WEYL’S TYPE THEOREMS FOR ALGEBRAICALLY $(p, k)$-QUASIHYPONORMAL OPERATORS

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Abstract. For a bounded linear operator $T$ we prove the following assertions: (a) If $T$ is algebraically $(p, k)$-quasihyponormal, then $T$ is $a$-isoloid, polaroid, reguloid and $a$-polaroid. (b) If $T^*$ is algebraically $(p, k)$-quasihyponormal, then $a$-Weyl’s theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$, where $Hol(\sigma(T))$ is the space of all functions that analytic in an open neighborhoods of $\sigma(T)$ of $T$. (c) If $T^*$ is algebraically $(p, k)$-quasihyponormal, then generalized $a$-Weyl’s theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$. (d) If $T$ is a $(p, k)$-quasihyponormal operator, then the spectral mapping theorem holds for semi-$B$-essential approximate point spectrum $\sigma_{SBF^+}(T)$, and for left Drazin spectrum $\sigma_{ID}(T)$ for every $f \in Hol(\sigma(T))$.

1. Introduction

Throughout this paper let $B(\mathcal{H})$, denote, the algebra of bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ for the null space and range of $T$, respectively. Also, let $\alpha(T) := \dim \ker(T)$, $\beta(T) := \dim \mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of $T$, respectively. An operator $T \in B(\mathcal{H})$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

$T$ is called Weyl if it is Fredholm of index 0, and Browder if it is Fredholm “of finite ascent and descent”.

Recall that the ascent, $a(T)$, of an operator $T$ is the smallest non-negative integer $p$ such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the descent, $d(T)$, of an operator $T$ is the smallest non-negative integer $q$ such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$.

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In the case of a normal operator defined by \( B T B \) studied this class of operators and has proved that an operator \( T \) is \( B \)-Fredholm if and only if \( \sigma(T) \cap \sigma_{\text{SF}}(T) = \emptyset \), where \( \sigma_n \) is a Fredholm operator and \( \sigma(T) \cap \sigma_{\text{SF}}(T) \) is a Fredholm operator of index 0. The \( B \)-Fredholm operator of index 0. The \( B \)-Fredholm operator of index 0. The essential spectrum \( E \) of \( T \) is defined as the index of the Fredholm operator \( E(T) \), where we write \( \text{acc}K \) for the accumulation points of \( K \subset \mathbb{C} \).

Following [13], we say that Weyl’s theorem holds for \( T \) if \( \sigma(T) \cap \sigma_{\text{SF}}(T) = E_0(T) \), where \( E_0(T) \) is the set of all eigenvalues \( \lambda \) of finite multiplicity isolated in \( \sigma(T) \). And Browder’s theorem holds for \( T \) if \( \sigma(T) \cap \sigma_{\text{SF}}(T) = \pi_0(T) \), where \( \pi_0 \) is the set of all poles of \( T \) of finite rank.

Let \( \Phi_+(\mathcal{H}) \) be the class of all upper semi-Fredholm operators, \( \Phi_-(\mathcal{H}) \) be the class of all \( T \in \Phi_+(\mathcal{H}) \) with \( i(T) \leq 0 \), and for any \( T \in \mathcal{B}(\mathcal{H}) \) let
\[
\sigma_{\text{SF}}^+(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \not\in \text{SF}^+(\mathcal{H}) \}.
\]
Let \( E_0^\mathcal{H} \) be the set of all eigenvalues of \( T \) of finite multiplicity which are isolated in \( \sigma_n(T) \). According to [27], we say that \( T \) satisfies a-Weyl’s theorem if \( \sigma_{\text{SF}}^+(T) = \sigma_n(T) \setminus E_0^\mathcal{H}(T) \). It follows from [27, Corollary 2.5] a-Weyl’s theorem implies Weyl’s theorem.

In [12] Berkani define the class of \( B \)-Fredholm operators as follows. For each integer \( n \), define \( T_n \) to be the restriction of \( T \) to \( \mathcal{R}(T^n) \) viewed as a map from \( \mathcal{R}(T^n) \) to \( \mathcal{R}(T^n) \) (in particular \( T_0 = T \) ). If for some \( n \) the range \( \mathcal{R}(T^n) \) is closed and \( T_n \) is a Fredholm (resp. semi-Fredholm) operator, then \( T \) is called a \( B \)-Fredholm (resp. semi-\( B \)-Fredholm) operator. In this case and from [6] \( T_n \) is a Fredholm operator and \( i(T_m) = i(T_n) \) for each \( m \geq n \). The index of a \( B \)-Fredholm operator \( T \) is defined as the index of the Fredholm operator \( T_n \), where \( n \) is any integer such that the range \( \mathcal{R}(T^n) \) is closed and \( T_n \) is a Fredholm operator (see [12]).

Let \( \mathcal{B}(\mathcal{H}) \) be the class of all \( B \)-Fredholm operators. In [6] Berkani has studied this class of operators and has proved that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is \( B \)-Fredholm if and only if \( T = T_0 \oplus T_1 \), where \( T_0 \) is a Fredholm operator and \( T_1 \) is a nilpotent operator.

Recall that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is called a \( B \)-Weyl operator (see [8]) if it is a \( B \)-Fredholm operator of index 0. The \( B \)-Weyl spectrum \( \sigma_{\text{BW}}(T) \) of \( T \) is defined by
\[
\sigma_{\text{BW}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B \text{-Weyl operator} \}.
\]
In the case of a normal operator \( T \) acting on a Hilbert space \( \mathcal{H} \), Berkani [12, Theorem 4.5] showed that \( \sigma_{\text{BW}}(T) = \sigma(T) \setminus E(T) \), where \( E(T) \) is the set of
all eigenvalues of $T$ which are isolated in the spectrum of $T$. This result gives a generalization of the classical Weyl’s theorem.

Let $SBF_+(\mathcal{H})$ be the class of all upper semi-$B$-Fredholm operators, and $SBF_+(\mathcal{H})$ the class of all $T \in SBF_+(\mathcal{H})$ such that $i(T) \leq 0$, and

$$\sigma_{SBF_+}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \notin SBF_+(\mathcal{H}) \}.$$  

Recall that an operator $T \in B(\mathcal{H})$ satisfies the generalized $a$-Weyl’s theorem if $\sigma_{SBF_+}(T) = \sigma_a(T) \setminus E^a(T)$, where $E^a(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma_a(T)$. Note that generalized $a$-Weyl’s theorem implies $a$-Weyl’s theorem (see [11]).

Recall that an operator $T \in B(\mathcal{H})$ is Drazin invertible if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where $T_0$ is a nilpotent operator and $T_1$ is an invertible operator (see [23, Proposition A]). The Drazin spectrum is given by

$$\sigma_D(T) := \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible} \}.$$  

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of all poles. An operator $T \in B(\mathcal{H})$ is called left Drazin invertible if $a(T) < \infty$ and $\mathcal{R}(T^a(T)^+) \text{ is closed}$ (see [9, Definition 2.4]). The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible} \}.$$  

Recall [9, Definition 2.5] that $\lambda \in \sigma_a(T)$ is a left pole of $T$ if $T - \lambda I$ is a left Drazin invertible operator and $\lambda \in \sigma_p(T)$ is a left pole of finite rank if $\lambda$ is a left pole of $T$ and $a(T - \lambda) < \infty$. We will denote $\pi^a(T)$ the set of all left poles of $T$, and by $\pi^*_a(T)$ the set of all left poles, of $T$ of finite rank. We have

$$\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T).$$

Note that if $\lambda \in \pi^a(T)$, then it is easily seen that $T - \lambda$ is an operator of topological uniform descent. Therefore it follows from ([11, Theorem 2.5]) that $\lambda$ is isolated in $\sigma_a(T)$.

