EXTENSIONS OF EXTENDED SYMMETRIC RINGS

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ABSTRACT. An endomorphism \( \alpha \) of a ring \( R \) is called right (left) symmetric if whenever \( abc = 0 \) for \( a, b, c \in R \), \( ac\alpha(b) = 0 \) \((\alpha(b)ac = 0)\). A ring \( R \) is called right (left) \( \alpha \)-symmetric if there exists a right (left) symmetric endomorphism \( \alpha \) of \( R \). The notion of an \( \alpha \)-symmetric ring is a generalization of \( \alpha \)-rigid rings as well as an extension of symmetric rings. We study characterizations of \( \alpha \)-symmetric rings and their related properties including extensions. The relationship between \( \alpha \)-symmetric rings and (extended) Armendariz rings is also investigated, consequently several known results relating to \( \alpha \)-rigid and symmetric rings can be obtained as corollaries of our results.

1. Introduction

Recall that a ring is reduced if it has no nonzero nilpotent elements. Lambek called a ring \( R \) symmetric [13] provided \( abc = 0 \) implies \( acb = 0 \) for \( a, b, c \in R \). Every reduced ring is symmetric ([16, Lemma 1.1]) but the converse does not hold by [2, Example II.5]. Cohn called a ring \( R \) reversible [5] if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \). Historically, some of the earliest known results about reversible rings (although not so called at the time) were due to Habeb [6]. It is obvious that commutative rings are symmetric and symmetric rings are reversible; but the converses do not hold by [2, Examples I.5 and II.5] and [14, Examples 5 and 7].

Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia called a ring \( R \) Armendariz [15] if whenever any polynomials \( f(x) = a_0 + a_1x + \cdots + a_mx^m \), \( g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( a_ib_j = 0 \) for each \( i \) and \( j \). This nomenclature was used by them since it was Armendariz who initially showed that a reduced ring always satisfies this condition ([3, Lemma 1]).

For a ring \( R \) with a ring endomorphism \( \alpha : R \to R \), a skew polynomial ring (also called an Ore extension of endomorphism type) \( R[x; \alpha] \) of \( R \) is the ring obtained by giving the polynomial ring over \( R \) with the new multiplication \( xr = \alpha(r)x \) for all \( r \in R \).

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The Armendariz property of a ring was extended to skew polynomial rings but with skewed scalar multiplication in [8, 9]: For an endomorphism $\alpha$ of a ring $R$, $R$ is called $\alpha$-Armendariz (resp. $\alpha$-skew Armendariz) if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha]$, $pq = 0$ implies $a_i b_j = 0$ (resp. $a_i \alpha(b_j) = 0$) for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

On the other hand, an endomorphism $\alpha$ of a ring $R$ is called rigid [12] if $a \alpha(a) = 0$ implies $a = 0$ for $a \in R$, and $R$ is an $\alpha$-rigid ring [7] if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a rigid endomorphism is a monomorphism, and $\alpha$-rigid rings are reduced rings by [7, Proposition 5]. Any $\alpha$-rigid ring is $\alpha$-Armendariz [7, Proposition 6], but the converse is not true, in general; every $\alpha$-Armendariz ring is $\alpha$-skew Armendariz, but the converse does not hold by [9, Theorem 1.7 and Example 1.8]. In [8, Proposition 3], $R$ is an $\alpha$-rigid ring if and only if $R[x; \alpha]$ is reduced.

Motivated by the above, in this paper we introduce the notion of an $\alpha$-symmetric ring for an endomorphism $\alpha$ of a ring $R$, as a generalization of $\alpha$-rigid rings and an extension of symmetric rings, and study $\alpha$-symmetric rings and their related properties. The relationship between $\alpha$-symmetric rings and extended Armendariz rings is also investigated. Consequently, several known results are obtained as corollaries of our results.

Throughout this paper $R$ denotes an associative ring with identity and $\alpha$ denotes a nonzero and non identity endomorphism, unless specified otherwise.

2. Properties of $\alpha$-symmetric rings

We begin with the following definition.

**Definition 2.1.** An endomorphism $\alpha$ of a ring $R$ is called right (left) symmetric if whenever $abc = 0$ for $a, b, c \in R$, $ac \alpha(b) = 0$ (resp. $\alpha(b)ac = 0$). A ring $R$ is called right (left) $\alpha$-symmetric if there exists a right (left) symmetric endomorphism $\alpha$ of $R$. $R$ is $\alpha$-symmetric if it is both right and left $\alpha$-symmetric.

