A LIOUVILLE TYPE THEOREM FOR HARMONIC MORPHISMS

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Abstract. Let $M$ be a complete Riemannian manifold and let $N$ be a Riemannian manifold of nonpositive scalar curvature. Let $\mu_0$ be the least eigenvalue of the Laplacian acting on $L^2$-functions on $M$. We show that if $\text{Ric}^M \geq -\mu_0$ at all $x \in M$ and either $\text{Ric}^M > -\mu_0$ at some point $x_0$ or $\text{Vol}(M)$ is infinite, then every harmonic morphism $\phi : M \to N$ of finite energy is constant.

1. Introduction

Let $(M, g)$ and $(N, h)$ be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map. For a compact domain $\Omega \subset M$, the energy $E$ of $\phi$ over $\Omega$ is defined by

$$E(\phi; \Omega) = \frac{1}{2} \int_{\Omega} |d\phi|^2 \mu_M,$$

where the differential $d\phi$ is a section of the bundle $T^*M \otimes \phi^{-1}TN \to M$ and $\phi^{-1}TN$ denotes the pull-back bundle via the map $\phi$. The bundle $T^*M \otimes \phi^{-1}TN \to M$ carries the connection $\nabla$ induced by the Levi-Civita connections on $M$ and $N$.

A map $\phi : M \to N$ is called harmonic if $\phi$ is a critical point of the energy functional defined by (1.1) on any compact domain $\Omega \subset M$, or equivalently the tension field $\tau(\phi) = \text{tr}_g \nabla d\phi$ is identically zero, where $\text{tr}_g$ denote the trace with respect to the metric $g$. Several studies are given for harmonic maps ([3]). For these harmonic maps, there are Liouville type theorems, which states that a harmonic map $\phi$ is constant under some conditions. The classical Liouville theorem says that any bounded harmonic function defined on the whole plane must be constant. In 1975, S. T. Yau ([10]) generalized the Liouville theorem to harmonic functions on Riemannian manifolds of nonnegative Ricci curvature. In 1976, R. M. Schoen and S. T. Yau ([8]) proved the following theorem.

**Theorem 1.1.** ([8]) Let $\phi : M \to N$ be a harmonic map from a complete, noncompact Riemannian manifold $M$ with nonnegative Ricci curvature to a...
complete Riemannian manifold \( N \) with nonpositive sectional curvature. If the energy of \( \phi \) is finite, then \( \phi \) is constant.

In 1997, S. D. Jung ([5]) improved Theorem 1.1 to harmonic maps on a complete Riemannian manifold \( M \), where the Ricci curvature \( \text{Ric}^M \) is bounded from below by a negative constant. In fact, let \( \mu_0 \) be the least eigenvalue of the Laplacian \( \Delta^M \) acting on \( L^2 \)-functions on the manifold \( M \). Then

**Theorem 1.2.** ([5]) Let \( \phi : M \to N \) be a harmonic map from a complete Riemannian manifold \( M \) to a Riemannian manifold \( N \) with nonpositive sectional curvature. Assume \( \text{Ric}^M \geq -\mu_0 \) at all \( x \in M \) and \( \text{Ric}^M > -\mu_0 \) at some point \( x_0 \). If the energy of \( \phi \) is finite, then \( \phi \) is constant.

A \( C^0 \) map \( \phi : M \to N \) is called a harmonic morphism if for any harmonic function \( f : U \to \mathbb{R} \) on an open set \( U \subset N \) such that \( \phi^{-1}(U) \) is nonempty, the composition \( f \circ \phi : \phi^{-1}(U) \to \mathbb{R} \) is also a harmonic function on \( \phi^{-1}(U) \).

