{k}{3} + \frac{p}{6})$-quasi Einstein manifold.

Example 2.4 ([7]). A conformally flat perfect fluid spacetime $(M^4, g)$ satisfying Einstein’s equation with cosmological constant is an $N(\frac{k}{3} + \frac{p}{6})$-quasi Einstein manifold, where $k$ is the gravitational constant, $\sigma$ is the energy density and $p$ is the isotropic pressure of the fluid.

3. The conformal curvature tensor of an $N(k)$-quasi Einstein manifold

Let $(M^n, g)$ be a Riemannian manifold, the conformal curvature tensor [5], is defined by

\[ C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \} \]
\[ + \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y], \]

where $Q$ is the Ricci operator. Also we have [8]

\[ \eta(C(X, Y)Z) = 0. \]  

Now, we prove the following theorem:

Theorem 3.1. If $M$ is an $N(k)$-quasi Einstein manifold, then $M$ satisfies the condition $C(\xi, X).S = 0$.

Proof. Assume that $M$ is an $(\xi)$-quasi Einstein manifold. Then we have

\[ C(\xi, X).S = -S(C(\xi, X)Y, Z) - S(Y, C(\xi, X)Z). \]  

In view of (1.1) in (3.3) we have

\[ C(\xi, X).S = b[\eta(C(\xi, X)Y)\eta(Z) + \eta(Y)\eta(C(\xi, X)Z)]. \]  

From (3.2) in (3.4) we get

\[ C(\xi, X).S = 0. \]  

This completes the proof of the theorem. \( \square \)
The pseudo projective curvature tensor $\hat{P}$ [10] and the projective curvature tensor [13], on a manifold $M$ of dimension $n$ are defined by

\[
\hat{P}(X, Y)Z = \alpha R(X, Y)Z + \beta \{S(Y, Z)X - S(X, Z)Y\}
\]

(3.6)

\[ - \frac{r}{n} \left[ \frac{\alpha}{n-1} + \beta \right] \{g(Y, Z)X - g(X, Z)Y\}
\]

and

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \{S(Y, Z)X - S(X, Z)Y\},
\]

respectively, where $\alpha$ and $\beta$ are constants such that $\alpha, \beta \neq 0$ and $r$ is the scalar curvature. If $\alpha = 1$ and $\beta = -\frac{1}{n-1}$, then the pseudo projective curvature tensor is reduced to the projective curvature tensor.

**Proposition 3.2.** In an $n$-dimensional $N(k)$-quasi Einstein manifold $M$, the pseudo projective curvature tensor $\hat{P}$ satisfies

\[
\hat{P}(\xi, X)Y = \left[ \frac{\alpha - \beta}{n} \right] g(X, Y)\xi - \eta (Y)X \]

for all vector fields $X, Y, Z$ on $M$.

**Proof.** From (1.1), (2.1), (2.2) and (3.6), Eq.(3.7) follows easily. \(\square\)

**Theorem 3.3.** Let $M$ be an $N(k)$-quasi Einstein manifold. Then $M$ satisfies the condition $\hat{P}(\xi, X).C = 0$ if and only if either $\alpha - \beta = 0$ or $M$ is conformally flat.

**Proof.** Assume that $M$ is an $n$-dimensional $N(k)$-quasi Einstein manifold and satisfies the condition $\hat{P}(\xi, X).C = 0$. Then we can write

\[
0 = \hat{P}(\xi, X)C(Y, Z)W - C(\hat{P}(\xi, X)Y, Z)W
\]

(3.8)

\[
- C(Y, \hat{P}(\xi, X)Z)W + C(Y, Z)\hat{P}(\xi, X)W
\]

for all vector fields $X, Y, Z, W$ on $M$.