Observe that every subring $S$ with $\alpha(S) \subseteq S$ of a right $\alpha$-symmetric ring is also right $\alpha$-symmetric; and any domain $R$ is $\alpha$-symmetric for any endomorphism $\alpha$ of $R$, but the converse does not hold (see Example 2.7(1) below).

The next example shows that the concept of $\alpha$-symmetric is not left-right symmetric.

**Example 2.2.** Let $\mathbb{Z}$ be the ring of integers. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$ 

Note that for $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB = O$ but $BA \neq O$. Thus $R$ is not reversible, and so $R$ is not symmetric.

(i) Let $\alpha : R \longrightarrow R$ be an endomorphism defined by
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\[ \alpha \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. \]

If \( ABC = O \) for \( A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \) and \( C = \begin{pmatrix} a'' & b'' \\ 0 & c'' \end{pmatrix} \in R \), then we get \( aa'a'' = 0 \) and so \( aa''a' = 0 \). Thus this yields \( AC\alpha(B) = O \), and hence \( R \) is right \( \alpha \)-symmetric. However, for \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \)

and \( C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R \) with \( ABC = O \), we have \( \alpha(B)AC \neq O \), and thus \( R \) is not left \( \alpha \)-symmetric.

(ii) Let \( \beta : R \rightarrow R \) be an endomorphism defined by

\[ \beta \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}. \]

By the similar method to (i), we can show that \( R \) is left \( \beta \)-symmetric. However, for \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)

we have \( AC\beta(B) \neq O \), and thus \( R \) is not right \( \beta \)-symmetric.

**Proposition 2.3.** (1) For a ring \( R \), \( R \) is right \( \alpha \)-symmetric if and only if \( ABC = 0 \) implies \( AC\alpha(B) = 0 \) for any three nonempty subsets \( A, B \) and \( C \) of \( R \).

(2) Let \( R \) be a reversible ring. \( R \) is right \( \alpha \)-symmetric if and only if \( R \) is left \( \alpha \)-symmetric.

**Proof.** (1) It suffices to show that \( ABC = 0 \) for any three nonempty subsets \( A, B \) and \( C \) of \( R \) implies \( AC\alpha(B) = 0 \), when \( R \) is right \( \alpha \)-symmetric. Let \( ABC = 0 \). Then \( abc = 0 \) for \( a \in A, b \in B \) and \( c \in C \), and hence \( ac\alpha(b) = 0 \) by the condition. Thus \( AC\alpha(B) = \sum_{a \in A, b \in B, c \in C} ac\alpha(b) = 0 \).

(2) Let \( abc = 0 \) for \( a, b, c \in R \). If \( R \) is right \( \alpha \)-symmetric, then \( ac\alpha(b) = 0 \). Since \( R \) is reversible, we have \( \alpha(b)ac = 0 \) and hence \( R \) is left \( \alpha \)-symmetric. The converse is similar. \( \square \)

Example 2.2 shows that the condition “\( R \) is reversible” in Proposition 2.3(2) cannot be dropped as well as there exists a right symmetric endomorphism \( \alpha \) of a ring \( R \) such that \( R \) is not symmetric. The next example provides that there exists a commutative reduced ring \( R \) which is not \( \alpha \)-symmetric for some endomorphism \( \alpha \) of \( R \).

**Example 2.4.** Let \( \mathbb{Z}_2 \) be the ring of integers modulo 2 and consider a ring \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) with the usual addition and multiplication. Then \( R \) is a commutative reduced ring, and so \( R \) is symmetric. Now, let \( \alpha : R \rightarrow R \) be defined by \( \alpha((a, b)) = (b, a) \). Then \( \alpha \) is an automorphism of \( R \). For \( a = (1, 0), b = (0, 1), c = (1, 1) \in R \), \( abc = 0 \) but \( ac\alpha(b) = (1, 0) \neq 0 \), and thus \( R \) is not right \( \alpha \)-symmetric.
Recently, the reversible property of a ring is extended to a ring endomorphism in [4] as follows: An endomorphism $\alpha$ of a ring $R$ is called right reversible if whenever $ab = 0$ for $a, b \in R$, $ba(a) = 0$. A ring $R$ is called right $\alpha$-reversible if there exists a right reversible endomorphism $\alpha$ of $R$. The notion of an $\alpha$-reversible ring is a generalization of $\alpha$-rigid rings as well as an extension of reversible rings.