For the sake of simplicity of notation we introduce the abbreviations $gaW$, $aW$, $gW$ and $W$ to signify that an operator $T \in B(\mathcal{H})$ obeys generalized $a$-Weyl’s theorem, $a$-Weyl’s theorem, generalized Weyl’s theorem and Weyl’s theorem, respectively. Analogous meaning is attached to the abbreviations $gaB$, $aB$, $gB$ and $B$ with respect to Browder’s theorem.

In the following diagram, arrows signify implications between various Weyl and Browder type theorems. It is known from [1, 3, 7, 11, 19, 20, 27] that if
Let \( T \in \mathcal{B}(\mathcal{H}) \), then we have:

\[
\begin{array}{c}
gW \longrightarrow gB \longrightarrow B \\
gaW \longrightarrow aW \longrightarrow aB \\
gaB
\end{array}
\]

The quasinilpotent part \( H_0(T - \lambda) \) and the analytic core \( K(T - \lambda) \) of \( T - \lambda \) are defined by

\[
H_0(T - \lambda) := \{ x \in \mathcal{H} : \lim_{n \to \infty} \|(T - \lambda)^nx\|^\frac{1}{2} = 0 \}.
\]

and

\[
K(T - \lambda) = \{ x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subset \mathcal{H} \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n\|x\| \text{ for all } n = 1, 2, \ldots \}.
\]

We note that \( H_0(T - \lambda) \) and \( K(T - \lambda) \) are generally non-closed hyper-invariant subspaces of \( T - \lambda \) such that \( (T - \lambda)^{-1}(0) \subset H_0(T - \lambda) \) for all \( p = 0, 1, \ldots \) and \( (T - \lambda)K(T - \lambda) = K(T - \lambda) \). Recall that if \( \lambda \in \text{iso}(\sigma(T)) \), then \( H_0(T - \lambda) = \chi_T(\{\lambda\}) \), where \( \chi_T(\{\lambda\}) \) is the glocal spectral subspace consisting of all \( x \in \mathcal{H} \) for which there exists an analytic function \( f : \mathbb{C} \setminus \{\lambda\} \to \mathcal{H} \) that satisfies \( (T - \mu)f(\mu) = x \) for all \( \mu \in \mathbb{C} \setminus \{\lambda\} \) (see [17]).

Let \( \text{Hol}(\sigma(T)) \) be the space of all functions that analytic in an open neighborhoods of \( \sigma(T) \). Following [18] we say that \( T \in \mathcal{B}(\mathcal{H}) \) has the single-valued extension property (SVEP) at point \( \lambda \in \mathbb{C} \) if for every open neighborhood \( U_\lambda \) of \( \lambda \), the only analytic function \( f : U_\lambda \to \mathcal{H} \) which satisfies the equation \( (T - \mu)f(\mu) = 0 \) is the constant function \( f \equiv 0 \). It is well-known that \( T \in \mathcal{B}(\mathcal{H}) \) has SVEP at every point of the resolvent \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). Moreover, from the identity theorem for analytic function it easily follows that \( T \in \mathcal{B}(\mathcal{H}) \) has SVEP at every point of the boundary \( \partial\sigma(T) \) of the spectrum. In particular, \( T \) has SVEP at every isolated point of \( \sigma(T) \). In [25, Proposition 1.8], Laursen proved that if \( T \) is of finite ascent, then \( T \) has SVEP.

**Proposition 1.1** ([24]). Let \( T \in \mathcal{B}(\mathcal{H}) \).

(i) If \( T \) has the SVEP, then \( i(T - \lambda I) \leq 0 \) for every \( \lambda \in \rho_{\text{SBF}}(T) \).

(ii) If \( T^* \) has the SVEP, then \( i(T - \lambda I) \geq 0 \) for every \( \lambda \in \rho_{\text{SBF}}(T) \).

(iii) If \( T^* \) has the SVEP, then

\[
\begin{align*}
(a) & \; \sigma_{\text{SBF}^+}(T) = \omega(T) \quad \text{and} \quad \sigma_{\text{SBF}^+}(T) = \sigma_{\text{SBF}^+}(T).
\end{align*}
\]

In [36] H. Weyl examined the spectra of all compact perturbations of a hermitian operator \( T \) on a Hilbert space and proved that their intersection coincides with the isolated point of the spectrum \( \sigma(T) \) which are the eigenvalues of finite multiplicity. Weyl's theorem has been extended to several classes of
Hilbert space operators including seminormal operators [4, 5]. In [7] M. Berkani introduced the concepts of the generalized Weyl’s theorem and generalized Browder’s theorem, and they showed that $T$ satisfies the generalized Weyl’s theorem whenever $T$ is a normal operator on Hilbert space. More recently, [10] extended this result to hyponormal operators. In [32] extended this result to log-hyponormal operators. Recently, Rashid et al. [31] showed that if $T$ is quasi-class $A$, then generalized Weyl’s theorem holds for every $f \in Hol(\sigma(T))$.

In this paper, we study generalized $a$-Weyl’s theorem for algebraically $(p, k)$-quasihyponormal operators. Among other things, we prove that the spectral mapping theorem holds for semi-$B$-essential approximate point spectrum $\sigma_{SBF^+}(T)$, and for left Drazin spectrum for every $f \in Hol(\sigma(T))$.

2. Properties of algebraically $(p, k)$-quasihyponormal operators

Definition 2.1 ([22]). An operator $T \in B(H)$ is said to be $(p, k)$-quasihyponormal if

$$T^k((T^*T)^p - (TT^*)^p)T^k \geq 0,$$

where $0 \leq p \leq 1$ and $k$ is a positive integer. Especially, when $p = 1, k = 1, p = k = 1$, $T$ is called $k$-quasihyponormal, $p$-quasihyponormal, quasihyponormal, respectively.

Definition 2.2. An operator $T \in B(H)$ is said to be algebraically $(p, k)$-quasihyponormal if there exists a non-constant complex polynomial $P$ such that $P(T)$ is a $(p, k)$-quasihyponormal operator.

In general, the following implications hold:

$p$-hyponormal $\Rightarrow p$-quasihyponormal $\Rightarrow$ algebraically $p$-quasihyponormal

$\Rightarrow$ algebraically $(p, k)$-quasihyponormal.

An operator $T \in B(H)$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. An operator $T \in B(H)$ is called normaloid if $r(T) = ||T||$, where $r(T)$ is the spectral radius of $T$. $X \in B(H)$ is called a quasiaffinity if $X$ has trivial kernel and dense range. $S \in B(H)$ is said to be a quasiaffine transform of $T \in B(H)$ (notation: $S \prec T$) if there is a quasiaffinity $X \in B(H)$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that $S$ and $T$ are quasisimilar.

The following facts follow from the above definition and some well known facts about $(p, k)$-quasihyponormal operators.

(i) If $T \in B(H)$ is an algebraically $(p, k)$-quasihyponormal operator, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

(ii) If $T \in B(H)$ is an algebraically $(p, k)$-quasihyponormal operator and $M$ is a closed $T$-invariant subspace of $H$, then $T|_M$ is an algebraically $(p, k)$-quasihyponormal operator.
Lemma 2.3. Let \( T \in \mathcal{B}(\mathcal{H}) \) be a \( p \)-quasihyponormal operator for \( 0 < p \leq 1 \). Then the following assertions hold.