Using (3.7), in (3.8) we obtain

\[
0 = b\left[ \frac{(\alpha - \beta)}{n} \right] \{C(Y, Z, W, X)\xi - \eta (C(Y, Z)W)X
\]

\[
- g(X, Y)C(\xi, Z)W + \eta (Y)C(X, Z)
\]

\[
- g(X, Z)C(Y, \xi)W + \eta (Z)C(Y, X)W
\]

\[
- g(X, W)C(Y, Z)\xi + \eta (W)C(Y, Z)X
\]

(3.9)

\[
+ \beta \{\eta (X)\eta (C(Y, Z)W)\xi - \eta (C(Y, Z)W)\eta (X)\eta (Y)C(X, Z)W
\]

\[
- \eta (X)\eta (Y)C(\xi, Z)W + \eta (Y)C(X, Z)W
\]

\[
- \eta (X)\eta (Z)C(Y, \xi)W + \eta (Z)C(Y, X)W
\]

\[
- \eta (X)\eta (W)C(Y, Z)\xi + \eta (W)C(Y, Z)X\}.
\]
Since \( b \neq 0 \) we have
\[
0 = \frac{\alpha - \beta}{n} [C(Y, Z, W, X)\xi - \eta(C(Y, Z)W)X
\]
\[
- g(X, Y)C(\xi, Z)W + \eta(Y)C(X, Z)W
\]
\[
- g(X, Z)C(Y, \xi)W + \eta(Z)C(Y, X)W
\]
\[
- g(X, W)C(Y, Z)\xi + \eta(W)C(Y, Z)X
\]
\[
+ \beta\{\eta(X)\eta(C(Y, Z)W)\xi - \eta(C(Y, Z)W)X
\]
\[
- \eta(X)\eta(Y)C(\xi, Z)W + \eta(Y)C(X, Z)W
\]
\[
- \eta(X)\eta(Z)C(Y, \xi)W + \eta(Z)C(Y, X)W
\]
\[
- \eta(X)\eta(W)C(Y, Z)\xi + \eta(W)C(Y, Z)X.\]
\[
(3.10)
\]
Taking the inner product of (3.9) by \( \xi \), we obtain
\[
0 = \frac{\alpha - \beta}{n} [C(Y, Z, W, X)\eta(X)
\]
\[
- g(X, Y)\eta(C(\xi, Z)W) + \eta(Y)\eta(C(X, Z)W)
\]
\[
- g(X, Z)\eta(C(Y, \xi)W) + \eta(Z)\eta(C(Y, X)W)
\]
\[
- g(X, W)\eta(C(Y, Z)\xi) + \eta(W)\eta(C(Y, Z)X)
\]
\[
+ \beta\{\eta(X)\eta(C(Y, Z)W) - \eta(C(Y, Z)W)\eta(X)
\]
\[
- \eta(X)\eta(Y)\eta(C(\xi, Z)W) + \eta(Y)\eta(C(X, Z)W)
\]
\[
- \eta(X)\eta(Z)\eta(C(Y, \xi)W) + \eta(Z)\eta(C(Y, X)W)
\]
\[
- \eta(X)\eta(W)\eta(C(Y, Z)\xi) + \eta(W)\eta(C(Y, Z)X).\]
\[
(3.11)
\]
From (3.2) in (3.10), we have
\[
0 = \frac{\alpha - \beta}{n} [C(Y, Z, W, X)].
\]
\[
(3.12)
\]
Then either \( \alpha - \beta = 0 \) or
\[
(3.13)
C(Y, Z, W, X) = 0,
\]
i.e., \( M \) is conformally flat. The converse statement is trivial. This completes the proof of the theorem. \( \square \)

**Corollary 3.4.** Let \( M \) be an \( N(k) \)-quasi Einstein manifold. Then \( M \) satisfies the condition \( P(\xi, X)C = 0 \) if and only if \( M \) is conformally flat.