**Theorem 2.5.** Let $R$ be a right $\alpha$-symmetric ring. Then we have the following.

1. For $a, b, c \in R$, $abc = 0$ implies $a\alpha^n(b) = 0$, $b\alpha^n(a) = 0$, and $aba^n(c) = 0$ for any positive integer $n$, especially, $R$ is a right $\alpha$-reversible ring.

2. Let $\alpha$ be a monomorphism of $R$. Then we have the following.

   i. $R$ is a symmetric ring.

   ii. For $a, b, c \in R$ $abc = 0$ implies $\alpha^n(a)bc = 0$ and $aa^n(b)c = 0$ for any positive integer $n$. Conversely, if $\alpha^n(a)bc = 0$, $aa^n(b)c = 0$, or $aba^n(c) = 0$ for some positive integer $m$, then $abc = 0$.

**Proof.** (1) Let $a, b, c \in R$ with $abc = 0$. Since $R$ is right $\alpha$-symmetric, $a\alpha^n(b) = 0$. Then $0 = a\alpha^n(b) = (ac)\alpha(b) \cdot 1$ implies $a\alpha^n(b) = 0$. Continuing this process, we have $a\alpha^n(b) = 0$ for any positive integer $n$. Similarly, $1 \cdot a(bc) = 0$ implies $b\alpha^n(a) = 0$. By the same method as above, we obtain $b\alpha^n(a) = 0$ for any positive integer $n$. Finally, $0 = abc = (ab)c \cdot 1$ implies $aba^n(c) = 0$, and thus $aba^n(c) = 0$ for any positive integer $n$.

(2) Suppose that $\alpha$ is a monomorphism. (i): Let $a, b, c \in R$ with $abc = 0$. Then $a\alpha^n(b) = 0$, and so $\alpha(b)\alpha(ac) = 0$ by (1). Since $\alpha$ is a monomorphism, $bac = 0$ and $acb = 0$. Thus $R$ is symmetric. (ii): Note that $R$ is symmetric and so reversible. Let $abc = 0$. Then $b\alpha^n(a) = 0$ by (1). Since $R$ is reversible, $\alpha^n(a)bc = 0$. Next, from $abc = 0$ we have $a\alpha^n(b) = 0$ by (1). Since $R$ is symmetric, $a\alpha^n(b)c = 0$. Conversely, if $\alpha^n(a)bc = 0$ for some positive integer $m$ then $\alpha^n(a)\alpha^n(bc) = \alpha^n(abc) = 0$ by (i), and thus $abc = 0$, since $\alpha$ is a monomorphism. Similarly, if $aa^n(b)c = 0$ then $a\alpha^n(b) = 0$, since $R$ is symmetric. Hence $\alpha^n(ac)\alpha^n(b) = 0$ by (i), and $acb = 0$ and so $abc = 0$. By the same method as above, we can obtain that $aba^n(c) = 0$ implies $abc = 0$. □

**Corollary 2.6.** Every symmetric ring is reversible.

Notice that for any positive integer $n$, “$\alpha^n(b) = 0$” is equivalent to “$\alpha R\alpha^n(b) = 0$”, when $R$ is a right $\alpha$-symmetric ring with $ab = 0$ for $a, b \in R$. For, $abr = 0$ implies $ara(b) = 0$ for any $r \in R$. This shows that $dra^n(b) = 0$ for any positive integer $n$ and any $r \in R$ from Theorem 2.5(1), and thus $a\alpha^n(b) = 0$.

We remark that the converse of Theorem 2.5(1) does not hold. For example, the ring $R$ with an endomorphism $\alpha$ in Example 2.2(1) is right $\alpha$-symmetric. However, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B \in R$, we have $A\alpha^n(B) = O = B\alpha^n(A)$ for any positive integer $n$ but $\overline{AB} \neq O$. 
In the next example, part (1) shows that there exists a right $\alpha$-symmetric ring $R$ for an automorphism $\alpha$, but $R$ is not semiprime and so not $\alpha$-rigid, and part (2) illuminates that there exists a commutative domain and an $\alpha$-symmetric ring $R$, but $R$ is not $\alpha$-rigid where $\alpha$ is not a monomorphism.

**Example 2.7.** (1) Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Clearly, $R$ is not semiprime and hence $R$ is not $\alpha$-rigid. Note that $\alpha$ is an automorphism. Moreover, $R$ is right $\alpha$-symmetric: Indeed, let $ABC = O$ for $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. $B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}$ and $C = \begin{pmatrix} a'' & b'' \\ 0 & a'' \end{pmatrix} \in R$, then we get $aa'a'' = 0$ and $aa'b'' + ab'a'' + ba'a'' = 0$. If $a = 0$ then $ba'a'' = 0$, if $a' = 0$ then $ab'a'' = 0$, and $a'' = 0$ then $ab'b'' = 0$. These imply that $aa''a' = 0$ and $-aa'b' + ab'a' + ba''a' = 0$. Thus $AC\alpha(B) = O$, and hence $R$ is right $\alpha$-symmetric.

