HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE PARALLEL NORMAL JACOBI OPERATOR

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Abstract. In this paper we give some non-existence theorems for Hopf hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator $R_N$ and totally geodesic $\mathcal{D}$ and $\mathcal{D}^\perp$ components of the Reeb vector.

0. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(M, \bar{g})$ play an important role in the study of differential geometry. It satisfies a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operators. That is, if $R$ is the curvature operator of $M$ and $X$ is any vector field tangent to $M$, the Jacobi operator with respect to $X$ at $x \in M$, $R_X \in \text{End}(T_x M)$, is defined as $R_X(Y)(x) = (\nabla_Y X)(x)$ for all $Y \in T_x M$, being a self-adjoint endomorphism of the tangent bundle $TM$ of $M$. Clearly, each vector field $X$ tangent to $M$ provides a Jacobi operator with respect to $X$ (See [7] and [9]).

If the structure vector field $\xi = -JN$ of a real hypersurface $M$ in complex projective space $P_n(\mathbb{C})$ is invariant under the shape operator, $\xi$ is said to be Hopf, where $J$ denotes a Kähler structure of $P_n(\mathbb{C})$, and $N$ is a unit normal vector field of $M$ in $P_n(\mathbb{C})$.

In the quaternionic projective space $\mathbb{H}P^m$ Pérez and Suh [10] classified the real hypersurfaces in $\mathbb{H}P^m$ with $\mathcal{D}^\perp$-parallel curvature tensor $\nabla_\xi R = 0$ for $\nu = 1, 2, 3$, where $R$ denotes the curvature tensor of $M$ in $\mathbb{H}P^m$ and $\mathcal{D}^\perp$ is a distribution defined by $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$. In this case they are congruent to a tube of radius $\frac{1}{2}$ over a totally geodesic quaternionic submanifold $\mathbb{H}^k$ in $\mathbb{H}P^m$, $2 \leq k \leq m - 2$.

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The vector fields \( \{\xi_1, \xi_2, \xi_3\} \) mentioned above, which are said to be *almost contact structure*, are defined by \( \xi_\nu = -J_\nu N \), \( \nu = 1, 2, 3 \), where \( \{J_1, J_2, J_3\} \) denote a local basis of a quaternionic Kähler structure of \( \mathbb{H}P^m \) and \( N \) is a unit normal vector field of \( M \) in \( \mathbb{H}P^m \).

In quaternionic space forms, Berndt [1] introduced the notion of *normal Jacobi operator*
\[ \hat{R}_N X = \hat{R}(X, N)N \in \text{End} (T_x M), \quad x \in M \]
for real hypersurfaces \( M \) in a quaternionic projective space \( \mathbb{H}P^m \) or in a quaternionic hyperbolic space \( \mathbb{H}H^m \), where \( \hat{R} \) denotes the curvature tensor of \( \mathbb{H}P^m \) and \( \mathbb{H}H^m \) respectively. Berndt [1] also showed that “*the curvature adaptedness*”, when the normal Jacobi operator \( \hat{R}_N \) commutes with the shape operator \( A \), is equivalent to the fact that the distributions \( \mathcal{D} \) and \( \mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\} \) are invariant under the shape operator \( A \) of \( M \), where \( T_x M = \mathcal{D} \oplus \mathcal{D}^\perp \), \( x \in M \).

Now let us consider a complex two-plane Grassmannian \( G_2(\mathbb{C}^{m+2}) \) which consists of all complex 2-dimensional linear subspaces in \( \mathbb{C}^{m+2} \). The situation for Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+1}) \) with parallel normal Jacobi operator \( \hat{R}_N \) is not so simple and will be quite different from the cases in \( \mathbb{H}P^m \).

In this paper the present authors consider a real hypersurface \( M \) in the complex two-plane Grassmannian \( G_2(\mathbb{C}^{m+2}) \) with Lie parallel normal Jacobi operator, that is, \( \mathcal{L}_X \hat{R}_N = 0 \) for any \( X \in T_x M, \ x \in M \), where \( \hat{R} \) and \( N \) respectively denote the curvature tensor of the ambient space \( G_2(\mathbb{C}^{m+2}) \) and a unit normal vector field of \( M \) in \( G_2(\mathbb{C}^{m+2}) \). The curvature tensor \( \hat{R}(X, Y)Z \) for any vector fields \( X, Y \) and \( Z \) on \( G_2(\mathbb{C}^{m+2}) \) is explicitly defined in Section 1. Then the normal Jacobi operator \( \hat{R}_N \) for the unit normal vector field \( N \) can be defined from the curvature tensor \( \hat{R}(X, N)N \) by putting \( Y = Z = N \).

The ambient space \( G_2(\mathbb{C}^{m+2}) \) is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure \( J \) and a quaternionic Kähler structure \( \mathfrak{J} \) not containing \( J \) (See Berndt [2]). From these two structures \( J \) and \( \mathfrak{J} \), we have geometric conditions naturally induced on a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) such that \( [\xi] = \text{Span}\{\xi\} \) or \( \mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\} \) is invariant under the shape operator. By these two conditions, Berndt and Suh [3] proved the following:

**Theorem A.** Let \( M \) be a connected real hypersurface in \( G_2(\mathbb{C}^{m+2}), m \geq 3 \). Then both \( [\xi] \) and \( \mathcal{D}^\perp \) are invariant under the shape operator of \( M \) if and only if

(A) \( M \) is an open part of a tube around a totally geodesic \( G_2(\mathbb{C}^{m+1}) \) in \( G_2(\mathbb{C}^{m+2}) \), or

(B) \( m \) is even, say \( m = 2n \), and \( M \) is an open part of a tube around a totally geodesic \( \mathbb{H}P^m \) in \( G_2(\mathbb{C}^{m+2}) \).