**4. The quasi-conformal curvature tensor of an \( N(k) \)-quasi Einstein manifold**

Let \( (M^n, g) \) be a Riemannian manifold, the quasi-conformal curvature tensor \[14\], is defined by
\[ \tilde{C}(X, Y)Z = \lambda R(X, Y)Z + \mu \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \} \]
\[ - \frac{r}{n} \left[ \frac{\lambda}{n-1} + 2\mu \right] \{ g(Y, Z)X - g(X, Z)Y \}, \]
where \( Q \) is the Ricci operator. Also we have [4]
\[ (4.2) \quad \eta(\tilde{C}(X, Y)Z) = \frac{b}{n} \left[ \mu(n - 2) + \lambda \right] \{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \}. \]

If \( \lambda = 1 \) and \( \mu = -\frac{1}{n-1} \), then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor [8].

**Theorem 4.1.** Let \( M \) be an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold. Then \( M \) satisfies the condition \( \tilde{C} \in \{ \xi, X \}S = 0 \) if and only if \( \mu(2 - n) = \lambda \).

**Proof.** Assume that \( M \) is an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold. The condition \( \tilde{C} \in \{ \xi, X \}S = 0 \) implies that
\[ (4.3) \quad S(\tilde{C}(\xi, X)Y, Z) + S(Y, \tilde{C}(\xi, X)Z) = 0. \]
In view of (1.1) in (4.3) we get
\[ (4.4) \quad \tilde{C}(\xi, X).S = b\eta(\tilde{C}(\xi, X)Y)\eta(Z) + \eta(Y)\eta(\tilde{C}(\xi, X)Z). \]
From (4.2) in (4.4) we have
\[ (4.5) \quad 0 = \frac{b^2}{n} \left[ \mu(n - 2) + \lambda \right] \{ g(X, Z)\eta(Y) + g(X, Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z) \}. \]
From (4.5), by a contraction, we get
\[ (4.6) \quad (n - 1)\frac{b^2}{n} \left[ \mu(n - 2) + \lambda \right] = 0. \]
Since \( b \neq 0 \), from (4.6) we have
\[ (4.7) \quad \mu(n - 2) + \lambda = 0. \]
From (4.7) we get \( \mu(2 - n) = \lambda \). The converse statement is trivial. This completes the proof of the theorem. \( \square \)

If \( P \) is a projective curvature tensor in an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold, we have [7]
\[ (4.8) \quad P(\xi, X)Y = \frac{b}{n-1} \left[ g(X, Y)\xi - \eta(X)\eta(Y)\xi \right]. \]
Next, we have the following theorem.

**Theorem 4.2.** Let \( M \) be an \( n \)-dimensional \( N(k) \)-quasi Einstein manifold. Then \( M \) satisfies the condition \( P(\xi, X)\tilde{C} = 0 \) if and only if \( \lambda + (n - 2)\mu = 0. \)
Assume that \( s \) satisfies the condition \( P(\xi, X) \). Then we can write
\[
0 = P(\xi, X) \tilde{C}(Y, Z) W - \tilde{C}(P(\xi, X) Y, Z) W
\]
for all vector fields \( X, Y, Z, W \) on \( M \).

Using (4.8), in (4.9) we obtain
\[
0 = \frac{b}{n-1} \left\{ \tilde{C}(Y, Z, W, X) \xi - \eta(X) \eta(\tilde{C}(Y, Z) W) \xi 
- g(X, Y) \tilde{C}(\xi, Z) W + \eta(X) \eta(Y) \tilde{C}(\xi, Z) W 
- g(X, Z) \tilde{C}(Y, \xi) W + \eta(X) \eta(Z) \tilde{C}(Y, \xi) W 
- g(X, W) \tilde{C}(Y, Z) \xi + \eta(X) \eta(W) \tilde{C}(Y, Z) \xi \right\}.
\]

Since \( b \neq 0 \) we have
\[
0 = \tilde{C}(Y, Z, W, X) \xi - \eta(X) \eta(\tilde{C}(Y, Z) W) \xi 
- g(X, Y) \tilde{C}(\xi, Z) W + \eta(X) \eta(Y) \tilde{C}(\xi, Z) W 
- g(X, Z) \tilde{C}(Y, \xi) W + \eta(X) \eta(Z) \tilde{C}(Y, \xi) W 
- g(X, W) \tilde{C}(Y, Z) \xi + \eta(X) \eta(W) \tilde{C}(Y, Z) \xi.
\]