(2) Let $R = F[x]$ be the polynomial ring over a field $F$. Define $\alpha : R \to R$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. Then $R$ is a commutative domain (and so reduced), but $\alpha$ is not a monomorphism. Since $R$ is a domain, $R$ is right $\alpha$-symmetric for any endomorphism $\alpha$ of $R$. However, $R$ is not $\alpha$-rigid by [8, Example 5(2)].

The class of semiprime rings and the class of right $\alpha$-symmetric rings do not depend on each other by Example 2.4 and Example 2.7(1). There exists a skew polynomial ring $R[x; \alpha]$ over a symmetric ring $R$ which is not a symmetric ring. For example, consider the commutative ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the automorphism $\alpha$ of $R$ defined by $\alpha((a, b)) = (b, a)$, as in Example 2.4. Then $R$ is a symmetric ring, but $R[x; \alpha]$ is not reversible hence not symmetric: Indeed, for $p = (1, 0), q = (0, 1)x \in R[x; \alpha]$, we get $pq = 0$ but $0 \neq (0, 1)q = qp$.

However, we have the following theorem.

**Theorem 2.8.** (1) For a ring $R$, $R$ is $\alpha$-rigid if and only if $R$ is semiprime and right $\alpha$-symmetric and $\alpha$ is a monomorphism.

(2) If the skew polynomial ring $R[x; \alpha]$ of a ring $R$ is a symmetric ring, then $R$ is $\alpha$-symmetric.

**Proof.** (1) Let $R$ be $\alpha$-rigid. Note that any $\alpha$-rigid ring is reduced and $\alpha$ is a monomorphism by [7, p. 218]. We show that $R$ is right $\alpha$-symmetric. Assume that $abc = 0$ for $a, b, c \in R$. Then we obtain $bac = 0$, since $R$ is reduced (and so symmetric). Thus $a\alpha(c)b\alpha(a\alpha(b)) = a\alpha(b\alpha(c))\alpha^2(b) = 0$. Since $R$ is $\alpha$-rigid, $a\alpha(b) = 0$ and thus $R$ is right $\alpha$-symmetric.

The converse follows from [4, Proposition 2.5(3)] and Theorem 2.5(1).
(2) Suppose that \( abc = 0 \) for \( a, b, c \in R \). Let \( p = a, q = b \) and \( h = cx \) in \( R[x; \alpha] \). Then \( pqh = abcx = 0 \in R[x; \alpha] \). Since \( R[x; \alpha] \) is symmetric, we get \( 0 = pqh = (ac)xb = ac\alpha(b)x \), and so \( ac\alpha(b) = 0 \). Thus \( R \) is right \( \alpha \)-symmetric and therefore \( R \) is \( \alpha \)-symmetric by Proposition 2.3(2).

**Corollary 2.9** ([10, Proposition 2.7(1)]). A ring \( R \) is reduced if and only if \( R \) is a semiprime and symmetric ring.

Observe that the class of right \( \alpha \)-symmetric rings and the class of \( \alpha \)-Armendariz rings do not depend on each other by Example 2.7(2) and [11, Example 14].

**Theorem 2.10.** Let \( R \) be an \( \alpha \)-Armendariz ring. The following statements are equivalent:

1. \( R[x; \alpha] \) is symmetric.
2. \( R \) is \( \alpha \)-symmetric.
3. \( R \) is right \( \alpha \)-symmetric.
4. \( R \) is symmetric.

**Proof.** (1)\( \Leftrightarrow \) (4) by [9, Theorem 3.6 (1)] and (1)\( \Rightarrow \) (2) by Theorem 2.8 (2). (2)\( \Rightarrow \) (3) is trivial. Now we show (3)\( \Rightarrow \) (4). Suppose \( abc = 0 \) for \( a, b, c \in R \).

Then \( ac\alpha(b) = 0 \), and so \( acb = 0 \) by [9, Proposition 1.3 (2)]. Thus \( R \) is symmetric. \( \square \)

The next result is a direct consequence of Theorem 2.10.