The structure vector field \( \xi \) of a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) is said to be a *Reeb* vector field. Moreover, the Reeb vector field \( \xi \) is said to be *Hopf* if it is invariant under the shape operator \( A \). The 1-dimensional foliation of \( M \) by
the integral manifolds of the Reeb vector field $\xi$ is said to be a Hopf foliation of $M$. We say that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the formulas in section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf.

The flow generated by the integral curves of the Reeb vector field is said to be a geodesic Reeb flow if $M$ becomes a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. We say that the Reeb vector field is Killing if the Lie derivative of the Riemannian metric $g$ for $M$ in $G_2(\mathbb{C}^{m+2})$ along the Reeb direction vanishes, that is, $\mathcal{L}_\xi g = 0$. This means that the Reeb flow is isometric. Using such a notion, Berndt and Suh [4] proved that a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow becomes an open part of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In [15], Suh also gave a characterization for this kind of hypersurfaces in terms of another geometric Lie invariant. Namely, he characterized them as the hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the shape operator $A$ is invariant under the Reeb flow.

Now by putting a unit normal vector field $N$ into the curvature tensor $\mathcal{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the normal Jacobi operator $\mathcal{R}_N$ can be defined in such a way that

$$\mathcal{R}_N X = \mathcal{R}(X, N) N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(X)\xi_{\nu} - \sum_{\nu=1}^{3} \{ \eta_{\nu}(\xi)(\phi_{\nu}\phi X - \eta(X)\xi_{\nu}) - \eta_{\nu}(\phi X)\phi_{\nu}\xi \}$$

for any tangent vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$.

In the paper [8] due to Jeong, Pérez and Suh, we classified real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting normal Jacobi operator, that is, $\mathcal{R}_N \circ \phi = \phi \circ \mathcal{R}_N$ or $\mathcal{R}_N \circ A = A \circ \mathcal{R}_N$. The fact that the normal Jacobi operator $\mathcal{R}_N$ commutes with the shape operator $A$ (or the structure tensor $\phi$) of $M$ in $G_2(\mathbb{C}^{m+2})$ means that the eigenspaces of the normal Jacobi operator are invariant under the shape operator $A$ (or the structure tensor $\phi$). Also, in [5], Jeong, Kim and Suh introduced the notion of parallel normal Jacobi operator for real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$. Such an operator is said to be parallel if $\nabla_X \mathcal{R}_N = 0$ for any tangent vector field $X$ on $M$. This means that the eigenspaces of the normal Jacobi operator $\mathcal{R}_N$ are parallel along any curve $\gamma$ in $M$. Here the eigenspaces of the normal Jacobi operator $\mathcal{R}_N$ are said to be parallel along $\gamma$ if they are invariant with respect to any parallel displacement along $\gamma$. Using this notion, they gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator.

Related to such a parallel normal Jacobi operator, in this paper the authors give a theorem for real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X \mathcal{R}_N = 0$ for any $X \in T_x M$, $x \in M$. This means that all the eigenspaces of the normal Jacobi operator $\mathcal{R}_N$ are invariant under
any parallel displacement $\phi_t^\ast$ generated from the flow $\phi_t$ such that $\phi_t(x) = \gamma(t)$ and $\gamma(0) = x$ for the integral curve $\gamma$ of $X$ in $T_xM$, $x \in M$.

Then the authors prove the following for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operators:

**Theorem 1.** Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $D$ and $D^\perp$ components of the Reeb vector field $\xi$ are totally geodesic, then $\xi$ belongs to either the distribution $\mathcal{D}$ or the distribution $\mathcal{D}^\perp$.

On the other hand, in the paper [6] of Jeong and Suh, we gave non-existence theorems for real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with Lie-parallel normal Jacobi operator, that is, $\mathcal{L}_\xi R_N = 0$ as follows:

**Theorem B.** There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi R_N = 0$ if the Reeb vector field $\xi \in \mathcal{D}^\perp$.

**Theorem C.** There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi R_N = 0$ if the Reeb vector field $\xi \in \mathcal{D}$.

Then as an application of Theorem 1 to Theorems B and C the authors can assert the following:

**Theorem 2.** There does not exist any Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $D$ and $D^\perp$ components of the Reeb vector field are totally geodesic.

1. **Riemannian geometry of $G_2(\mathbb{C}^{m+2})$**

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [2], [3], and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = SL(2) \times U(m) \subset G$. The space $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = \mathfrak{k} \oplus \mathfrak{m}$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{k}$, negative $B$ restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximum sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $CP^2$ with constant holomorphic sectional curvature eight. When $m = 2$, $G_2(\mathbb{C}^{m+2})$ is isometric to the four-dimensional complex projective space $CP^3$ with constant holomorphic sectional curvature four.
we note that the isomorphism $\text{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G^+_2(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. From now on, in this paper we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition

$$\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_\nu$, $\nu = 1, 2, 3$, is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and $JJ_\nu$ is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\text{tr}(JJ_\nu) = 0$.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index $\nu$ is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\hat{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\hat{R}(X, Y) Z = g(Y, Z) X - g(X, Z) Y + g(JY, Z) JX - g(JX, Z) JY - 2g(JX, Y) JZ$$

$$+ \sum_{\nu=1}^3 \{g(J_\nu Z J_\nu X - g(J_\nu X, Z) J_\nu Y - 2g(J_\nu X, Y) J_\nu Z)\}$$

$$+ \sum_{\nu=1}^3 \{g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY\},$$

where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive some fundamental formulas which will be used in the proof of our theorems and the equation of Codazzi for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [12], [13], and [14]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. More explicitly, we can define a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and its dual 1-form $\eta$ on $M$ by $g(\phi X, Y) = g(JX, Y)$.
that a tensor $\phi$ defined by $\phi_M$ induces an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ in such a way that a tensor field $\phi$, vector field $\xi$, and its dual 1-form $\eta$ on $M$ defined by $g(\phi\xi, X) = g(J\xi, X)$ and $\eta\xi = \xi X$ for any tangent vector fields $X$ and $Y$ on $M$. Then they also satisfy the following

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0 \quad \text{and} \quad \eta(\xi) = 1
\]

for any tangent vector field $X$.

Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$ in such a way that a tensor field $\phi_\nu$ of type $(1, 1)$, a vector field $\xi_\nu$ and its dual 1-form $\eta_\nu$ on $M$ defined by $g(\phi_\nu\xi_\nu, X) = g(J_\nu\xi_\nu, X)$ and $\eta_\nu\xi_\nu = \xi_\nu X$ for any tangent vector fields $X$ and $Y$ on $M$. Then they

\[
\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu = 0, \quad \eta_\nu(\phi_\nu X) = 0 \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1
\]

for any vector field $X$ tangent to $M$ and $\nu = 1, 2, 3$.

Using the above expression (1.2) for the curvature tensor $R$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

\[
(\nabla_X Y - \nabla_Y X) = \eta(Y)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu=1}^{3} \{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \}
\]

\[
+ \sum_{\nu=1}^{3} \{ \eta_\nu(\phi X)\phi_\nu Y - \eta_\nu(\phi Y)\phi_\nu X \}
\]

\[
+ \sum_{\nu=1}^{3} \{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \} \xi_\nu.
\]

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

\[
\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},
\]

\[
\phi\xi_\nu = \phi\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X),
\]

\[
\phi_\nu\phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu,
\]

\[
\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}.
\]

Now let us note that

\[
JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N
\]

for any vector field $X$ tangent to $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

\[
(\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,
\]

\[
(\nabla_X \xi_\nu = \phi_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,
\]

and $\eta(X) = g(X, \xi)$ for any tangent vector fields $X$ and $Y$ on $M$. Then they satisfy the following

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0 \quad \text{and} \quad \eta(\xi) = 1
\]

for any tangent vector field $X$.
where the terms in the right side can be given respectively as follows:

\begin{equation}
(\nabla_X \phi_\nu) Y = - q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y + \eta_\nu(Y) AX - g(AX, Y) \xi_\nu.
\end{equation}

Summing up these formulas, we find the following

\begin{align}
\nabla_X (\phi_\nu \xi) &= \nabla_X (\phi \xi) \\
&= (\nabla_X \phi) \xi + \phi(\nabla_X \xi) \\
&= q_{\nu+2}(X) \phi_{\nu+1} \xi - q_{\nu+1}(X) \phi_{\nu+2} \xi + \phi_\nu \phi AX \\
&- g(AX, \xi) \xi_\nu + \eta(\xi_\nu) AX.
\end{align}

Moreover, from \( JJ_\nu = J_\nu J, \nu = 1, 2, 3, \) it follows that

\begin{equation}
\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X) \xi - \eta(X) \xi_\nu.
\end{equation}

3. Lie parallel normal Jacobi operator

Let \( M \) be a real hypersurface in \( G_2(C^{m+2}) \) with Lie parallel normal Jacobi operator, that is, \( L_X \bar{R}_N = 0 \) for any vector field \( X \) tangent to \( M \). Then first of all, we write the normal Jacobi operator \( \bar{R}_N \), which is given by

\begin{equation}
\bar{R}_N(X) = \bar{R}(X, N) N = X + 3\eta(X) \xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X) \xi_\nu \\
- \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi) J_\nu(\phi X + \eta(X) N) - \eta_\nu(\phi X)(\phi_\nu \phi X + \eta(\xi) N) \right\} \\
= X + 3\eta(X) \xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X) \xi_\nu \\
- \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi)(\phi_\nu \phi X - \eta(X) \xi_\nu) - \eta_\nu(\phi X) \phi_\nu \xi \right\}
\end{equation}

where we have used the following

\begin{align*}
g(J_\nu J N, N) &= -g(J N, J_\nu N) = -g(\xi, \xi_\nu) = -\eta_\nu(\xi), \\
g(J_\nu J X, N) &= g(X, J J_\nu N) = -g(X, J \xi_\nu) \\
&= -g(X, \phi \xi_\nu + \eta(\xi_\nu) N) = -g(X, \phi \xi_\nu),
\end{align*}

and

\[ J_\nu J N = -J_\nu \xi = -\phi_\nu \xi - \eta_\nu(\xi) N. \]

Of course, by (2.7) we know that the normal Jacobi operator \( \bar{R}_N \) is a symmetric endomorphism of \( T_x M, x \in M \).

Now let us consider the Lie derivative of the normal Jacobi operator along any direction. Then for any vector fields \( X \) and \( Y \) tangent to \( M \) it is given by

\begin{equation}
(L_X \bar{R}_N) Y = L_X (\bar{R}_N Y) - \bar{R}_N (L_X Y) \\
= [X, \bar{R}_N Y] - \bar{R}_N [X, Y] \\
= (\nabla_X \bar{R}_N) Y - \nabla_{\bar{R}_N Y} X + \bar{R}_N (\nabla_X Y)
\end{equation}

where the terms in the right side can be given respectively as follows:

\begin{align*}
(\nabla_X \bar{R}_N) Y &= 3(\nabla_X \eta)(Y) \xi + 3\eta(Y) \nabla_X \xi + 3 \sum_{\nu=1}^{3} (\nabla_X \eta_\nu)(Y) \xi_\nu \\
\end{align*}
Then by the formulas given in section 2, (3.2) gives the following for a real and
\[ \nabla_R \nabla Y X = \nabla Y X + 3\eta(Y)\nabla_\xi X + 3 \sum_{\nu=1}^3 \eta_\nu(Y)\nabla_\xi_\nu X - \sum_{\nu=1}^3 \eta_\nu(\xi)\nabla_{\phi_\nu Y} X \]
and
\[ \bar{R}(\nabla Y X) = \nabla Y X + 3\eta(\nabla Y X)\xi + 3 \sum_{\nu=1}^3 \eta_\nu(\nabla Y X)\xi_\nu \]
\[ - \sum_{\nu=1}^3 \{ \eta_\nu(\xi)(\phi_\nu Y - g(\phi_\nu Y)\xi) \} \]