Taking the inner product of (4.10) by \( \xi \), we obtain
\[
0 = \tilde{C}(Y, Z, W, X) - \eta(X) \eta(\tilde{C}(Y, Z) W)
- g(X, Y) \eta(\tilde{C}(\xi, Z) W) + \eta(X) \eta(Y) \eta(\tilde{C}(\xi, Z) W)
- g(X, Z) \eta(\tilde{C}(Y, \xi) W) + \eta(X) \eta(Z) \eta(\tilde{C}(Y, \xi) W)
- g(X, W) \eta(\tilde{C}(Y, Z) \xi) + \eta(X) \eta(W) \eta(\tilde{C}(Y, Z) \xi).
\]

From (4.2) in (4.11), we get
\[
0 = \tilde{C}(Y, Z, W, X) - \frac{b}{n} \left\{ \mu(n-2) + \lambda \right\} \left\{ g(X, Y) g(Z, W)
- g(X, Z) g(Y, W) + g(X, Z) \eta(Y) \eta(W) - g(X, Y) \eta(Z) \eta(W) \right\}.
\]

Now using (4.1) in (4.12), we have
\[
0 = \lambda R(Y, Z, W, X) + \mu \left\{ S(Z, W) g(Y, X) - S(Y, W) g(Y, X) + g(Z, W) S(X, Y) - g(Y, W) S(X, Z) \right\}
- r \left[ \frac{\lambda}{n-1} + 2 \mu \right] \left\{ g(Z, W) g(X, Y) - g(Y, W) g(X, Z) \right\}
- \frac{b}{n} \left\{ \mu(n-2) + \lambda \right\} \left\{ g(X, Y) g(Z, W) - g(X, Z) g(Y, W)
+ g(X, Z) \eta(Y) \eta(W) - g(X, Y) \eta(Z) \eta(W) \right\}.
\]
Also from (4.13), by contraction we have

\[
(4.14) \quad 0 = [\mu(n - 2) + \lambda] \left\{ S(Z, W) - (a + b)g(Z, W) - \frac{b(n - 1)}{n} \eta(Z)\eta(W) \right\}.
\]

Then either \(\mu(n - 2) + \lambda = 0\) or

\[
(4.15) \quad S(Z, W) - (a + b)g(Z, W) - \frac{b(n - 1)}{n} \eta(Z)\eta(W) = 0.
\]

Assume that \(\mu(n - 2) + \lambda \neq 0\). Then from (4.15) we get

\[
(4.16) \quad S(Z, \xi) = \left[ (a + b) + \frac{b(n - 1)}{n} \right] \eta(Z).
\]

Then from (1.4) and (4.15) we have \(\frac{b(n - 1)}{n} = 0\). Since \(M\) is an \(N(k)\)-quasi Einstein manifold this is not possible. The converse statement is trivial. This completes the proof of the theorem. \(\square\)

**Theorem 4.3.** Let \(M\) be an \(N(k)\)-quasi Einstein manifold. If \(M\) satisfies the condition \(\tilde{P}(\xi, X)\tilde{C} = 0\), then \(\alpha - \beta = 0\) or \(\alpha - \beta = 0\) or \(\mu(n - 2) + \lambda = 1\).

**Proof.** Assume that \(M\) is an \(n\)-dimensional \(N(k)\)-quasi Einstein manifold and satisfies the condition \(\tilde{P}(\xi, X)\tilde{C} = 0\). Then we can write

\[
(4.17) \quad 0 = \tilde{P}(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(\tilde{P}(\xi, X)Y, Z)W
- \tilde{C}(Y, \tilde{P}(\xi, X)Z)W - \tilde{C}(Y, Z)\tilde{P}(\xi, X)W
\]

for all vector fields \(X, Y, Z, W\) on \(M\).