**Corollary 2.11** ([10, Proposition 3.4]). Let \( R \) be an Armendariz ring. \( R \) is symmetric if and only if \( R[x] \) is symmetric.

Notice that the converse of Theorem 2.8(2) does not hold and the condition “\( R \) is an \( \alpha \)-Armendariz ring” in Theorem 2.10 are not superfluous by Example 2.7(2): Indeed, consider \( A = R[y; \alpha] = F[x][y; \alpha] \). Now, let \( p = 1, q = xy \) and \( h = x \in A \). Then \( pqh = 0 \), but \( phq = x^2y \not= 0 \). Hence \( A \) is not symmetric.

Note that \( R \) is not \( \alpha \)-Armendariz by [9, Example 1.9].

### 3. Extensions of \( \alpha \)-symmetric rings

Given a ring \( R \) and an \((R,R)\)-bimodule \( M \), the **trivial extension** of \( R \) by \( M \) is the ring \( T(R,M) = R \oplus M \) with the usual addition and the following multiplication:

\[
(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]

This is isomorphic to the ring of all matrices \( \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \), where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

For an endomorphism \( \alpha \) of a ring \( R \) and the trivial extension \( T(R,R) \) of \( R \), \( \bar{\alpha} : T(R,R) \longrightarrow T(R,R) \) defined by

\[
\bar{\alpha} \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}
\]


is an endomorphism of $T(R,R)$. Since $T(R,0)$ is isomorphic to $R$, we can identify the restriction of $\bar{\alpha}$ on $T(R,0)$ to $\alpha$.

Note that the trivial extension of a reduced ring is symmetric by [10, Corollary 2.4]. For a right $\alpha$-symmetric ring $R$, $T(R,R)$ needs not to be an $\bar{\alpha}$-symmetric ring by the next example.

**Example 3.1.** Consider the right $\alpha$-symmetric ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$ 

in Example 2.7(1) where $\alpha$ is defined by

$$\alpha \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$ 

For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in T(R,R),$$

and

$$ABC = O$$

$AC\bar{\alpha}(B) \neq O$. Thus $R$ is not $\bar{\alpha}$-symmetric.

Recall that another generalization of a symmetric ring is a semicommutative ring. A ring $R$ is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about semicommutative rings (although not so called at the time) was due to Shin [16]. He proved that any symmetric ring is semicommutative ([16, Proposition 1.4]) but the converse does not hold ([16, Example 5.4(a)]). Semicommutative rings were also studied under the name **zero insertive** by Habeb [6].

**Proposition 3.2.** Let $R$ be a reduced ring. If $R$ is an $\alpha$-symmetric ring, then $T(R,R)$ is an $\bar{\alpha}$-symmetric ring.

**Proof.** Let $ABC = O$ for

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a'' & b'' \\ 0 & a'' \end{pmatrix} \in T(R,R).$$

Then we have

1. $aa'a'' = 0$; and
2. $aa'b'' + ab'a'' + ba'a'' = 0$.

In the following, we freely use the fact that $R$ is a reduced ring if and only if for any $a, b \in R$, $ab^2 = 0$ (or, $a^2b = 0$) implies $ab = 0$; and every reduced ring
Let \( R \) be a symmetric ring. For any \( n \geq 3 \) and an endomorphism \( \alpha \) of a ring \( R \) is also extended to the endomorphism \( \overline{\alpha} : T_n \to T_n \) defined by \( \overline{\alpha}((a)) = (\alpha(a)) \).

The following example shows that \( T_n \) cannot be \( \overline{\alpha} \)-symmetric for any \( n \geq 3 \), even if \( R \) is an \( \alpha \)-rigid ring.

**Example 3.4.** Let \( \alpha \) be an endomorphism of an \( \alpha \)-rigid ring \( R \). Note that if \( R \) is an \( \alpha \)-rigid ring, then \( \alpha(e) = e \) for \( e^2 = e \in R \) by [7, Proposition 5]. In particular \( \alpha(1) = 1 \). First, we show that \( T_3 \) is not \( \overline{\alpha} \)-symmetric. For

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \in T_3,
\]

\( ABC = O \). But we have \( AC\overline{\alpha}(B) = CB \neq O \in T_3 \).

In case of \( n \geq 4 \), we can also prove that \( T_n \) is not \( \overline{\alpha} \)-symmetric by the same method as the above.