As a generalization of Riemannian submersions, a horizontally weakly conformal map is a map \( \phi : (M, g) \to (N, h) \) with the property that for each \( x \in M \) at which \( d\phi_x \neq 0 \), the restriction \( d\phi_x|_{H_x} : H_x \to T_{\phi(x)}N \) is conformal and surjective, where \( H_x \) denotes the orthogonal complement of \( V_x = \ker d\phi_x \) in \( T_xM \). We call \( H_x \) the horizontal and \( V_x \) the vertical space of \( \phi \) at \( x \). Thus \( T_xM = V_x \oplus H_x \). Let \( C_\phi = \{ x \in M | d\phi_x = 0 \} \). Trivially, \( \phi \) is horizontally weakly conformal if and only if there exists a function \( \lambda : M \setminus C_\phi \to \mathbb{R}^+ \) such that

\[
h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y) \quad \forall X, Y \in H_x.
\]

Note that at the point \( x \in C_\phi \) we can let \( \lambda(x) = 0 \) and obtain a continuous function \( \lambda : M \to \mathbb{R}^+ \cup \{0\} \) which is called the dilation of a horizontally weakly conformal map \( \phi \).

It is well-known ([4]) that a smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is a harmonic morphism if any only if it is harmonic and horizontally weakly conformal. It is also well-known ([4]) that if \( \dim(M) < \dim(N) \), then every harmonic morphism must be constant.

For the Liouville type theorem for harmonic morphisms in case of \( \dim(M) \geq \dim(N) \), G. Choi and G. Yun ([2]) recently proved the following theorem.

**Theorem 1.3.** ([2]) Let \( \phi : M \to N \) be a harmonic morphism from a complete, noncompact Riemannian manifold \( M \) of nonnegative Ricci curvature to a complete Riemannian manifold \( N \) with nonpositive scalar curvature. If the energy of \( \phi \) is finite, then \( \phi \) is constant.

In this paper, we give extension of Theorem 1.3 to manifolds, where the Ricci curvature of \( M \) is bounded from below by \( -\mu_0 \). That is, our main theorem is the following:

**Theorem 1.4.** Let \( \phi : M \to N \) be a harmonic morphism from a complete, noncompact Riemannian manifold \( M \) to a complete Riemannian manifold \( N \)
with nonpositive scalar curvature. Assume that $\text{Ric}^M \geq -\mu_0$ at all $x \in M$ and either $\text{Ric}^M > -\mu_0$ at some point $x_0$ or $\text{Vol}(M)$ is infinite. If the energy of $\phi$ is finite, then $\phi$ is constant.

2. The Weitzenböck formula

In this section, we review the Weitzenböck formula (see [7, 9]). Let $(M^m, g)$ and $(N^n, h)$ be Riemannian manifolds and let $\nabla^M$ and $\nabla^N$ be their Levi-Civita connections respectively. Let $\phi : M \to N$ be a smooth map and $E = \phi^{-1}TN$ be the induced bundle over $M$. Then $E$ has a naturally induced metric connection $\nabla \equiv \phi^{-1}\nabla^N$ and $\text{d}\phi$ is a cross section of $\text{Hom}(TM, E)$ over $M$. Since $\text{Hom}(TM, E)$ is canonically identified with $T^*M \otimes E$, $\text{d}\phi$ is regarded as an $E$-valued 1-form. Let $d\gamma : \Lambda^r(E) \to \Lambda^{r+1}(E)$ be an anti-derivation and $\delta\gamma$ the formal adjoint of $d\gamma$, where $\Lambda^r(E)$ is the space of $E$-valued $r$-forms with an inner product $\langle \cdot, \cdot \rangle$ on $M$. Let $\{e_i\}_{i=1,...,m}$ and $\{v_a\}_{a=1,...,n}$ be local orthonormal frame fields on $M$ and $N$ respectively, and let $\{\omega^i\}$ and $\{\theta^a\}$ be their dual coframe fields respectively. Locally, the operators $d\gamma$ and $\delta\gamma$ are expressed by

$$d\gamma = \sum_{j=1}^m \omega^j \wedge \nabla e_j$$
$$\delta\gamma = -\sum_{j=1}^m i(e_j)\nabla e_j,$$

respectively, where $i(X)$ is the interior product. The Laplacian $\Delta$ on $\Lambda^r(E)$ is defined by

$$\Delta = d\gamma \delta\gamma + \delta\gamma d\gamma.$$ 

Then the Weitzenböck formula is given by

$$\Delta = -\sum_j \nabla^2_{e_j e_j} + \sum_{k,j} \omega^k \wedge i(e_j)R(e_j, e_k),$$

where $\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla^XY}$ and $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any $X, Y \in TM$. From (2.2), we have that for any $\Phi \in \Lambda^r(E),$

$$-\frac{1}{2} \Delta M|\Phi|^2 = |\nabla \Phi|^2 + \sum_j \nabla^2_{e_j e_j} \Phi, \Phi).$$