Using (3.7), in (4.17) we obtain

\[
0 = b\left[ \frac{(\alpha - \beta)}{n} \{ \tilde{C}(Y, Z, W, X)\xi - \eta(\tilde{C}(Y, Z)W)X
- g(X, Y)\tilde{C}(\xi, Z)W + \eta(Y)\tilde{C}(X, Z)W
- g(X, Z)\tilde{C}(Y, \xi)W + \eta(Z)\tilde{C}(Y, X)W
- \tilde{C}(W, Z)\xi + \eta(W)\tilde{C}(Y, Z)X
+ {\beta(\eta(X)\eta(\tilde{C}(Y, Z)W)\xi - \eta(\tilde{C}(Y, Z)W)X}
- \eta(X)\eta(Y)\tilde{C}(\xi, Z)W + \eta(Y)\tilde{C}(X, Z)W
- \eta(X)\eta(Z)\tilde{C}(Y, \xi)W + \eta(Z)\tilde{C}(Y, X)W
- \eta(X)\eta(W)\tilde{C}(Y, Z)\xi + \eta(W)\tilde{C}(Y, Z)X \}ight].
\]
Since \( b \neq 0 \) we have

\[
0 = \frac{(\alpha - \beta)}{n} \{ \tilde{C}(Y, Z, W, X)\xi - \eta(\tilde{C}(Y, Z)W)X \\
- g(X, Y)\tilde{C}(\xi, Z)W + \eta(Y)\tilde{C}(X, Z)W \\
- g(X, Z)\tilde{C}(\xi, X)W + \eta(Z)\tilde{C}(Y, X)W \\
- g(X, W)\tilde{C}(Y, Z)\xi + \eta(W)\tilde{C}(Y, Z)X \}
\]

\[(4.18)\]

Taking the inner product of (3.9) by \( \xi \), we obtain

\[
0 = \frac{(\alpha - \beta)}{n} \{ \tilde{C}(Y, Z, W, X) - \eta(\tilde{C}(Y, Z)W)\eta(X) \\
- g(X, Y)\eta(\tilde{C}(\xi, Z)W) + \eta(Y)\eta(\tilde{C}(X, Z)W) \\
- g(X, Z)\eta(\tilde{C}(\xi, X)W) + \eta(Z)\eta(\tilde{C}(Y, X)W) \\
- g(X, W)\eta(\tilde{C}(Y, Z)\xi) + \eta(W)\eta(\tilde{C}(Y, Z)X) \}
\]

\[(4.19)\]

From (4.2) in (4.19), we have

\[
0 = \frac{b}{n} [\mu(n - 2) + \lambda] \left( \frac{\alpha - \beta}{n} \right) \{ \tilde{C}(Y, Z, W, X) + g(Y, W)g(X, Z) \\
- g(Z, W)g(X, Y) \} + \beta \{ g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W) \}.
\]

Taking \( X = Y = \xi \) in (4.20) we obtain

\[
0 = \frac{b}{n} [\mu(n - 2) + \lambda] \left( \frac{\alpha - \beta}{n} \right) \left( \frac{b}{n} [\mu(n - 2) + \lambda] - 1 \right) \{ g(Z, W) - \eta(Z)\eta(W) \}.
\]

(4.21)

Since \( M \) is an \( N(k) \)-quasi Einstein manifold then \( b \neq 0 \) and \( g(Z, W) \neq \eta(Z)\eta(W) \). Then from (4.21) we have

\[
0 = [\mu(n - 2) + \lambda] \left( \frac{\alpha - \beta}{n} \right) \left( \frac{b}{n} [\mu(n - 2) + \lambda] - 1 \right).
\]

(4.22)

From (4.22), it follows that \( \mu(n - 2) + \lambda = 0 \) or \( \alpha - \beta = 0 \) or \( \mu(n - 2) + \lambda = 1 \). This completes the proof of the theorem. \( \Box \)
References