Recall that if \( \alpha \) is an endomorphism of a ring \( R \), then the map \( \overline{\alpha} : R[x] \to R[x] \) defined by \( \overline{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i) x^i \) is an endomorphism of the polynomial ring \( R[x] \) and clearly this map extends \( \alpha \). The Laurent polynomial ring \( R[x, x^{-1}] \) with an indeterminate \( x \), consists of all formal sums \( \sum_{i=k}^{\infty} a_i x^i \), where \( a_i \in R \) and \( k, n \) are (possibly negative) integers. The map \( \overline{\alpha} : R[x, x^{-1}] \to R[x, x^{-1}] \) defined by \( \overline{\alpha}(\sum_{i=k}^{\infty} a_i x^i) = \sum_{i=k}^{\infty} \alpha(a_i) x^i \) extends \( \alpha \) and also is an endomorphism of \( R[x, x^{-1}] \). Multiplication is subject to \( xr = \alpha(r)x \) and \( rx^{-1} = x^{-1}\alpha(r) \).

The following results extend the class of right \( \alpha \)-symmetric rings.

**Theorem 3.5.** Let \( R \) be a ring.

(1) \( R[x] \) is right \( \overline{\alpha} \)-symmetric if and only if \( R[x, x^{-1}] \) is right \( \overline{\alpha} \)-symmetric.
(2) If $R$ is an Armendariz ring, then $R$ is right $\alpha$-symmetric if and only if $R[x]$ is right $\bar{\alpha}$-symmetric.

Proof. (1) It is sufficient to show necessity. Let $f(x), g(x)$ and $h(x) \in R[x; x^{-1}]$ with $f(x)g(x)h(x) = 0$. Then there exists a positive integer $n$ such that $f_1(x) = f(x)x^n, g_1(x) = g(x)x^n$ and $h_1(x) = h(x)x^n \in R[x]$, and so $f_1(x)g_1(x)h_1(x) = 0$. Since $R[x]$ is right $\bar{\alpha}$-symmetric, we obtain $f_1(x)h_1(x)\bar{\alpha}(g_1(x)) = 0$. Hence $f(x)h(x)\bar{\alpha}(g(x)) = x^{-3n}f_1(x)h_1(x)\bar{\alpha}(g_1(x)) = 0$. Thus $R[x; x^{-1}]$ is right $\bar{\alpha}$-symmetric.

(2) It also suffices to establish necessity. Let $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j$ and $h(x) = \sum_{k=0}^{l} c_kx^k \in R[x]$ with $f(x)g(x)h(x) = 0$. By [1, Proposition 1], $a_ib_jc_k = 0$ for all $i, j$ and $k$, and so $a_i\bar{\alpha}(b_j) = 0$ since $R$ is Armendariz and right $\alpha$-symmetric. This yields $f(x)h(x)\bar{\alpha}(g(x)) = 0$, and thus $R[x]$ is right $\bar{\alpha}$-symmetric. \hfill $\square$

Corollary 3.6. (1) [10, Lemma 3.2(2)] For a ring $R$, $R[x]$ is symmetric if and only if so is $R[x; x^{-1}]$.

(2) [10, Proposition 3.4] Let $R$ be an Armendariz ring. $R$ is symmetric if and only if $R[x]$ is symmetric.

Note that Example 2.2(i) and Example 2.4 show that Armendariz rings and right $\alpha$-symmetric rings do not depend on each other.

For an ideal $I$ of $R$, if $\alpha(I) \subseteq I$ then $\bar{\alpha}: R/I \rightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring $R/I$. The homomorphic image of a symmetric ring may not necessarily be symmetric by [10, p.163]. One may conjecture that $R$ is $\alpha$-symmetric if for any right $\alpha$-symmetric nonzero proper ideal $I$ of $R$, $R/I$ is $\bar{\alpha}$-symmetric, where $I$ is considered as a ring without identity. However, the next example erases the possibility.

Example 3.7. For a field $F$, consider a ring $R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$ and an endomorphism $\alpha$ of $R$ defined by

$$\alpha \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) = \left( \begin{array}{cc} a & -b \\ 0 & c \end{array} \right).$$

For a right ideal $I = \left( \begin{array}{cc} 0 & F \\ 0 & 0 \end{array} \right)$ of $R$, it can be easily checked that $I$ is right $\alpha$-symmetric and the factor ring

$$R/I = \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & c \end{array} \right) + I \mid a, c \in F \right\}$$

is reduced. Observe that $R/I$ is $\bar{\alpha}$-symmetric, where $\bar{\alpha}$ is an identity map on $R/I$.

