VIABILITY FOR SEMILINEAR DIFFERENTIAL EQUATIONS
OF RETARDED TYPE

QIXIANG DONG AND GANG LI
VIABILITY FOR SEMILINEAR DIFFERENTIAL EQUATIONS OF RETARDED TYPE

QIXIANG DONG and GANG LI

Abstract. Let $X$ be a Banach space, $A : D(A) \subset X \to X$ the generator of a compact $C_0$-semigroup $S(t) : X \to X, t \geq 0, D$ a locally closed subset in $X$, and $f : (a, b) \times C([−q, 0]; X) \to X$ a function of Caratheodory type. The main result of this paper is that a necessary and sufficient condition in order that $D$ be a viable domain of the semilinear differential equation of retarded type

$$u'(t) = Au(t) + f(t, u_t), t \in [t_0, t_0 + T], u_{t_0} = φ \in C([−q, 0]; X)$$

is the tangency condition

$$\liminf_{h \downarrow 0} h^{-1}d(S(h)v(0) + hf(t, v); D) = 0$$

for almost every $t \in (a, b)$ and every $v \in C([−q, 0]; X)$ with $v(0) \in D$.

1. Introduction

Let $X$ be a real Banach space, $A : D(A) \subset X \to X$ the infinitesimal generator of a $C_0$-semigroup $S(t) : X \to X, t \geq 0, D$ a nonempty subset in $X$. Let $q$ and $T$ be positive numbers and $−\infty \leq a < b \leq +\infty$. Given $t_0 \in (a, b)$, a function $x : [t_0 − q, t_0 + T] \to X$ and $t \in [t_0, t_0 + T]$, define $x_t : [−q, 0] \to X$ by $x_t(θ) = x(t + θ)$ for all $θ \in [−q, 0]$. In this paper we discuss the semilinear differential equation of retarded type:

$$u'(t) = Au(t) + f(t, u_t), \quad t \in [t_0, t_0 + T]$$

with the initial condition

$$u_{t_0} = φ \in C([−q, 0]; X),$$

where $C([−q, 0]; X)$ denotes the Banach space of continuous $X$-valued functions on $[−q, 0]$ with supremum norm, $f : (a, b) \times C([−q, 0]; X) \to X$ and $t_0 \in (a, b)$.

We say that $D$ is viable domain for (1.1) if for each $t_0 \in (a, b)$, and $φ \in C([−q, 0]; X)$ with $φ(0) \in D$, there exists at least one mild solution $u : [t_0 −...
$q, t_0 + T] \to X$ to (1.1) and (1.2) with $T = T(t_0, \phi) > 0, t_0 + T < b$, such that $u(t) \in D$ for all $t \in [t_0, t_0 + T]$. We recall that by mild solution to (1.1) and (1.2) we mean a continuous function $u : [t_0 - q, t_0 + T] \to X$, satisfying $u_{t_0} = \phi$, and

$$u(t) = S(t - t_0)\phi(0) + \int_{t_0}^{t} S(t - s)f(s, u_s)ds$$

for $t \in [t_0, t_0 + T]$.

The viability problem for the differential equation

$$(1.4) \quad u'(t) = Au(t) + F(t, u(t)), t \in [t_0, t_0 + T]$$

$$(1.5) \quad u(t_0) = x_0 \in D$$

has been studied by many authors by using various frameworks and techniques. In this respect it should be noted the pioneering work of Nagumo [15] who considered the finite dimensional case, $A = 0$ and $F$ is continuous. In this context he showed that a necessary and sufficient condition in order that $D$ be a viable domain for (1.3) is the following tangency condition:

$$\lim h \to 0 h^{-1}d(x + hF(t, x); D) = 0$$

for each $(t, x) \in (a, b) \times D$. It is interesting to note that Nagumo’s result (or some variant of it) has been rediscovered several times among others by Brezis [4], Crandall [7], Hartman [9], and Martin [14]. For the development in this area, we refer the readers to Ursescu [22], Pavel [19], Cârjă and Marques [5], Cârjă and Vrabie [6]. Brief reviews of the main contributions in this area can be found in [5] and [6]. We emphasize Pavel’s main contribution who was the first who formulated the corresponding tangency condition applying to the semilinear case. More precisely, Pavel [19] showed that, whenever $A$ generates a compact $C_0$-semigroup and $F$ is continuous on $(a, b) \times D$, where $D$ is locally closed in $X$, a sufficient condition for viability is

$$\lim h \to 0 h^{-1}d(S(h)x + hF(t, x); D) = 0$$

for each $(t, x) \in (a, b) \times D$.