1. \( \| T^n x \|^2 \leq \| T^{n-1} x \| \| T^{n+1} x \| \) for all unit vector \( x \in \mathcal{H} \) and all positive integer \( n \).
2. \( \| T^n \|^n \leq \| T^{n-1} \|^n r(T^n) \) for all positive integer \( n \), where \( r(T^n) \) denote the spectral radius of \( T^n \). Hence \( T \) is normaloid.
3. \( T \) is a paranormal operator.

Proof. (1) It is obvious that if \( T \) is \( p \)-quasihyponormal, then it is a \((p, n)\)-quasihyponormal operator for each positive integer \( n \), since
\[
\langle T^n (TT^n)^p T^n x, x \rangle = \langle (T^* T)^p T^n x, T^n x \rangle \leq \| T^n x \|^{2p} \| (TT^n)^p T^n x \| \] (by Hőlder-McCarthy inequality)
\[
= \| T^n x \|^{2p} \| T^{n+1} x \|^{2p+2}
\]
and
\[
\langle T^n (TT^n)^p T^n x, x \rangle = \langle (T^* T)^p T^n x, T^n x \rangle \leq \| T^n x \|^{2p} \| (TT^n)^p T^n x \| \] (by Hőlder-McCarthy inequality)
\[
= \| T^n x \|^{2p} \| T^{n+1} x \|^{2p+2}
\]
But \( T \) is a \( p \)-quasihyponormal operator. Then
\[
\langle T^n ((T^* T)^p - (TT^n)^p) T^n x, x \rangle \geq 0
\]
Hence
\[
\| T^n x \|^2 \leq \| T^{n-1} x \| \| T^{n+1} x \|
\]
(2) If \( T^n = 0 \) for some \( n > 1 \), then \( T = 0 \), and in this case \( r(T) = 0 \). Hence (2) is obvious. Hence we may assume \( T^n \neq 0 \) for all \( n \geq 1 \). Then
\[
\| T^n \| \leq \| T^{n+1} \| \leq \cdots \leq \| T^{mn} \|
\]
by (1), and we have
\[
\left( \frac{\| T^n \|}{\| T^{n-1} \|} \right)^{mn-n-1} \leq \frac{\| T^{n+1} \|}{\| T^n \|} \times \cdots \times \frac{\| T^{mn} \|}{\| T^{mn-1} \|} = \| T^{mn} \|. \]
Hence
\[
\left( \frac{\| T^n \|}{\| T^{n-1} \|} \right)^{\frac{n-1}{m-1}} \leq \frac{\| T^{mn} \|^{\frac{1}{m}}}{\| T^{mn-1} \|^{\frac{1}{m}}}
\]
Now letting \( m \to \infty \). We get
\[
\| T^n \|^n \leq \| T^{n-1} \|^n r(T^n).
\]
Put \( n = 1 \), we have \( \| T \| \leq r(T) \). So \( \| T \| = r(T) \), i.e., \( T \) is normaloid.
(3) Put \( n = 1 \) in (1), we have \( \|Tx\|^2 \leq \|T^2x\| \), that is, \( T \) is paranormal. \( \square \)

**Definition 2.4** ([17]). An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be totally hereditarily normaloid, \( T \in THN \) if every part of \( T \) (i.e., its restriction to an invariant subspace), and \( T_p^{-1} \) for every invertible part \( T_p \) of \( T \), is normaloid.

**Lemma 2.5.** Let \( T \in THN \), let \( \lambda \in \mathbb{C} \). Assume that \( \sigma(T) = \{\lambda\} \). Then \( T = \lambda I \).

**Proof.** We consider two cases:

- case I. \( (\lambda = 0) \): Since \( T \) is normaloid. Therefore \( T = 0 \),
- case II. \( (\lambda \neq 0) \): Here \( T \) is invertible, and since \( T \in THN \), we see that \( T, T^{-1} \) are normaloid. On the other hand \( \sigma(T^{-1}) = \{\frac{1}{\lambda}\} \), so \( \|T\|||T^{-1}|| = |\lambda|\frac{1}{|\lambda|} = 1 \).

It follows that \( T \) is convexoid, so \( W(T) = \{\lambda\} \). Therefore \( T = \lambda I \). \( \square \)

In [14], Curto and Han proved that quasinilpotent algebraically paranormal operators are nilpotent. We now establish a similar result for algebraically \((p, k)\)-quasihyponormal operators.

**Proposition 2.6.** Let \( T \) be a quasinilpotent \((p, k)\)-quasihyponormal operator. Then \( T \) is nilpotent.

**Proof.** Assume that \( p(T) \) is a totally hereditarily normaloid operator for some nonconstant polynomial \( p \). Since \( \sigma(p(T)) = p(\sigma(T)) \), the operator \( p(T) - p(0) \) is quasinilpotent. Thus Lemma 2.5 would imply that

\[
 cT^m(T - \lambda_1 I) \cdots (T - \lambda_n I) \equiv p(T) - p(0) = 0,
\]

where \( m \geq 1 \). Since \( T - \lambda_j I \) is invertible for every \( \lambda_j \neq 0 \), we must have \( T^m = 0 \).

**Lemma 2.7.** Let \( T \) be an invertible \( p \)-quasihyponormal operator. Then \( \mathcal{H} = \mathcal{R}(T) \oplus \ker(T) \). Moreover \( T_1 \), the restriction of \( T \) to \( \mathcal{R}(T) \) is one-one and onto.

**Proof.** Suppose that \( y \in \mathcal{R}(T) \cap \ker(T) \) then \( y = Tx \) for some \( x \in \mathcal{H} \) and \( Ty = 0 \). It follows that \( T^2x = 0 \). However, \( d(T) = 1 \) and so \( x \in \ker(T^2) = \ker(T) \).

Hence \( y = Tx = 0 \) and so \( \mathcal{R}(T) \cap \ker(T) = \{0\} \). Also, \( TR(T) = \mathcal{R}(T) \).

If \( x \in \mathcal{H} \), there is \( u \in \mathcal{R}(T) \) such that \( Tu = Tx \). Now if \( z = x - u \), then \( Tz = 0 \). Hence

\[
 \mathcal{H} = \mathcal{R}(T) \oplus \ker(T).
\]

Since \( a(T) = 1 \), \( T \) maps \( \mathcal{R}(T) \) onto itself. If \( y \in \mathcal{R}(T) \) and \( Ty = 0 \), then \( y \in \mathcal{R}(T) \cap \ker(T) = \{0\} \). Hence \( T_1 \) is one-one and onto. \( \square \)

Observe that \( \{\lambda_0\} \) is a clopen subset of \( \sigma(T) \). Let \( T \in \mathcal{B}(\mathcal{H}) \). The \( R_\lambda(T) = (T - \lambda)^{-1} \) is analytic on \( p(T) \), and an isolated point \( \lambda_0 \) of \( \sigma(T) \) is an isolated singular point of the resolvent of \( T \). Here there is a Laurent expansion of this function in powers of \( \lambda - \lambda_0 \). We write this in the form

\[
 (T - \lambda)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n.
\]
The coefficients $A_n$ and $B_n$ are members of $\mathcal{B}(\mathcal{H})$ and given by the standard formulas

$$A_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \lambda_0)^{-n-1}(\lambda - T)^{-1} d\lambda,$$

$$B_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \lambda_0)^{n-1}(\lambda - T)^{-1} d\lambda,$$

where $\Gamma$ is any circle $|\lambda - \lambda_0| = \rho$ with $0 < \rho < \delta$ described once counterclockwise.