However, for $A = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$, $B = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$ and $C = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \in R$, we have $AC\alpha(B) \neq O$ and $ABC = O$. Thus $R$ is not right $\alpha$-symmetric.
Theorem 3.8. Let $R$ be a reduced ring and $n$ be any positive integer. If $R$ is right $\alpha$-symmetric with $\alpha(1) = 1$, then $R[x]/(x^n)$ is a right $\alpha$-symmetric ring, where $(x^n)$ is the ideal generated by $x^n$.

Proof. Let $S = R[x]/(x^n)$. If $n = 1$, then $S \cong R$. If $n = 2$, then $S$ is $\alpha$-symmetric by Proposition 3.2, since $S \cong T(R, R)$. Now, we assume $n \geq 3$.

Let $f = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}, g = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ and $h = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in S$ with $fgf = 0$, where $x = x + (x^n)$. Note that $a_ib_jc_kx^{i+j+k} = 0$ for all $i, j, k$ with $i + j + k \geq n$. Hence it suffices to show the cases $i + j + k \leq n - 1$. Since $fgf = 0$, we have the following equations:

$(1)$ $a_0b_0c_0 = 0$.

$(2)$ $a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 = 0$.

$(3)$ $a_0b_0c_2 + a_0b_1c_1 + a_0b_2c_0 + a_1b_1c_0 + a_2b_0c_0 = 0$.

Inductively we assume that $a_ib_jc_k = 0$ for $i + j + k = 0, 1, \ldots, (n-2)$.

We apply the above method to Eq.$(n-1)$. First, the induction hypotheses and Eq.$(n-1)$ give $a_0b_0c_0 = 0$ and $a_0b_0c_1 + a_0b_1c_0 = 0$; multiplying $b_1c_0$ gives $0 = a_0b_1(c_0)^2 = a_0b_1c_0$, so we have

$$(2)' \quad a_0b_0c_1 = 0, a_0b_1c_0 = 0 \quad \text{and} \quad a_1b_0c_0 = 0.$$

From Eqs.$(1)$, $(2)'$ and $(3)$, we get $a_2b_0c_0 = 0$ and

$$(3)' \quad a_0b_0c_2 + a_0b_1c_1 + a_0b_2c_0 + a_1b_1c_1 + a_2b_0c_1 + a_3b_1c_0 = 0,$$

in a similar way. If we multiply Eq.$(3)'$ on the right side by $b_1c_0, b_0c_1, b_2c_0$ and $b_1c_1$ respectively, then we obtain $a_1b_1c_0 = 0, a_1b_0c_1 = 0, a_0b_2c_0 = 0, a_0b_1c_1 = 0$, and $a_0b_0c_2 = 0$ in turn.

Inductively we assume that $a_ib_jc_k = 0$ for $i + j + k = 0, 1, \ldots, (n-2)$. We apply the above method to Eq.$(n-1)$. First, the induction hypotheses and Eq.$(n-1)$ give $a_0b_0c_0 = 0$ and

$$(n-1)' \quad a_0b_0c_{n-1} + a_0b_1c_{n-2} + \cdots + a_{n-2}b_0c_1 + a_{n-3}b_1c_0 = 0.$$

If we multiply Eq.$(n-1)'$ on the right side by $b_1c_0, b_0c_1, \ldots$, and $b_1c_{n-2}$ respectively, then we obtain $a_{n-2}b_1c_0 = 0, a_{n-3}b_2c_1 = 0, \ldots, a_{n-3}b_2c_1 = 0$ and so $a_0b_0c_{n-1} = 0$, in turn. This shows that $a_ib_jc_k = 0$ for all $i, j, k$ with $i + j + k = n - 1$. Consequently, $a_ib_jc_k = 0$ for all $i, j, k$ with $i + j \leq n - 1$, and thus $a_{n-k}c_{n-k}^i(b_j) = 0$ for any positive integer $t$ by Theorem 2.5(1). This yields $f\alpha(g) = 0$, and therefore $S$ is right $\alpha$-symmetric. □

Corollary 3.9 ([10, Theorem 2.3]). If $R$ is a reduced ring, then $R[x]/(x^n)$ is a symmetric ring for any positive integer $n$.

Let $R$ be an algebra over a commutative ring $S$. Recall that the Dorroh extension of $R$ by $S$ is the ring $D = R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$.
and \( s_i \in S \). For an endomorphism \( \alpha \) of \( R \) and the Dorroh extension \( D \) of \( R \) by \( S, \alpha : D \to D \) defined by \( \bar{\alpha}(r,s) = (\alpha(r), s) \) is an \( S \)-algebra homomorphism.