Then by the formulas given in section 2, (3.2) gives the following for a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with Lie parallel normal Jacobi operator \( \bar{R}_N \):
\[ (\mathcal{L}_X \bar{R}_N)Y = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3 \sum_{\nu=1}^3 g(\phi_\nu AX, Y)\xi_\nu \]
\[ + 3 \sum_{\nu=1}^3 \eta_\nu(Y)\phi_\nu AX \]
\[ - \sum_{\nu=1}^3 \{ X(\eta_\nu(\xi))(\phi_\nu Y - g(\phi_\nu Y)\xi) \} \]
\[ + \eta_\nu(\xi) \left\{ - q_{\nu+1}(X)\phi_{\nu+2} Y + q_{\nu+2}(X)\phi_{\nu+1} Y \right\} \]
\[ + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu \]
\[ + \eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu \xi - g(\phi AX, Y)\xi_\nu \]
\[ - \eta(\nu)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu AX) \]
\[ - g(\phi_\nu AX, \phi Y)\phi_\nu \xi - \eta(\nu)\eta_\nu(AX)\phi_\nu \xi + g(AX, Y)\eta_\nu(\xi)\phi_\nu \xi \]
\[ - \eta_\nu(\phi Y)\left\{ \eta_\nu(\xi)AX - g(AX, \xi)\xi_\nu + \phi_\nu AX \right\} \]
\[ - 3\eta(\nu)\nabla_\xi X - 3 \sum_{\nu=1}^3 \eta_\nu(Y)\nabla_\xi_\nu X \]
\[ + \sum_{\nu=1}^3 \{ \eta_\nu(\xi)(\phi_\nu Y - g(\phi_\nu Y)\xi) \} \]
\[ + 3\eta(\nabla Y X)\xi + 3 \sum_{\nu=1}^3 \eta_\nu(\nabla Y X)\xi_\nu \]
\[ - \sum_{\nu=1}^3 \{ \eta_\nu(\xi)(\phi_\nu Y - g(\phi_\nu Y)\xi) \} \]
\[ = 0, \]
where in the first equality we have used the following formulas
\[
3 \sum_{\nu=1}^{3} g(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}, Y)\xi_{\nu} \\
+ 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y)\left\{ q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} \right\} = 0
\]
and
\[
\sum_{\nu=1}^{3} \{ \eta_{\nu+1}(\phi Y)q_{\nu+2}(X)\phi_{\nu}\xi - \eta_{\nu+2}(\phi Y)q_{\nu+1}(X)\phi_{\nu}\xi \\
- \eta_{\nu}(\phi Y)q_{\nu+1}(X)\phi_{\nu+2}\xi + \eta_{\nu}(\phi Y)q_{\nu+2}(X)\phi_{\nu+1}\xi \} = 0.
\]
In particular by putting \( X = \xi \) in (3.3) we have the following
\[
(L_{\xi} R_{Y})Y = 3g(\phi A\xi, Y)\xi + 3\sum_{\nu=1}^{3} g(\phi_{\nu} A\xi, Y)\xi_{\nu} \\
+ 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi_{\nu} A\xi \\
- \sum_{\nu=1}^{3} \left[ \xi(\eta_{\nu}(\xi))(\phi_{\nu}\phi Y - \eta(Y))\xi_{\nu} \right. \\
+ \left. \eta_{\nu}(\xi)\left\{ - q_{\nu+1}(\xi)\phi_{\nu+2}\phi Y + q_{\nu+2}(\xi)\phi_{\nu+1}\phi Y \\
+ \eta_{\nu}(\phi Y)A\xi - g(A\xi, \phi Y)\xi_{\nu} \\
+ \eta(Y)\phi_{\nu} A\xi - g(A\xi, Y)\phi_{\nu}\xi - g(\phi A\xi, Y)\xi_{\nu} \right. \\
- \eta(Y)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_{\nu} A\xi) \right. \\
- g(\phi_{\nu} A\xi, \phi Y)\phi_{\nu}\xi - \eta(Y)\eta_{\nu}(A\xi)\phi_{\nu}\xi + g(A\xi, Y)\eta_{\nu}(\xi)\phi_{\nu}\xi \\
- \eta_{\nu}(\phi Y)\left\{ \eta_{\nu}(A\xi)A\xi - g(\phi_{\nu} A\xi, \xi_{\nu} + \phi_{\nu} \phi A\xi) \right\} \\
- 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi A\xi_{\nu} + 3\sum_{\nu=1}^{3} \eta_{\nu}(\phi AY)\xi_{\nu} \\
+ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi)\left\{ \phi A\phi_{\nu}\phi Y - \eta(Y)\phi A\xi_{\nu} \right. - \left. \eta_{\nu}(\phi Y)\phi A\phi_{\nu}\xi \right\} \\
+ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi)\left\{ \phi_{\nu} AY - \eta(AY)\phi_{\nu}\xi \right. \right. - \left. \eta_{\nu}(AY)\phi_{\nu}\xi \right. \\
+ \left. \eta(AY)\eta_{\nu}(\xi)\phi_{\nu}\xi \right]\right] \\
= 0,
\]
where in the first equality we have used the second formula of (2.3). From this, by putting \( Y = \xi \) in (3.4) we have the following
\[
(L_{\xi} R_{Y})\xi = 6\sum_{\nu=1}^{3} g(\phi_{\nu} A\xi, \xi)\xi_{\nu} + 4\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu} A\xi
\]
\[ (3.5) \quad + \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi) \eta_{\nu} + \eta_{\nu}(\xi) \{ q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2} \} \right] \]
\[ - 4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \phi A_{\xi_{\nu}} \]
\[ = 0. \]

4. Lie parallel normal Jacobi operator

In this section we want to prove the following:

**Proposition 4.1.** Let \( M \) be a Hopf real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with Lie parallel normal Jacobi operator. If the integral curves of \( D \) and \( D^\perp \) components of the Reeb vector field \( \xi \) are totally geodesic, then \( \xi \) belongs to either the distribution \( D \) or the distribution \( D^\perp \).