The function $f_n$ defined by

$$f_n(\lambda) = \begin{cases} 
(\lambda - \lambda_0)^{n-1}, & \text{if } |\lambda - \lambda_0| \leq \rho < \delta, \\
0, & \text{otherwise.}
\end{cases}$$

is in $Hol(\sigma(T))$ and moreover

$$B_n = f_n(T), \ n = 1, 2, \ldots.$$

For each positive integer $n$, we have

$$(\lambda - \lambda_0)f_n(\lambda) = f_{n+1}.$$

So

$$T - \lambda_0)B_n = B_{n+1}$$

and by induction

$$T - \lambda_0)^n B_1 = B_{n+1}.$$

We note in passing that

$$B_1 = E(\lambda_0)$$

the spectral projection corresponding to the clopen set $\lambda_0$ of $\sigma(T)$.

Consider for each non-negative integer $n$ the function $g_n$ defined by

$$g_n(\lambda) = \begin{cases} 
0, & \text{if } |\lambda - \lambda_0| \leq \rho < \delta, \\
(\lambda - \lambda_0)^{-n-1}, & \text{otherwise.}
\end{cases}$$

is in $Hol(\sigma(T))$. Moreover,

$$A_n = -g_n(T)$$

for each non-negative integer $n$. We have

$$(\lambda - \lambda_0)g_{n+1}(\lambda) = g_n(\lambda)$$

and so

$$(\lambda - \lambda_0)A_{n+1} = A_n.$$

Similarly $(\lambda - \lambda_0)g_0(\lambda) + f_1(\lambda) = 1$ and so

$$(T - \lambda)A_0 = B_0 - 1.$$
Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda_0$ is an isolated point of $\sigma(T)$, then $\lambda_0$ is called a pole of order $m$ if and only if $E(\lambda_0)(\lambda_0 - T)^m = 0$ and $E(\lambda_0)(\lambda_0 - T)^{m-1} \neq 0$.

**Lemma 2.8.** Let $T$ be a $(p,k)$-quasihyponormal operator and $\lambda_0 \in \text{iso}\sigma(T)$. Let $\tau = \sigma(T) \setminus \{\lambda_0\}$. Then $\lambda_0$ is an eigenvalue of $T$. The ascent and descent of $T - \lambda_0$ are both equal to $k$. Also

$$\mathcal{R}(E(\lambda_0)) = \ker((T - \lambda_0)^k),$$

$$\mathcal{R}(E(\tau)) = \mathcal{R}((T - \lambda_0)^k).$$

**Proof.** For convenience we denote the null-space and range of $(\lambda_0 - T)^k$ by $\ker_k$ and $\mathcal{R}_k$, respectively. If $x \in \ker_k$, where $k \geq 1$, we see by (2.7), induction and (2.8) that

$$0 = A_{k-1}(T - \lambda_0)^k x = (T - \lambda_0)^k A_{k-1} x = (T - \lambda_0)A_0 x = B_1 x - x.$$ 

So that by (2.5), we have $x = B_1 x \in \mathcal{R}(E(\lambda_0))$. Thus $\ker_k \subseteq \mathcal{R}(E(\lambda_0))$ if $k \geq 1$. On the other hand, it follows from (2.4) that if $x \in \mathcal{R}(E(\lambda_0))$, then $x = B_1 x$ and $(T - \lambda_0)^k x = B_{k+1} x$. Since $B_{n+1} x = 0$ if $n \geq k$. It follows that $\mathcal{R}(E(\lambda_0)) \subseteq \ker_k$ and $\ker_n = \mathcal{R}(E(\lambda_0))$ if $n \geq k$. However, $\ker_{k-1}$ is a proper subset of $\ker_k$ because $B_k \neq 0$. The equations $\ker_{k-1} = \ker_k = \mathcal{R}(E(\lambda_0))$ imply that $B_k = 0$ in view of the relation $B_k = (T - \lambda_0)^{k-1} B_1$. We have now proved that the ascent of $\lambda_0 - T$ is $k$ and $\ker_k = \mathcal{R}(E(\lambda_0))$. In particular, since $k > 0$, $\lambda_0$ is an eigenvalue of $T$.

Now let $T_1$ and $T_2$ be the restrictions of $T$ to $\mathcal{R}(E(\tau))$ and $\mathcal{R}(E(\lambda_0))$, respectively. $\lambda_0 \in \sigma(T_2)$ but $\lambda_0 \notin \sigma(T_1)$. Hence, the descent of $\lambda_0 - T_1$ is 0 and $\mathcal{R}((\lambda_0 - T_1)^k) = \mathcal{R}(E(\tau))$ when $k \geq 1$. Thus $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_k$. Now if $n \geq k$, the only point common to $\mathcal{R}_n$ and $\ker_n$ is 0. For, if $x \in \mathcal{R}_n \cap \ker_n$, then $(\lambda_0 - T)^n x = 0$ and there is $y \in \mathcal{H}$ such that $x = \lambda_0 - T)^n y$. Hence $y \in \ker_{2n}$ is ker and so $x = 0$. Now suppose that $n \geq k$ and $x \in \mathcal{R}_n$. Let $x_1 = E(\tau)x$ and $x_2 = E(\lambda_0)$, then $x_2 = x - x_1 \in \mathcal{R}_n$ because $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_n$. However, $x_2 \in \mathcal{R}(E(\lambda_0)) = \ker_n$, and so $x_2 = 0$ whence $x = x_1 \in \mathcal{R}(E(\tau))$. Thus $\mathcal{R}_n \subseteq \mathcal{R}(E(\tau))$ if $n \geq k$ and therefore that the descent of $\lambda_0 - T$ is less that or equal to $k$. Then by [15, Proposition 1.49] shows that the descent is exactly $k$, which know to be the ascent. \hfill \Box

**Corollary 2.9.** Let $T \in \mathcal{B}(\mathcal{H})$ be a $(p,k)$-quasihyponormal operator. Then $T$ is of finite ascent.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be polaroid if $\text{iso}\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of all poles of $T$. In general, if $T$ is polaroid, then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in C^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

Then $T$ is a compact quasinilpotent operator with $\alpha(T) = 1$, and so $T$ is isoloid. However, since $T$ does not have finite ascent, $T$ is not polaroid.
Proposition 2.10. Let $T$ be an algebraically $(p, k)$-quasihyponormal operator. Then $T$ is polaroid.

Proof. Suppose $T$ is an algebraically $(p, k)$-quasihyponormal operator. Then $p(T)$ is $(p, k)$-quasihyponormal for some nonconstant polynomial $p$. Let $\lambda \in \text{iso}(\sigma(T))$. Using the spectral projection $P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where $D$ is a closed disk of center $\lambda$ which contains no other points of $\sigma(T)$, we can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and } \sigma(T_1) = \{\lambda\} \quad \text{and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $T_1$ is algebraically $(p, k)$-quasihyponormal and $\sigma(T_1) = \{\lambda\}$. But $\sigma(T_1 - \lambda I) = \{0\}$, it follows from Proposition 2.6 that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \lambda I$ has finite ascent and descent. On the other hand, since $T_2 - \lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda I$ has finite ascent and descent. Therefore $\lambda$ is a pole of the resolvent of $T$. Thus if $\lambda \in \text{iso}(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $\text{iso}(\sigma(T)) \subset \pi(T)$. Hence $T$ is polaroid.

Corollary 2.11. Let $T$ be an algebraically $(p, k)$-quasihyponormal operator. Then $T$ is isoloid.

For $T \in \mathcal{B}(H)$, $\lambda \in \sigma(T)$ is said to be a regular point if there exists $S \in \mathcal{B}(H)$ such that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$. $T$ is is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well known [19, Theorems 4.6.4 and 8.4.4] that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$ for some $S \in \mathcal{B}(H) \iff T - \lambda I$ has a closed range.