Equivalently,

$$-\frac{1}{2} \Delta M|\Phi|^2 = |\nabla \Phi|^2 - \langle \Delta \Phi, \Phi \rangle + \sum_{k,j} \omega^k \wedge i(e_j)R(e_j, e_k)\Phi, \Phi).$$

Let $R^E$ be the curvature tensor of $\nabla$ on $E$. Then $R^E$ is related to the curvature tensor $R^N$ of $\nabla^N$ in the following way: let $X, Y \in T_x M$ and $s \in \Gamma E$, then

$$R^E(X, Y)s = R^N(d\phi_x(X), d\phi_x(Y))s.$$

When a function $f$ is given on $N$, we shall identify it throughout this paper with the function $f \circ \phi$ induced on $M$. Let $f^a \equiv \phi^* \theta^a$. Then $d\phi$ is expressed by
\begin{equation}
\tag{2.6}
d\phi = \sum_{a=1}^{n} f^a \otimes v_a.
\end{equation}

Since a direct calculation gives
\begin{equation}
\tag{2.7}
R(e_j, e_k) d\phi = \sum_{a} R^M(e_j, e_k) f^a \otimes v_a + \sum_{a} f^a \otimes R^E(e_j, e_k) v_a,
\end{equation}
we have
\[
\sum_{k,j} \langle \omega^k \wedge i(e_j) R(e_j, e_k) d\phi, d\phi \rangle = \sum_{k,j,a,b} \langle \omega^k \wedge i(e_j) R^M(e_j, e_k) f^a \otimes v_a, f^b \otimes v_b \rangle + \sum_{k,j,a,b} g(\omega^k \wedge i(e_j) f^a, f^b) h(R^E(e_j, e_k) v_a, v_b).
\]

Since $d\phi(e_l) = \sum_a f^a(e_l) v_a$, we have
\begin{equation}
\tag{2.8}
\sum_{k,j,a} g(\omega^k \wedge i(e_j) R^M(e_j, e_k) f^a, f^a) = \sum_{k} h(d\phi(Ric^M(e_k)), d\phi(e_k)).
\end{equation}

From (2.5) and (2.8), we have
\begin{equation}
\tag{2.9}
\sum_{k,j} \langle \omega^k \wedge i(e_j) R(e_j, e_k) d\phi, d\phi \rangle = \sum_{k} h(d\phi(Ric^M(e_k)), d\phi(e_k)) + \sum_{k,j} h(R^N(d\phi(e_j), d\phi(e_k)) d\phi(e_j), d\phi(e_k))
\end{equation}

Hence we have the following lemma.

**Lemma 2.1.** ([7]) Let $\phi : (M, g) \rightarrow (N, h)$ be an arbitrary smooth map. Then the Weitzenböck formula is given by
\begin{equation}
\tag{2.10}
-\frac{1}{2} \Delta^M |d\phi|^2 = |\nabla d\phi|^2 - \langle d\phi, \Delta d\phi \rangle + F(\phi),
\end{equation}
where
\begin{equation}
\tag{2.11}
F(\phi) = \sum_{k=1}^{m} h(d\phi(Ric^M(e_k)), d\phi(e_k))
- \sum_{k,j=1}^{m} h(R^N(d\phi(e_j), d\phi(e_k)) d\phi(e_j), d\phi(e_k)).
\end{equation}
3. Proof of Theorem 1.4

Assume that dim\(M = m \geq n = \text{dim}N\). Let \(\phi : M \rightarrow N\) be a harmonic morphism and \(\lambda\) the dilation of \(\phi\). Let \(\{e_i\}_{i=1,\ldots,m}\) be a local orthonormal frame field on \(M\) such that \(\{e_i\} \in H_x \ (i = 1,\ldots,n)\) and \(\{e_{n+i}\} \in V_x \ (i = 1,\ldots,m-n)\). Note that for any harmonic map, \(\nabla \phi(\nabla \phi) = \nabla \phi(\nabla \phi) = 0 \ (\[3\])\).