In the following, we give some other example of right \( \alpha \)-symmetric rings.

**Proposition 3.10.** (1) If \( e \) is a central idempotent of a ring \( R \) with \( \alpha(e) = e \) and \( \alpha(1-e) = 1-e \), then \( eR \) and \( (1-e)R \) are right \( \alpha \)-symmetric if and only if \( R \) is right \( \alpha \)-symmetric.

(2) If \( R \) is a right \( \alpha \)-symmetric ring with \( \alpha(1) = 1 \) and \( S \) is a domain, then the Dorroh extension \( D \) of \( R \) by \( S \) is \( \bar{\alpha} \)-symmetric.

**Proof.** (1) It is enough to show the necessity. Suppose that \( eR \) and \( (1-e)R \) are right \( \alpha \)-symmetric. Let \( abc = 0 \) for \( a, b, c \in R \). Then \( 0 = cab = a(eb)c \) and \( 0 = (1-e)abc = a((1-e)b)c \). By hypothesis, we get \( 0 = aco(\alpha(b)) = \alpha(aco(b)) = \alpha(1-e)b = \alpha(1-e)\alpha(b) = (1-e)\alpha(b) \). Thus \( \alpha(b) = (1-e)\alpha(b) = 0 \), and therefore \( R \) is right \( \alpha \)-symmetric.

(2) Let \( (r_1, s_1), (r_2, s_2), (r_3, s_3) \in D \) with \( (r_1, s_1)(r_2, s_2)(r_3, s_3) = 0 \). Then \( r_1r_2r_3 + s_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_1s_2r_3 + s_1s_3r_2 + s_2s_3r_1 = 0 \) and \( s_1s_2s_3 = 0 \).

Since \( S \) is a domain, we get \( s_1 = 0, s_2 = 0 \) or \( s_3 = 0 \). In the following computations, we freely use the assumption that \( R \) is right \( \alpha \)-symmetric with \( \alpha(1) = 1 \). If \( s_1 = 0 \), then \( 0 = r_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_2s_3r_1 \) and so \( 0 = r_1(r_3 + s_3)\alpha(r_2 + s_2) + r_1s_3\alpha(r_2) + r_1r_3s_2 + r_1s_3s_2 \). This yields \( (r_1, s_1)(r_3, s_3)\bar{\alpha}(r_2, s_2) = 0 \). Similarly, let \( s_2 = 0 \). Then \( r_1s_1\alpha(r_2 + s_3) = 0 \), and so \( r_1s_1(r_3 + s_2)\alpha(r_2) = 0 \), and hence \( r_1s_2\alpha(r_2) + s_1s_2r_2 + s_3r_1\alpha(r_2) + s_3s_2s_2 = 0 \). Thus we have \( (r_1, s_1)(r_3, s_3)\bar{\alpha}(r_2, s_2) = 0 \). Finally, let \( s_3 = 0 \). Then \( (r_1 + s_1)(r_2 + s_2)r_3 = 0 \), and so \( 0 = (r_1 + s_1)(r_3, s_3)\alpha(r_2) + s_2(r_1r_3 + s_1r_3) \). This imply \( (r_1, s_1)(r_3, s_3)\bar{\alpha}(r_2, s_2) = 0 \). Therefore the Dorroh extension \( D \) is \( \bar{\alpha} \)-symmetric.

**Corollary 3.11.** (1) [10, Proposition 3.6(2)] For an abelian ring \( R \), \( R \) is symmetric if and only if \( eR \) and \( (1-e)R \) are symmetric for every idempotent \( e \) of \( R \) if and only if \( eR \) and \( (1-e)R \) are symmetric for some idempotent \( e \) of \( R \).

(2) [10, Proposition 4.2(1)] Let \( R \) be an algebra over a commutative ring \( S \), and \( D \) be the Dorroh extension of \( R \) by \( S \). If \( R \) is symmetric and \( S \) is a domain, then \( D \) is symmetric.

Note that the condition “\( \alpha(1) = 1 \)” in Proposition 3.10(2) cannot be dropped by the next example.

**Example 3.12.** Let \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( \alpha : R \to R \) be defined by \( \alpha((a,b)) = (0,b) \). Consider the Dorroh extension \( D \) of \( R \) by the ring of integers \( \mathbb{Z} \). Then we have \( ((1,0),0)((1,0),-1)((1,0),0) = 0 \) in \( D \), but \( ((1,0),0)((1,0),0)\bar{\alpha}((1,0),-1) = (0,1,0) \neq 0 \) in \( D \).
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References


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