Theorem 2.12. Let $T$ be an algebraically $(p, k)$-quasihyponormal operator. Then $T$ is reguloid.

Proof. Suppose $T$ is an algebraically $(p, k)$-quasihyponormal operator. Then $p(T)$ is a $(p, k)$-quasihyponormal operator for some nonconstant polynomial $p$. Let $\lambda \in \text{iso}(\sigma(T))$. Using the spectral projection $P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where $D$ is a closed disk of center $\lambda$ which contains no other points of $\sigma(T)$, we can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and } \sigma(T_1) = \{\lambda\} \quad \text{and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since $T_1$ is algebraically $(p, k)$-quasihyponormal and $\sigma(T_1) = \{\lambda\}$, it follows from Lemma 2.5 that $T_1 = \lambda I$. Therefore by [34, Theorem 6],

$$\mathcal{H} = \mathcal{E}(\mathcal{H}) \oplus \mathcal{E}(\mathcal{H})^\perp = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^\perp.$$

Relative to decomposition 2.9, $T = \lambda I \oplus T_2$. Therefore $T - \lambda I = 0 \oplus T - \lambda I$ and hence $\text{ran}(T - \lambda I) = (T - \lambda I)(\mathcal{H}) = 0 \oplus (T_2 - \lambda I)(\ker(T - \lambda I)^\perp)$. Since $T_2 - \lambda I$ is invertible, $T - \lambda I$ has closed range.

Theorem 2.13. Let $T^* \in \mathcal{B}(H)$ be an algebraically $(p, k)$-quasihyponormal operator. Then $T$ is a-isoloid.
Proof. Suppose $T^*$ is algebraically $(p,k)$-quasihyponormal. Since $T^*$ has SVEP, then $\sigma(T) = \sigma_a(T)$. Let $\lambda \in \text{iso}(\sigma_a(T)) = \text{iso}(\sigma(T))$. But $T^*$ is polaroid, hence $T$ is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus $T$ is a-isoloid. \hfill \Box

3. Weyl’s type theorem

Lemma 3.1. If $T$ is a $(p,k)$-quasihyponormal operator and $S \prec T$, then $S$ has SVEP.

Proof. Since $T$ is a $(p,k)$-quasihyponormal operator, then it has a SVEP. So the result follows from [14, Lemma 3.1]. \hfill \Box

Theorem 3.2. Let $S, T \in \mathcal{B}(\mathcal{H})$. If $T$ has SVEP and $S \prec T$, then $f(S) \in gaB$ for every $f \in Hol(\sigma(T))$. In particular, if $T$ has SVEP, then $T \in gaB$.

Proof. Suppose that $T$ has SVEP. Since $S \prec T$, it follows from the proof of [14] that $S$ has SVEP. We now show that $S \in gaB$. Let $\lambda \in \sigma_a(S) \setminus \sigma_{SBF^+}(S)$; then $S = \lambda I \in SBF^+(S)$ but not bounded below. Since $S - \lambda I \in SBF^+(S)$, it follows from [11, Corollary 2.10] that $S - \lambda I = S_1 \oplus S_2$, where $S_1$ is an upper semi-Fredholm operator with $i(S_1) \leq 0$, and $S_2$ is nilpotent. Since $S$ has SVEP, $S_1$ and $S_2$ also have SVEP. Therefore $a$-Browder’s theorem holds for $S_1$, and hence $\sigma_{ab}(S_1) = \sigma_{SBF^-}(S_1)$. Since $S_1$ is semi-Fredholm with $i(S_1) \leq 0$, $S_1$ is $a$-Browder’s. Hence $\lambda$ is an isolated point of $\sigma_a(S)$. It follows that $S \in gaB$.

Now let $f \in Hol(\sigma(T))$. Since the SVEP is stable under the functional calculus, then $f(S)$ has the SVEP. Therefore $f(S) \in gaB$, by the first part of the proof. \hfill \Box

We now recall that the generalized $a$-Weyl’s theorem may not hold for quasinilpotent operators, and that it does not necessarily transfer to or from adjoints.

Example 3.3. Let $T \in \mathcal{B}(\mathcal{H})$ defined on $\ell^2$ by

$$T(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

Then $T$ is a quasinilpotent operator and $\sigma(T) = \sigma_{SBF^-}(T) = E^0(T) = \{0\}$. Thus $T$ does not obey generalized $a$-Weyl’s theorem.

Now $\sigma(T^*) = \sigma_{SBF^-}(T^*) = \{0\}$ and $E^0(T^*) = \emptyset$. Therefore $T^* \in gaW$.

As a consequence of [17, Theorem 2.4] and [16, Lemma 2.5] we have:

Theorem 3.4. Let $T \in \mathcal{B}(\mathcal{H})$ be a $(p,k)$-quasihyponormal operator. Then $T$ is of stable index.

Let $T \in \mathcal{B}(\mathcal{H})$. It is well known that the inclusion $\sigma_{SBF^-}(f(T)) \subseteq f(\sigma_{SBF^-}(T))$ holds for every $f \in Hol(\sigma(T))$ with no restriction on $T$ [29]. The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically $(p,k)$-quasihyponormal operator.
Theorem 3.5. Suppose $T^*$ or $T$ is an algebraically $(p,k)$-quasihyponormal operator. Then
\[
\sigma_{SF^-}(f(T)) = f(\sigma_{SF^-}(T)).
\]

Proof. Assume first that $T$ is an algebraically $(p,k)$-quasihyponormal operator and let $f \in Hol(\sigma(T))$. It suffices to show that $\sigma_{SF^-}(f(T)) \supseteq f(\sigma_{SF^-}(T))$. Suppose that $\lambda \notin \sigma_{SF^-}(f(T))$. Then $f(T) - \lambda I \notin SF^- (\mathcal{H})$ and
\[
f(T) - \lambda I = c(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)g(T),
\]
where $c, \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{C}$, and $g(T)$ is invertible. Since $T$ is an algebraically $(p,k)$-quasihyponormal operator, it has SVEP. It follows from [2, Theorem 2.6] that $i(T - \mu_j) \leq 0$ for each $j = 1, 2, \ldots, n$. Therefore $\lambda \notin f(\sigma_{SF^-}(T))$, and hence $\sigma_{SF^-}(f(T)) = f(\sigma_{SF^-}(T))$. Suppose now that $T^*$ is an algebraically $(p,k)$-quasihyponormal operator. Then $T^*$ has SVEP, and so by [2, Theorem 2.6] $i(T - \mu_j I) \geq 0$ for each $j = 1, 2, \ldots, n$. Since
\[
0 \leq \sum_{j=1}^{n} i(T - \mu_j I) = i(f(T) - \lambda I) \leq 0,
\]
$T - \mu_j I$ is Weyl for each $j = 1, 2, \ldots, n$. Hence $\lambda \notin f(\sigma_{SF^-}(T))$, and so $\sigma_{SF^-}(f(T)) = f(\sigma_{SF^-}(T))$. This completes the proof. \hfill \Box

Theorem 3.6. Suppose $T^*$ is an algebraically $(p,k)$-quasihyponormal operator. Then $\alpha$-Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.

Proof. Suppose $T^*$ is an algebraically $(p,k)$-quasihyponormal operator. We first show that $\alpha$-Weyls theorem holds for $T$. Suppose that $\lambda \in \sigma_{a}(T) \setminus \sigma_{SF^-}(T)$. Then $T - \lambda I$ is upper semi-Fredholm and $i(T - \lambda I) \leq 0$. Since $T^*$ is an algebraically $(p,k)$-quasihyponormal operator, $T^*$ has SVEP. Therefore by [2, Theorem 2.6] that $i(T - \lambda I) \geq 0$, and hence $T - \lambda I$ is Weyl. Since $T^*$ has SVEP, it follows from [18, Corollary 7] that $\sigma_{a}(T) = \sigma(T)$. Also, since Weyls theorem holds for $T$ by [26], $\lambda \in \pi_{0}^*(T)$.