**Proof.** When the function \( \alpha = g(A\xi, \xi) \) identically vanishes, the proposition was proved directly by Pérez and Suh [11]. Thus we consider only the case that the function \( \alpha \) is non-vanishing in this proof.

By putting \( A\xi = \alpha \xi \) into (3.5) we have

\[ (4.1) \quad \sum_{\nu=1}^{3} \eta_{\nu}(\xi) (\alpha \phi_{\nu} \xi - \phi A_{\xi_{\nu}}) = 0, \]

where we have used the following formula

\[ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi) \eta_{\nu} + \eta_{\nu}(\xi) \{ q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2} \} \right] = 0. \]

Now let us put \( \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \) for some unit \( X_0 \in D \) and \( \xi_1 \in D^\perp \). Then naturally we know that \( \eta(\xi_2) = \eta(\xi_3) = 0 \). Hereafter, unless otherwise stated, let us assume \( \eta(X_0)\eta(\xi_1) \neq 0 \).

Then (4.1) reduces to

\[ \alpha \phi_{\xi} \xi - \phi A_{\xi_{1}} = 0. \]

From this, by taking the structure tensor \( \phi \) and also using that \( \xi \) is principal, we have

\[ (4.2) \quad A\xi_{1} = \alpha \xi_{1} \quad \text{and} \quad A X_0 = \alpha X_0. \]

Then putting \( X = X_0 \) and \( Y = \xi \) into (3.3) and using (4.2) gives

\[ 0 = (\mathcal{L}_{X_0} \tilde{R}_N) \xi \]
\[ = 3\alpha \phi X_0 + 3\alpha \sum_{\nu=1}^{3} g(\phi_{\nu} X_0, \xi) \xi_{\nu} + 3\alpha \eta_1(\xi) \phi_{\xi_1} X_0 \]
\[ + \eta_1(\xi) \{ q_3(X_0)\xi_2 - q_2(X_0)\xi_3 \} - 3 \nabla_{\xi} X_0 - 4 \eta_1(\xi) \nabla_{\xi} X_{0} \]
\[ + 3 \eta(\nabla_{\xi} X_0) \xi + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{\xi} X_0) \xi_{\nu} - \eta_1(\xi) \phi_{\phi} \nabla_{\xi} X_0 \]
\[ + \eta_1(\xi)(\phi \nabla_{\xi} X_0) \xi_1 + \sum_{\nu=1}^{3} \eta_{\nu}(\phi \nabla_{\xi} X_0) \phi_{\nu}, \]

where we have used

\[ X_0(\eta_1(\xi)) \xi_1 = g(\nabla_{X_0} \xi_1, \xi) \xi_1 + g(\xi_1, \nabla_{X_0} \xi) \xi_1 \]
\[ \begin{align*}
&\vDash g(\phi_1 AX_0, \xi_1) + g(\xi_1, \phi AX_0) \\
&\quad = \vDash g(X_0, \phi_1 \xi_1) - \alpha g(\phi_1 \xi_1, X_0) \\
&\quad = -\alpha g(\phi_1 (\eta(X_0) X_0 + \eta(\xi_1)) \\
&\quad = -2\alpha g(X_0, \phi_1 X_0) \\
&\quad = 0.
\end{align*} \]

From this, together with (2.3) and (2.4), and using \( \phi X_0 \in \mathfrak{D} \), \( \nabla_\xi X_0 \in \mathfrak{D} \) and \( \eta(\nabla_\xi X_0) = 0 \), we have
\[
\begin{align*}
0 &= (L_{X_0} \tilde{R}_N)\xi \\
&\quad = 3\alpha (\phi X_0 + \eta_1(\xi) \phi_1 X_0) + \eta_1(\xi) (q_2(X_0) \xi_2 - q_2(X_0) \xi_3) \\
&\quad - 3\nabla_\xi X_0 - 4\eta_1(\xi) \nabla_\xi X_0 - \eta_1(\xi) \phi_1 \phi \nabla_\xi X_0 \\
&\quad + \sum_{\nu=1}^3 \eta_\nu(\phi \nabla_\xi X_0) \phi_\nu \xi, \\
\end{align*}
\]
because we know the following
\[
\begin{align*}
&g(\phi X_0, \xi_\nu) = -g(X_0, \phi_\nu) = -g(X_0, \phi_\nu \xi) = 0, \\
&\eta(\nabla_\xi X_0) = g(\nabla_\xi X_0, \xi_\nu) = g(\nabla_\xi X_0, n(X_0) X_0 + \eta(\xi_1) \xi_1) = 0.
\end{align*}
\]
and
\[
\begin{align*}
g(\nabla_\xi X_0, \xi_\nu) &= -g(X_0, \nabla_\xi \xi_\nu) \\
&= -\alpha g(X_0, \phi_\nu \xi_\nu) \\
&= -\alpha g(X_0, \phi_\nu \xi) \\
&= \alpha g(\phi X_0, \xi_\nu) \\
&= 0
\end{align*}
\]
for any \( \nu = 1, 2, 3 \).