From (1.2) and (2.10), we have the following lemma.

**Lemma 3.1.** ([6]) If \(\phi : M \rightarrow N\) is a harmonic morphism, then
\[
-\frac{n}{2} \Delta^M \lambda^2 = |\nabla \phi|^2 + \lambda^2 \text{tr} \text{Ric}^M |_\mathcal{H} - \lambda^4 r_N \circ \phi,
\]
where \(\lambda\) denotes the dilation, \(\text{tr} \text{Ric}^M |_\mathcal{H}\) the trace of the Ricci tensor of \(M\) on the horizontal distribution \(\mathcal{H}\), and \(r_N\) the scalar curvature of \(N\).

Let \(\mu_0\) be the least eigenvalue of \(\Delta^M\) acting on \(L^2\)-functions on \(M\). Then we have the following lemma.

**Lemma 3.2.** Let \(M\) be a complete Riemannian manifold such that \(\text{Ric}^M \geq -\mu_0\) at all \(x \in M\) and let \(N\) be a Riemannian manifold of nonpositive scalar curvature. If \(\phi : M \rightarrow N\) is a harmonic morphism, then
\[
n \lambda \geq -\frac{\mu_0}{n} \lambda_0 \lambda.
\]

**Proof.** Since \(\Delta^M \lambda^2 = 2\lambda \Delta^M \lambda - 2|\nabla^M \lambda|^2\), we have from (3.1),
\[
n \lambda \Delta^M \lambda = n|\nabla^M \lambda|^2 - |\nabla \phi|^2 - \lambda^4 \text{tr} \text{Ric}^M |_\mathcal{H} + \lambda^4 r_N \circ \phi.
\]
Since \(|\nabla \phi|^2 = n\lambda^2\), we have \(n|\nabla^M \lambda|^2 = n \lambda \nabla^M \lambda\) and
\[
|\nabla^M |\nabla \phi||^2 = n|\nabla^M \lambda|^2.
\]
By the first Kato’s inequality ([1]), i.e., \(|\nabla^M |\nabla \phi||^2 \leq |\nabla \phi|^2\), (3.4) yields
\[
n |\nabla^M \lambda|^2 \leq |\nabla \phi|^2.
\]
Since the scalar curvature \(r_N\) of \(N\) is nonpositive, the first inequality of (3.2) follows from (3.3) and (3.5). The second inequality of (3.2) is trivial from \(\text{Ric}^M \geq -\mu_0\). \(\square\)

**Proof of Theorem 1.4.** We choose a Lipschitz continuous function \(\omega_\ell\) on \(M\) such that \(\omega_\ell \in C_0^{\infty}(M)\) and \(\omega_\ell \equiv 1\) on \(B(x_0, \ell)\), \(\lim_{\ell \rightarrow \infty} \omega_\ell = 1\), \(\text{supp} \omega_\ell \subset B(x_0, 2\ell)\) and \(|d\omega_\ell| \leq C/\ell\) for some constant \(C\), where \(\ell \in \mathbb{R}^+\) and \(B(x_0, \ell)\) is the Riemannian open ball with radius \(\ell\).

Multiplying (3.2) by \(\omega_\ell^2 \lambda\) and integrating by parts, we obtain
\[
n \int_M \langle d\lambda, d(\omega_\ell^2 \lambda) \rangle \leq - \int_M \omega_\ell^2 \lambda^2 \text{tr} \text{Ric}^M |_\mathcal{H} \leq n\mu_0 \int_M (\omega_\ell \lambda)^2.
\]
By a direct calculation, we have
\[
\langle d\lambda, d(\omega_\ell^2 \lambda) \rangle = 2\omega_\ell \lambda \langle d\lambda, d\omega_\ell \rangle + |\omega_\ell d\lambda|^2 = |d(\omega_\ell \lambda)|^2 - \lambda^2 |d\omega_\ell|^2.
\]
From (3.6) and (3.7), we have
\[
\int_M |d(\omega_t \lambda)|^2 \leq -\frac{1}{n} \int_M \omega_t^2 \lambda^2 \text{tr} \text{Ric}^M|_\mathcal{H} + \int_M \lambda^2 |d\omega_t|_\mathcal{H}^2 \\
\leq \mu_0 \int_M (\omega_t \lambda)^2 + \int_M \lambda^2 |d\omega_t|_\mathcal{H}^2.
\]