Conversely, suppose that $\lambda \in \pi_{0}^*(T)$. Since $T^*$ has SVEP, it follows from [18, Corollary 7] that $\sigma_{a}(T) = \sigma(T)$. Therefore $\lambda$ is an isolated point of $\sigma(T)$, and hence $\lambda$ is an isolated point of $\sigma(T^*)$. But $T^*$ is an algebraically $(p,k)$-quasihyponormal operator, hence by Proposition 2.10 that $\lambda \in \pi(T^*)$. Therefore there exists a natural number $n_0$ such that $n_0 = a(T^* - \lambda I) = d(T^* - \lambda I)$. Hence we have $\mathcal{H} = \ker((T^* - \lambda I)^{n_0}) \oplus \ker((T^* - \lambda I)^{n_0})$ and $\ker((T^* - \lambda I)^{n_0}) \oplus \ker((T - \lambda I)^{n_0})$ is closed. Therefore $\ker((T^* - \lambda I)^{n_0})$ is closed and $\mathcal{H} = \ker((T^* - \lambda I)^{n_0}) \oplus \ker((T^* - \lambda I)^{n_0}) \oplus \ker((T - \lambda I)^{n_0}) \oplus \ker((T - \lambda I)^{n_0})$. So $\lambda \in \sigma_{a}(T)$, and hence $T - \lambda I$ is Weyl. Consequently, $\lambda \in \sigma_{a}(T) \setminus \sigma_{SF^-}(T)$. Thus $\alpha$-Weyls theorem holds for $T$. 

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Now we show that $T$ is $a$-isoloid. Let $\lambda$ be an isolated point of $\sigma_a(T)$. Since $T^*$ has SVEP, $\lambda$ is an isolated point of $\sigma(T)$. But $T^*$ is polaroid, hence $T$ is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus $T$ is $a$-isoloid.

Finally, we shall show that $a$-Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$. Let $f \in Hol(\sigma(T))$. Since $a$-Weyls theorem holds for $T$, it satisfies $a$-Browders theorem. Therefore $\sigma_{ab}(T) = \sigma_{SF+}(T)$. It follows from Theorem 3.5 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{SF+}(T)) = \sigma_{SF+}(f(T)),$$

and hence $a$-Browders theorem holds for $f(T)$. So $\sigma_a(f(T)) \setminus \sigma_{SF+}(f(T)) \subset \pi_0^a(T)$. Conversely, suppose that $\lambda \in \pi_0^a(f(T))$. Then $\lambda$ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda I) < 1$. Since $\lambda$ is an isolated point of $\sigma_a(T)$, $\mu_j \in \sigma_a(T)$, then $\mu_j$ is an isolated point of $\sigma_a(T)$. Since $T$ is $a$-isoloid, $0 < \alpha(T - \mu_j) < 1$ for each $j = 1, 2, \ldots, n$. Since $a$-Weyls theorem holds for $T$, $T - \mu_j$ is upper semi-Fredholm and $i(T - \mu_j) \leq 0$ for each $j = 1, 2, \ldots, n$. Therefore $f(T) - \lambda I$ is upper semi-Fredholm and $f(T) - \lambda I = \sum_{j=1}^n i(T - \mu_j) I \leq 0$. Hence $\lambda \in \sigma_a(f(T)) \setminus \sigma_{SF+}(f(T))$, and so $a$-Weyls theorem holds for $f(T)$ for each $f \in Hol(\sigma(T))$. This completes the proof. □

**Theorem 3.7.** Let $T$ be an algebraically $(p, k)$-quasihyponormal operator. Then $\sigma_{SD}(T) = \sigma_{SBF+}(T) \cup \text{acc}(\sigma_a(T))$.

**Proof.** Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{SD}(T)$. Then $T - \lambda I$ is left Drazin invertible but not bounded below. In particular, $T - \lambda I$ is semi-B-Fredholm. Therefore $d = a(T - \lambda) < \infty$ and $\text{ran}((T - \lambda)^{d+1})$ is closed. On the other hand, since $d = a(T - \lambda) < \infty$ and $(\text{ran}(T - \lambda)^{d+1})$ is closed, $\lambda$ is an isolated point of $\sigma_a(T)$. Hence $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF+}(T) \cup \text{acc}(\sigma_a(T)))$.

Conversely, suppose that $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF+}(T) \cup \text{acc}(\sigma_a(T)))$. Then $T - \lambda I$ is semi-B-Fredholm and $\lambda$ is an isolated point of $\sigma_a(T)$. Since $T - \lambda I$ is upper semi-Fredholm, it follows from [11, Corollary 2.10] that $T - \lambda I$ can be decompose as $T - \lambda I = T_1 \oplus T_2$, where $T_1$ is an upper semi-Fredholm operator with $i(T_1) \leq 0$ and $T_2$ is nilpotent. We consider two cases.

Case I. Suppose that $T_1$ is bounded below. Then $T - \lambda I$ is left Drazin invertible, and so $\lambda \notin \sigma_{SD}(T)$.

Case II. Suppose that $T_1$ is not bounded below. Then $0$ is an isolated point of $\sigma_a(T_1)$. But $T_1$ is an upper semi-Fredholm operator, hence it follows from the punctured neighborhood theorem that $T_1$ is $a$-Browder. Therefore there exists a finite rank operator $S_1$ such that $T_1 + S_1$ is bounded below and $T_1 S_1 = S_1 T_1$. Put $F := S_1 \oplus 0$. Then $F$ is a finite rank operator, $TF = FT$ and $T - \lambda I + F = T_1 \oplus T_2 + S_1 \oplus 0 = (T_1 + S_1) \oplus T_2$ is left Drazin invertible. Hence $\lambda \notin \sigma_{SD}(T)$. □
As shown in [12] that the spectral mapping theorem holds for the Drazin spectrum. We prove here the spectral mapping theorem holds for left Drazin spectrum.

**Theorem 3.8.** Let $T$ be an algebraically $(p,k)$-quasihyponormal operator and let $f \in Hol(\sigma(T))$. Then $\sigma_{lD}(f(T)) = f(\sigma_{lD}(T))$.

**Proof.** Suppose that $\mu \notin f(\sigma_{lD}(T))$ and set $h(\lambda) = f(\lambda) - \mu I$. Then $h$ has no zeros in $\sigma_{lD}(T)$. Since $\sigma_{lD}(T) = \sigma_{SBF_{+}}(T) \cup acc(\sigma_{a}(T))$ by Theorem 3.7, we conclude that $h$ has finitely many zeros in $\sigma_{a}(T)$. Now we consider two cases.

Case I. Suppose that $h$ has no zeros in $\sigma_{a}(T)$. Then $h(T) = f(T) - \mu I$ is bounded below, and so $\mu \notin \sigma_{lD}(f(T))$.