On the other hand, we know that
\[
\nabla_\xi X_0 \in \mathfrak{D},
\]
because
\[
\begin{align*}
g(\nabla_\xi X_0, \xi_\nu) &= -g(X_0, \nabla_\xi \xi_\nu) \\
&= -g(X_0, q_{\nu+2}(\xi_1) \xi_{\nu+1} - q_{\nu+1}(\xi_1) \xi_{\nu+2} + \phi_\nu A \xi_1) \\
&= -\alpha g(X_0, \phi_\nu \xi_1) \\
&= 0.
\end{align*}
\]

Moreover, the following formulas hold
\[
\begin{align*}
g(\phi \nabla_\xi X_0, \xi_2) &= 0 \quad \text{and} \quad g(\phi \nabla_\xi X_0, \xi_3) = 0.
\end{align*}
\]
In fact, differentiating \( g(\phi X_0, \xi_2) = 0 \) gives
\[
\begin{align*}
0 &= g((\nabla_\xi \phi) X_0, \xi_2) + g(\phi \nabla_\xi X_0, \xi_2) + g(\phi X_0, \nabla_\xi \xi_2) \\
&= g(\phi \nabla_\xi X_0, \xi_2) + \alpha g(\phi X_0, \phi \xi_2)
\end{align*}
\]
and similarly the latter term comes from \( g(\phi X_0, \xi_3) = 0 \).

By taking the inner product (4.3) with \( \xi_3 \), and using the facts that \( \phi X_0 \), \( \phi_1 X_0 \), \( \nabla_\xi X_0 \) and \( \nabla_{\xi_1} X_0 \) belong to the distribution \( \mathcal{D} \), we have

\[
0 = -\eta_1(\xi) q_2(X_0) - \eta_1(\xi) g(\phi_1 \phi \nabla_\xi X_0, \xi_3) + \eta_1(\phi \nabla_\xi X_0) g(\phi_1 \xi, \xi_3) \\
= -\eta_1(\xi) q_2(X_0).
\]

Similarly, by taking the inner product with \( \xi_2 \) to (4.3), we have the following relations

\[
q_2(X_0) = 0 \quad \text{and} \quad q_3(X_0) = 0
\]

under the assumption of \( \eta_1(\xi) \neq 0 \). Then (4.4), (4.5) and (4.6) give

\[
0 = (L_{X_0} R_N) \xi
\]

\[
= 3\alpha(\phi X_0 + \eta(\xi_1) \phi_1 X_0) - 3\nabla_\xi X_0 - 4\eta(\xi) \nabla_{\xi_1} X_0 \\
- \eta_1(\xi) \phi_1 \phi \nabla_\xi X_0 + \eta_1(\phi \nabla_\xi X_0) \phi_1 \xi.
\]

On the other hand, by the assumption of \( M \) being Hopf and using (4.2), we have

\[
\nabla_\xi \xi = \phi A \xi
\]

\[
= \phi A(\eta(X_0) X_0 + \eta(\xi_1) \xi_1)
\]

\[
= \alpha(\eta(X_0) \phi X_0 + \eta(\xi_1) \eta(X_0) \phi_1 X_0)
\]

\[
= \alpha \eta(X_0)(\phi X_0 + \eta(\xi_1) \phi_1 X_0)
\]

\[
= 0.
\]

Consequently, we see

\[
\phi X_0 + \eta(\xi_1) \phi_1 X_0 = 0.
\]

from the assumption of \( \alpha \neq 0 \) and \( \eta(X_0) \neq 0 \).

Substituting (4.8) into (4.7), we have

\[
0 = (L_{X_0} R_N) \xi
\]

\[
= -3\nabla_\xi X_0 - 4\eta(\xi) \nabla_{\xi_1} X_0 - \eta_1(\xi) \phi_1 \phi \nabla_\xi X_0 + \eta_1(\phi \nabla_\xi X_0) \phi_1 \xi.
\]

Now, by applying the operator \( \phi_1 \) to (4.8) we have

\[
(4.9) \quad \phi_1 \phi X_0 = \eta(\xi_1) X_0.
\]

Then by differentiating (4.9) along the direction of the Reeb vector field \( \xi \) and using (2.1), (2.3), (2.4), (2.5) and (4.8), we have

\[
(4.10) \quad q_2(\xi) \eta(\xi_1) \phi_2 X_0 + q_3(\xi) \eta(\xi_1) \phi_3 X_0 + \phi_1 \phi \nabla_\xi X_0 = \eta(\xi_1) \nabla_\xi X_0.
\]

By taking the inner product (4.10) with \( \xi_2 \) and \( \xi_3 \) respectively and using the fact that \( \nabla_\xi X_0 \), \( \phi_2 X_0 \in \mathcal{D} \), \( \nu = 1, 2, 3 \), we have the following respectively

\[
(4.11) \quad g(\nabla_\xi X_0, \phi_3 X_0) = 0 \quad \text{and} \quad g(\nabla_\xi X_0, \phi_2 X_0) = 0.
\]
On the other hand, the assumption that $\mathcal{D}^\perp$-component of $\xi$ is totally geodesic and (4.2) give

\begin{equation}
q_2(\xi_1) = q_3(\xi_1) = 0.
\end{equation}

Let us differentiate the formula (4.9) along the direction of $\xi_1$. Then by virtue of the formulas (2.3), (2.4), (2.5) and (4.12), we have

\begin{equation}
\phi_1 \phi \nabla_{\xi_1} X_0 = \eta(\xi_1) \nabla_{\xi_1} X_0.
\end{equation}

On the other hand, by taking the inner product (4.10) with $\phi_2 X_0$, $\phi_3 X_0$ respectively and using (2.1), (2.7) and (4.11) respectively we have

\begin{equation}
q_2(\xi) = 0 \quad \text{and} \quad q_3(\xi) = 0.
\end{equation}

Then (4.10) implies that

\begin{equation}
\phi_1 \phi \nabla_{\xi_1} X_0 = \eta(\xi_1) \nabla_{\xi_1} X_0.
\end{equation}