Since \(\mu_0\) is the infimum of the spectrum of the Laplacian \(\Delta^M\) acting on \(L^2\) functions on \(M\), the Rayleigh theorem implies
\[
\int_M |d(\omega_t \lambda)|^2 \geq \mu_0 \int_M (\omega_t \lambda)^2.
\]

If we let \(\ell \to +\infty\) in (3.8) with (3.9), then we have
\[
(3.10) \quad \mu_0 \int_M \lambda^2 \lambda \leq -\frac{1}{n} \int_M \lambda^2 \text{tr} \text{Ric}^M|_\mathcal{H} \leq \mu_0 \int_M \lambda^2.
\]

This means that
\[
(3.11) \quad 0 = \int_M (n \mu_0 + \text{tr} \text{Ric}^M|_\mathcal{H}) \lambda^2 = \frac{1}{n} \int_M (n \mu_0 + \text{tr} \text{Ric}^M|_\mathcal{H}) |d\phi|^2.
\]

If \(\text{Ric}^M \geq -\mu_0\) at all \(x\) and \(\text{Ric}^M > -\mu_0\) at some \(x_0\), then \(n \mu_0 + \text{tr} \text{Ric}^M|_\mathcal{H} \geq 0\) for all \(x\) and \(n \mu_0 + \text{tr} \text{Ric}^M|_\mathcal{H} > 0\) at some point \(x_0\), respectively. The unique continuation property for sections implies \(|d\phi| = 0\), i.e., \(\phi\) is constant.

Now we study Theorem 1.4 under the assumption \(\text{Ric}^M \geq -\mu_0\) and \(\text{Vol}(M) = \infty\). We first note that for any real number \(\delta > 0\)
\[
(3.12) \quad |2 \int_M \omega_t \lambda |d\lambda, d\omega_t| \leq \delta^2 \int_M \omega_t^2 |d\lambda|^2 + \frac{1}{\delta^2} \int_M \lambda^2 |d\omega_t|^2.
\]

From (3.6), (3.7) and (3.12), we have
\[
(1 - \delta^2) \int_M \omega_t^2 |d\lambda|^2 - \frac{1}{\delta^2} \int_M \lambda^2 |d\omega_t|^2 \leq -\frac{1}{n} \int_M \omega_t^2 \lambda^2 \text{tr} \text{Ric}^M|_\mathcal{H} \leq \mu_0 \int_M (\omega_t \lambda)^2.
\]

From (3.13), Fatou’s lemma implies that \(d\lambda\) is \(L^2\)-section. Hence if we choose \(\delta = \frac{1}{\sqrt{\ell}}\) and let \(\ell \to +\infty\), then
\[
(3.14) \quad \int_M |d\lambda|^2 \leq -\frac{1}{n} \int_M \lambda^2 \text{tr} \text{Ric}^M|_\mathcal{H} \leq \mu_0 \int_M \lambda^2.
\]

On the other hand, from (3.7) and (3.12) we similarly obtain
\[
(3.15) \quad (1 + \delta^2) \int_M \omega_t^2 |d\lambda|^2 \geq \int_M |d(\omega_t \lambda)|^2 - (1 + \frac{1}{\delta^2}) \int_M \lambda^2 |d\omega_t|^2.
\]

If we put \(\delta = \frac{1}{\sqrt{\ell}}\) and let \(\ell \to +\infty\), then we have from (3.9)
\[
(3.16) \quad \int_M |d\lambda|^2 \geq \mu_0 \int_M \lambda^2.
\]
From (3.14) and (3.16), we have \( \int_M (\Delta^M \lambda - \mu_0 \lambda) \lambda = 0 \). Hence (3.2) implies that \( \Delta^M \lambda = \mu_0 \lambda \). This means that \( \lambda \) is nonnegative \( L^2 \)-subharmonic function. By the maximum principle ([11]), \( \lambda \) is constant. Since \( \text{Vol}(M) = \infty \), it is trivial that \( \lambda = 0 \), which yields that \( \phi \) is constant.

\( \square \)

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