Case II. Suppose that $h$ has at least one zero in $\sigma_{a}(T)$. Then

$$h(\lambda) = c(\lambda - \lambda_{1})(\lambda - \lambda_{2})\cdots(\lambda - \lambda_{n})g(\lambda),$$

where $c, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{n} \in \mathbb{C}$ and $g(\lambda)$ is a nonvanishing analytic function on an open neighborhood. Therefore

$$h(T) = c(T - \lambda_{1}I)(T - \lambda_{2}I)\cdots(T - \lambda_{n}I)g(T),$$

where $g(T)$ is bounded below. Since $\mu \notin f(\sigma_{lD}(T))$, $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n} \notin \sigma_{lD}(T)$. Therefore $T - \lambda_{j}I$ is left Drazin invertible, and hence each $T - \lambda_{j}I \in SBF_{+}(r), j = 1, 2, \ldots, n$. But each $\lambda_{j}$ is an isolated point of $\sigma_{a}(T)$, it follows from [11, Theorem 2.8] that each $\lambda_{j}$ is a left pole of the resolvent of $T$. Therefore $a(T - \lambda_{j}I) = d < \infty$ and $ran(T - \lambda_{j}I)^{d+1}$ is closed $(j = 1, 2, \ldots, n)$, so $a((T - \lambda_{1})(T - \lambda_{2})\cdots(T - \lambda_{n})) = s < \infty$ and $ran((T - \lambda_{1})(T - \lambda_{2})\cdots(T - \lambda_{n}))^{s+1}$ is closed. Since $g(T)$ is bounded below, $a(h(T)) = t < \infty$ and $ran(h(T))^{t+1}$ is closed. Therefore $h(T)$ is left Drazin invertible, and so $\mu \notin \sigma_{lD}(h(T))$. Hence $\mu \notin \sigma_{lD}(f(T))$. It follows from Cases I and II that $\sigma_{lD}(f(T)) \subseteq f(\sigma_{lD}(T))$.

Conversely, suppose that $\lambda \notin \sigma_{lD}(f(T))$. Then $f(T) - \lambda I$ is left Drazin invertible. We again consider two cases.

Case I. Suppose that $f(T) - \lambda I$ is bounded below. Then $\lambda \notin \sigma_{a}(f(T)) = f(\sigma_{a}(T))$, and hence $\lambda \notin f(\sigma_{lD}(T))$.

Case II. Suppose that $\lambda \in \sigma_{a}(f(T)) \setminus \sigma_{lD}(f(T))$. Write

$$f(T) = c(T - \lambda_{1}I)(T - \lambda_{2}I)\cdots(T - \lambda_{n}I)g(T),$$

where $c, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{n} \in \mathbb{C}$ and $g(T)$ is bounded below. Since $f(T) - \lambda I$ is left Drazin invertible, $f(T) = c(T - \lambda_{1}I)(T - \lambda_{2}I)\cdots(T - \lambda_{n}I)g(T)$ has finite ascent say $r$ and $ran(f(T))^{r+1}$ is closed. Hence $T - \lambda_{j}I$ has finite ascent say $r_{j}$ and $ran(T - \lambda_{j}I)^{r_{j}+1}$ is closed for every $j = 1, 2, \ldots, n$. Therefore each $T - \lambda_{j}I$ is left Drazin invertible, and so $\lambda_{1}, \ldots, \lambda_{n} \notin \sigma_{lD}(T)$.

We now wish to prove that $\lambda \notin f(\sigma_{lD}(T))$. Assume not; then there exists $\mu \in \sigma_{a}(T)$ such that $f(\mu) = \lambda$. Since $g(\mu) \neq 0$, we must have $\mu = \mu_{j}$ for some $j = 1, 2, \ldots, n$, which implies $\mu_{j} \in \sigma_{lD}(T)$, a contradiction. Hence $\lambda \notin f(\sigma_{lD}(T))$, and so $f(\sigma_{lD}(T)) \subseteq \sigma_{lD}(f(T))$. This completes the proof. □
Theorem 3.9. Suppose $T$ or $T^*$ is an algebraically $(p,k)$-quasihyponormal operator. Then $f(\sigma_{SBF^+_+}(T)) = \sigma_{SBF^+_+}(f(T))$ for all $f \in \text{Hol}(\sigma(T))$.

Proof. Let $\lambda \notin \sigma_{SBF^+_+}(f(T))$. Then $f(T) - \lambda I \in SBF^+_+(T)$ and
\[
f(T) - \lambda I = \prod_{j=1}^{m} (T - \lambda_j I)g(T),
\]
where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T) - \lambda I$ is an upper semi-$B$-Fredholm operator, it follows from [7, Theorem 3.2] that $T - \lambda_j I$ is upper semi-$B$-Fredholm for each $1 \leq j \leq m$. Hence
\[
i(f(T) - \lambda I) = \sum_{j=1}^{m} i(T - \lambda_j I) \leq 0.
\]
Now from [7, Remark A] there exists some integer $k$ such that for each, $1 \leq j \leq m$, $T - (\lambda_j + \frac{1}{k})I$ is an upper semi-$B$-Fredholm operator and $i(T - (\lambda_j + \frac{1}{k})I) = i(T - \lambda_j I)$. If $T$ is an algebraically $(p,k)$-quasihyponormal operator, then it follows from Proposition 1.1 that $i(T - \lambda_j I) \leq 0$. Hence $\lambda \notin f(\sigma_{SBF^+_+}(T))$.

Now if $T^*$ is an algebraically $(p,k)$-quasihyponormal operator, then we have from Proposition 1.1 that $i(T - \lambda_j I) = 0$ and so $T - \lambda_j I$ is a $B$-Fredholm operator of index 0. Thus $\lambda \notin f(\sigma_{SBF^+_+}(T))$.

For the converse inclusion. Let $\lambda \in \sigma_{SBF^+_+}(f(T)) \setminus f(\sigma_{SBF^+_+}(T))$. Suppose that
\[
f(T) - \lambda I = \prod_{j=1}^{m} (T - \lambda_j I)g(T),
\]
where $\lambda_1, \ldots, \lambda_m \in \mathbb{C} \setminus \sigma_{SBF^+_+}(T)$ and $g(T)$ is invertible. Hence $f(T) - \lambda I$ is upper semi-$B$-Fredholm and $i(f(T) - \lambda I) = \sum_{j=1}^{m} i(T - \lambda_j I) \leq 0$. Therefore $\lambda \notin \sigma_{SBF^+_+}(f(T))$, so a contradiction. $\square$

Lemma 3.10. Suppose that $T \in B(H)$ is algebraically $(p,k)$-quasihyponormal. Then for any $f \in \text{Hol}(\sigma(T))$ we have
\[
\sigma_a(f(T)) \setminus E^a(f(T)) = f(\sigma_a(T)) \setminus E^a(T).
\]

Proof. Let $\lambda \in \sigma_a(f(T)) \setminus E^a(f(T))$. Then $\lambda \in \sigma_a(f(T)) = \sigma_a(T)$. We distinguish two cases:

Case I. $\lambda \notin \text{iso}(f(\sigma_a(T)))$, then there is an infinite sequence $\{\eta_n\}_{n \in \mathbb{N}} \in \sigma_a(T)$ such that $\lambda = f(\eta_0)$ and $\eta_n \rightarrow \eta_0$. But $f \in \text{Hol}(\sigma(T))$, therefore $f(\eta_n) \rightarrow f(\eta_0) = \lambda$ and $\lambda \in f(\sigma_a(T) \setminus E^a(T))$.

Case II. $\lambda \in \text{iso}(f(\sigma_a(T)))$, since $\lambda \notin E^a(f(T))$ then $\lambda$ is not an eigenvalue of $f(T)$. Then
\[
f(T) - \lambda I = (T - \eta_1 I)^{\ell_1}(T - \eta_2 I)^{\ell_2} \cdots (T - \eta_m I)^{\ell_m}g(T),
\]
where \( \eta_1, \ldots, \eta_m \) are scalars and \( g \) is invertible. Since \( \lambda \) is not an eigenvalue of \( f(T) \), then for each \( j \in \{1, \ldots, m\} \), \( \eta_j \) is not an eigenvalue of \( T \). Hence \( \eta_j \in \sigma_a(T) \setminus \mathbb{E}_a(T) \) and \( \lambda = f(\eta_j) \in f(\sigma_a(T) \setminus \mathbb{E}_a(T)) \).