Moreover, by differentiating (4.8) along the direction of $\xi$ and using (2.3), (2.4), (2.5) and (4.14), we have

\begin{equation}
\phi \nabla_{\xi} X_0 = \alpha \eta(\xi) \eta(X_0) \xi_1 - \eta_1(\xi) \phi_1 \nabla_{\xi} X_0.
\end{equation}

From this, by applying $\phi$ and using (4.15) we have

\begin{equation}
\nabla_{\xi} X_0 = -\alpha \eta(\xi_1) \phi_1 X_0.
\end{equation}

Now differentiating (4.8) along the direction $\xi_1$ and using (2.3) and (2.5), we have

\begin{equation}
\alpha \eta(X_0) \xi_1 + \phi \nabla_{\xi_1} X_0 = -\eta_1(\xi) \phi_1 \nabla_{\xi_1} X_0.
\end{equation}

Similarly, by applying $\phi$ to above equation and using (4.13) we have

\begin{equation}
\nabla_{\xi_1} X_0 = \alpha \phi_1 X_0.
\end{equation}

Then (4.16) and (4.17) give

\begin{equation}
\nabla_{\xi} X_0 = -\eta(\xi_1) \nabla_{\xi_1} X_0.
\end{equation}

On the other hand, we know that

\begin{equation}
\begin{aligned}
\nabla_{\xi} X_0 &= \nabla_{\eta(X_0) X_0 + \eta(\xi_1) \xi_1} X_0 \\
&= \eta(X_0) \nabla_{X_0} X_0 + \eta(\xi_1) \nabla_{\xi_1} X_0 \\
&= \eta(\xi_1) \nabla_{\xi_1} X_0,
\end{aligned}
\end{equation}

because the $\mathcal{D}$-component of the Reeb vector field $\xi$ is totally geodesic. From (4.18) and (4.19) we see that $\eta(\xi_1) \nabla_{\xi_1} X_0 = 0$. This means that $\nabla_{\xi_1} X_0 = 0$. From this together with (4.17), it follows that $\phi_1 X_0 = 0$. This gives a contradiction. So we only have $\xi \in \mathcal{D}$ or $\xi \in \mathcal{D}^\perp$. \qed
5. Lie parallel normal Jacobi operator for $\xi \in D^\perp$

In order to give a complete proof of Theorem 2, first we consider the case that the Reeb vector field $\xi$ belongs to the distribution $D^\perp$. Now in this direction we introduce some lemmas given in Jeong and Suh [6] as follows:

**Lemma 5.A.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in D^\perp$. Then $A\xi = \alpha \xi + \beta U$, where $U$ is a unit vector field orthogonal to $\xi$ and belongs to $D$.

Moreover, from Lemma 5.A, they proved the following lemmas:

**Lemma 5.B.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in D^\perp$. Then $\beta$ identically vanishes, that is, the Reeb vector field $\xi$ is principal.

**Lemma 5.C.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in D^\perp$. Then $g(A D, D^\perp) = 0$.

From these lemmas we assert:

**Lemma 5.1.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie parallel normal Jacobi operator and $\xi \in D^\perp$. Then the Reeb vector $\xi$ is principal and $g(A D, D^\perp) = 0$.

Before going to give our proof of Theorem 2 in the introduction, let us check “What kind of model hypersurfaces given in Theorem A satisfy Lie parallel normal Jacobi operator.” In other words, it will be an interesting problem to know whether there exist real hypersurfaces in $G_2(C^{m+2})$ satisfying the condition $L_X R_N = 0$ for $\xi \in D^\perp$.

Then by virtue of Lemmas 5.1, we are able to recall the proposition given by Berndt and Suh [3] as follows:

**Proposition A.** Let $M$ be a connected real hypersurface of $G_2(C^{m+2})$. Suppose that $A D \subset D$, $A \xi = \alpha \xi$, and $\xi$ is tangent to $D^\perp$. Let $J_1$ be the almost Hermitian structure such that $JN = J_1 N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures

$$
\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0
$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are $m(\alpha) = 1$, $m(\beta) = 2$, $m(\lambda) = 2m - 2 = m(\mu)$, and the corresponding eigenspaces we have

$$
T_\alpha = \mathbb{R} \xi = \mathbb{R} JN = \mathbb{R} \xi_1, \\
T_\beta = C_+ \xi = C_+ N = \mathbb{R} \xi_2 \oplus \mathbb{R} \xi_3, \\
T_\lambda = \{ X | X \perp \mathbb{R} \xi, JX = J_1 X \}, \\
T_\mu = \{ X | X \perp \mathbb{R} \xi, JX = -J_1 X \},
$$
where \( \mathbb{R}_\xi, \mathbb{C}_\xi \) and \( \mathbb{H}_\xi \) respectively denotes real, complex and quaternionic span of the structure vector \( \xi \) and \( \mathbb{C}^\perp \xi \) denotes the orthogonal complement of \( \mathbb{C}_\xi \) in \( \mathbb{H}_\xi \).

In the proof of Lemma 5.C (See Section 4 in [6]) we have asserted that \( A\xi_2 = 0 \) and \( A\xi_3 = 0 \). But the principal curvature \( \beta = \sqrt{2} \cot \left( \sqrt{2} r \right) \) given in Proposition A is never vanishing for any \( r \in (0, \frac{\pi}{4}) \). So this gives a contradiction. Accordingly, we completed the proof of our Theorem 2 for the case \( \xi \in \mathcal{D}^\perp \).

### 6. Lie parallel normal Jacobi operator for \( \xi \in \mathcal{D} \)

Next we consider the case that the Reeb vector field \( \xi \) belongs to the distribution \( \mathcal{D} \). Then in this section we introduce the following lemmas due to Jeong and Suh [6] for hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with Lie \( \xi \)-parallel normal Jacobi operator.

**Lemma 6.A.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie \( \xi \)-parallel normal Jacobi operator and \( \xi \in \mathcal{D} \). Then the Reeb vector \( \xi \) is principal.

Then by using Lemma 6.A, Jeong and Suh [6] also verified the following:

**Lemma 6.B.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie \( \xi \)-parallel normal Jacobi operator and \( \xi \in \mathcal{D} \). Then \( g(AD, D^\perp) = 0 \).

By virtue of these Lemmas 6.A and 6.B we have

**Lemma 6.C.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie parallel normal Jacobi operator and \( \xi \in \mathcal{D} \). Then the Reeb vector field \( \xi \) is principal and \( g(AD, D^\perp) = 0 \).

From this Lemma 6.1, together with Theorem A due to Berndt and Suh [3], we have that \( M \) is locally a tube over a totally geodesic and totally real quaternionic projective space \( HP^m, m = 2n \). So for the geometrical structure of such a tube we recall the following proposition.

**Proposition B.** Let \( M \) be a connected real hypersurface of \( G_2(\mathbb{C}^{m+2}) \). Suppose that \( AD \subset \mathcal{D} \), \( A\xi = \alpha \xi \), and \( \xi \) is tangent to \( \mathcal{D} \). Then the quaternionic dimension \( m \) of \( G_2(\mathbb{C}^{m+2}) \) is even, say \( m = 2n \), and \( M \) has five distinct constant principal curvatures

\[
\alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)
\]

with some \( r \in (0, \pi/4) \). The corresponding multiplicities are

\[
m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)
\]

and the corresponding eigenspaces are

\[
T_\alpha = \mathbb{R}_\xi, \quad T_\beta = \mathfrak{J}_\xi, \quad T_\gamma = \mathfrak{K}_\xi, \quad T_\lambda, \quad T_\mu,
\]

where

\[
T_\lambda \oplus T_\mu = (H\mathbb{C}_\xi)^\perp, \quad \mathfrak{J}_\lambda = T_\lambda, \quad \mathfrak{J}_\mu = T_\mu, \quad JT_\lambda = T_\mu.
\]
Now, using the assumption that \( M \) is Hopf in (3.4), we have the following
\[
(\mathcal{L}_\xi \tilde{R}_N)Y = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, Y) \xi_\nu + 4\alpha \sum_{\nu=1}^{3} \eta_\nu(Y) \phi_\nu \xi \\
-3 \sum_{\nu=1}^{3} \eta_\nu(Y) \phi A \xi_\nu + 3 \sum_{\nu=1}^{3} \eta_\nu(\phi AY) \xi_\nu \\
+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi)(\phi A \phi_\nu Y - \eta(Y) \phi A \xi_\nu) - \eta_\nu(\phi Y) \phi A \phi_\nu \xi \right\} \\
+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi)(\phi_\nu AY - \alpha \eta(Y) \phi_\nu \xi) - \eta_\nu(AY) \phi_\nu \xi + \eta(AY) \eta_\nu(\xi) \phi_\nu \xi \right\} \\
= 0.
\]
Moreover, using the fact that the Reeb vector field \( \dot{\xi} \) belongs to the distribution \( \mathcal{D} \), we have
\[
(\mathcal{L}_\xi \tilde{R}_N)Y = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, Y) \xi_\nu + 4\alpha \sum_{\nu=1}^{3} \eta_\nu(Y) \phi_\nu \xi \\
-3 \sum_{\nu=1}^{3} \eta_\nu(Y) \phi A \xi_\nu + 3 \sum_{\nu=1}^{3} \eta_\nu(\phi AY) \xi_\nu \\
- \sum_{\nu=1}^{3} \eta_\nu(\phi Y) \phi A \phi_\nu \xi - \sum_{\nu=1}^{3} \eta_\nu(AY) \phi_\nu \xi \\
= 0
\]
for any \( Y \in T_x M, \ x \in M \).

Let us construct a sub-distribution \( \mathcal{D}_0 \) of the distribution \( \mathcal{D} \) in such a way that
\[
[\xi] \oplus \mathcal{D}_0 = \mathcal{D},
\]
where \([\xi]\) denotes an one-dimensional vector subspace spanned by the Reeb vector field \( \xi \). Then \( \mathcal{D}_0 \) becomes \( \mathcal{D}_0 = \{Y \in \mathcal{D} | Y \perp \xi \} \). Here, if we substitute any \( Y \in \mathcal{D}_0 \) in (6.1) and use \( \xi \in \mathcal{D} \), the left side of (6.1) becomes
\[
(\mathcal{L}_\xi \tilde{R}_N)Y = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, Y) \xi_\nu + 3 \sum_{\nu=1}^{3} \eta_\nu(\phi AY) \xi_\nu \\
- \sum_{\nu=1}^{3} \eta_\nu(\phi Y) \phi A \phi_\nu \xi - \sum_{\nu=1}^{3} \eta_\nu(AY) \phi_\nu \xi.
\]
From this, putting \( Y = \phi_\mu \xi \in T_\xi \gamma \), and using \( A \phi_\mu \xi = 0, \mu = 1, 2, 3 \), given in Proposition \( B \), it becomes
\[
(\mathcal{L}_\xi \tilde{R}_N)\phi_\mu \xi = 4\alpha \phi_\mu \xi.
\]
From this, with the assumption of \( \mathcal{L}_\xi \tilde{R}_N = 0 \), we have \( \alpha = 0 \). But the principal curvature \( \alpha = -2 \tan(2\varphi) \) in Proposition \( B \) never vanishes for \( r \in (0, \frac{\pi}{2}) \). This gives a contradiction for the case \( \xi \in \mathcal{D} \). Accordingly, we complete the proof of our Theorem 2 for \( \xi \in \mathcal{D} \) in the introduction.

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