Conversely, let \( \lambda \in f(\sigma_a(T) \setminus \mathbb{E}_a(T)) \) then \( \lambda \in \sigma_a(f(T)) = f(\sigma_a(T)) \). Assume that \( \lambda \in \mathbb{E}_a(f(T)) \). Then

\[
f(T) - \lambda I = (T - \eta_1 I)^{t_1} (T - \eta_2 I)^{t_2} \cdots (T - \eta_m I)^{t_m} g(T),
\]

where \( \eta_1, \ldots, \eta_m \) are scalars and \( g \) is invertible. If \( \eta_j \in \sigma_a(T) \), then \( \eta_j \in iso(\sigma_a(T)) \). Since \( T \) is a-isoloid, \( \eta_j \) is an eigenvalue of \( T \). Hence \( \eta_j \in \mathbb{E}_a(T) \). So \( \lambda = f(\eta_j) \) this leads a contradiction to the fact that \( \lambda \in f(\sigma_a(T) \setminus \mathbb{E}_a(T)) \).

**Theorem 3.11.** Let \( T^* \in \mathcal{B}(\mathcal{H}) \) be algebraically \((p, k)\)-quasihyponormal. Then generalized a-Weyl’s theorem holds for \( f(T) \), for every \( f \in Hol(\sigma(T)) \).

**Proof.** If \( T^* \) is an algebraically \((p, k)\)-quasihyponormal operator, then \( T^* \) has SVEP \( \sigma(T) = \sigma_a(T) \) and consequently \( E(T) = \mathbb{E}_a(T) \).

Let \( \lambda \notin \sigma_{\text{SBF}_+}(T) \) be given. Then \( T - \lambda \) is semi-B-Fredholm and \( i(T - \lambda) \leq 0 \). Then Proposition 1.1 implies that \( i(T - \lambda) = 0 \) and consequently \( T - \lambda \) is B-Weyl’s. Hence \( \lambda \notin \sigma_{\text{BHF}}(T) \). Hence it follows from [37, Theorem 3.1] that \( \lambda \in E(T) = \mathbb{E}_a(T) \).

For the converse, let \( \lambda \in \mathbb{E}_a(T) \). Then \( \lambda \in iso(\sigma_a(T)) \). Since \( T^* \) has SVEP, we have \( \sigma(T) = \sigma_a(T) \). Hence \( \lambda \in \sigma(T^*) \). Now we represent \( T^* \) as the direct sum \( T^* = T_1 \oplus T_2 \), where \( \sigma(T_1) = \{\lambda\} \) and \( \sigma(T_2) = \sigma(T) \setminus \{\lambda\} \). Since \( T \in \mathcal{Y} \) then so does \( T_1 \), and so we have two cases:

Case I. \((\lambda = 0)\): then \( T_1 \) is quasinilpotent. Hence it follows that \( T_1 \) is nilpotent. Since \( T_2 \) is invertible, Then \( T^* \) is B-Weyl’s.

Case II. \((\lambda \neq 0)\): Since \( \sigma(T_1) = \{\lambda\} \), then \( T_1 - \lambda \) is nilpotent and \( T_2 - \lambda \) is invertible, it follows from [37, Theorem 3.1] that \( T^* - \lambda \) is B-Weyl’s. Thus in any case \( \lambda \in \sigma_a(T) \setminus \sigma_{\text{SBF}_+}(T) \).

Let \( f \in Hol(\sigma(T)) \). Since \( T \) is a-isoloid, then it follows from Theorem 3.9 that \( \sigma_{\text{SBF}_+}(f(T)) = f(\sigma_{\text{SBF}_+}(T)) = f(\sigma_a(T) \setminus \mathbb{E}_a(T)) = \sigma_a(f(T)) \setminus \mathbb{E}_a(f(T)) \).

Thus generalized a-Weyl’s theorem holds for \( f(T) \). \( \square \)

**Corollary 3.12.** Let \( T^* \in \mathcal{B}(\mathcal{H}) \) be an algebraically \((p, k)\)-quasihyponormal. Then \( \mathbb{E}_a(T) = \pi_a(T) \).

**Proof.** If \( T^* \) is an algebraically \((p, k)\)-quasihyponormal operator, then \( \sigma_a(T) \setminus \sigma_{\text{SBF}_+}(T) = \mathbb{E}_a(T) \). Let \( \lambda \in \mathbb{E}_a(T) \). Then \( \lambda \) is isolated in \( \sigma_a(T) \), and \( \lambda \notin \sigma_{\text{SBF}_+}(T) \). So \( T - \lambda I \) is in \( \text{SBF}_+(\mathcal{H}) \). It follows from [11, Theorem 2.8] that \( \lambda \) is a left pole of \( T \), and so \( \lambda \in \pi_a(T) \). As we have always \( \pi_a(T) \subset \mathbb{E}_a(T) \), then \( \mathbb{E}_a(T) = \pi_a(T) \).

\( \square \)

**Definition 3.13.** Let \( T \in \mathcal{B}(\mathcal{H}) \) and let \( k \in \mathbb{N} \). Then \( T \) has a uniform descent for \( n \geq k \) if \( R(T) + \ker(T^n) = R(T) + \ker(T^k) \) for all \( n \geq k \). If, in addition,
\( \mathcal{R}(T) + \ker(T^k) \) is closed, then \( T \) is said to have a topological uniform descent for \( n \geq k \).

An operator \( T \in \mathcal{B}(\mathcal{H}) \) is called \( a \)-polaroid if \( \text{iso} \sigma_a(T) \subset \pi^n(T) \). In general, if \( T \) is \( a \)-polaroid, then it is polaroid. However, the converse is not true. Consider the following example.

**Example 3.14.** Let \( R \) be the unilateral right shift on \( \ell^2(\mathbb{N}) \) and define
\[
U(x_1, x_2, \ldots) := (0, x_2, x_3, \ldots) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).
\]
Clearly, \( U \) is a quasi-nilpotent operator.

**Theorem 3.15.** Let \( T^* \in \mathcal{B}(\mathcal{H}) \) be an algebraically \((p,k)\)-quasihyponormal operator. Then \( T \) is \( a \)-polaroid.

**Proof.** Suppose \( T^* \) is algebraically \((p,k)\)-quasihyponormal. Since \( T^* \) has the SVEP, then \( \sigma_a(T^*) = \sigma(T^*) \). Let \( \lambda \in \text{iso}(\sigma_a(T^*)) = \text{iso}(\sigma(T^*)) \). Since \( a \)-Weyl’s theorem holds for \( T \) by Theorem 3.6, then \( \lambda \) is a left pole of finite rank of \( T \). Therefore \( T - \lambda I \) has a finite ascent \( k = a(T - \lambda I) \) and \( \mathcal{R}(T - \lambda I)^{k+1} \) is closed. Since \( T - \lambda I \) is also an operator of topological uniform descent for \( n \geq 0 \), then it follows from [9, Lemma 2.8] that \( T - \lambda I \) is injective. So \( a(T - \lambda I) = 0 \) and \( \mathcal{R}(T - \lambda I) \) is closed. Since \( \pi^n(T) = E^n(T) \), we see that \( \lambda \) is a left pole of \( T \).

That is, all isolated points of the approximate point spectrum of \( T \) are left poles of the resolvent of \( T \).

\[ \square \]

**References**


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