Concerning the differential equations of retarded type, the development was initiated about existence and stability by Travis and Webb [20], [21] and Webb [23], [24]. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors (see, e.g., [1], [2], [11], [13] and references therein). Iacob and Pavel [10] discussed viability problem for semilinear differential equations of retarded type. They proved that, whenever $A$ generates a compact $C_0$-semigroup and $f$ is continuous from $(a, b) \times C([-q, 0]; X)$ into $X$ a necessary and sufficient condition for viability for (1.1) is

$$\lim h \to 0 h^{-1}d(S(h)v(0) + hf(t, v); D) = 0$$
for each $t \in (a, b)$, each $v \in C([-q, 0]; X)$ with $v(0) \in D$, where $D$ is a locally closed subset in $X$.

The aim of this paper is to discuss the viable problem of the semilinear differential equation of retarded type (1.1). We prove that a necessary and sufficient condition in order that $D$ be a viable domain of (1.1) is the tangency condition. We only suppose that $f$ is of Caratheodory type. Our result extends and improves that of Iacob and Pavel [10] who considered the case in which $f$ is continuous, and also extends the well-known existence result of Hale [8] who considered the case in which $X$ is finite dimensional and $A = 0$. Moreover, using a standard argument based on Zorn’s Lemma, we get the existence of noncontinuable(saturated) mild solutions.

2. Preliminaries

Let $X$ be a real Banach space, $A : D(A) \subset X \to X$ generates a $C_0$-semigroup $S(t) : X \to X$, $t \geq 0$. It is well known that in this case $S(t), t \geq 0$ is exponentially bounded, i.e., there are constants $C \geq 1$ and $\omega > 0$ such that $\|S(t)\| \leq C e^{\omega t}$, $\forall t \geq 0$.

Moreover, if $S(t), t \geq 0$ is a compact semigroup (i.e., $S(t)$ maps bounded subsets into relatively compact subsets for $t > 0$), then $S(t)$ is continuous in the uniformly operator topology for $t > 0$ (see Pazy [19]) and $X$ is separable (see [5]). For more details of semigroups of linear operators, we refer the readers to Pazy [19].

For convenience of future reference, we list the following conditions:

(A1) for each $v \in C([-q, 0]; X)$, the function $f(\cdot, v) : (a, b) \to X$ is measurable on $(a, b)$;

(A2) for almost every (a.e.) $t \in (a, b)$, $f(t, \cdot) : C([-q, 0]; X) \to X$ is continuous on $C([-q, 0]; X)$;

(A3) for every $r > 0$, there is a function $m_r \in L(a, b; X)$ such that $\|f(t, v)\| \leq m_r(t)$ for a.e. $t \in (a, b)$ and every $v \in C([-q, 0]; X)$ with $\|v\| \leq r$.

(T) (Tangency condition)

\[
\liminf_{h \downarrow 0} h^{-1} d(S(h)v(0) + hf(t, v); D) = 0
\]

for a.e. $t \in (a, b)$ and all $v \in C([-q, 0]; X)$ with $v(0) \in D$, where $d(x, D)$ denotes the distance from $x \in X$ to the subset $D \subset X$.

Since the distance is non-expansive, i.e.,

$|d(x, D) - d(y, D)| \leq \|x - y\|, \quad \forall x, y \in X,$

by standard arguments (see [10], [17]), Condition (T) is equivalent to

\[
\liminf_{h \downarrow 0} h^{-1} d(S(h)v(0) + \int_b^{t+h} S(t + h - s)f(t, v)ds; D) = 0
\]

for a.e. $t \in (a, b)$ and all $v \in C([-q, 0]; X)$ with $v(0) \in D$. 
Let $f$ be Caratheodory type if $f$ satisfies (A1)-(A3). A Caratheodory type function has the following Scorza Dragoni property which is nothing but the special case of [3], [12]. We denote by $\lambda$ the Lebesgue measure on $\mathbb{R}$ and by $\mathcal{L}$, the collection of all Lebesgue measurable sets in $\mathbb{R}$.

**Theorem 2.1.** Let $X, Y$ be separable metric spaces and $I = (a, b) or I \in \mathcal{L}((a, b))$. Let $f : I \times X \to Y$ be a function such that $f(\cdot, x)$ is measurable for every $x \in X$ and $f(t, \cdot)$ is continuous for almost every $t \in I$. Then, for each $\varepsilon > 0$, there exists a compact subset $K \subset I$ such that $\lambda(I \setminus K) < \varepsilon$ and the restriction of $f$ to $K \times X$ is continuous.

Suppose that $u : (a - q, b) \to X$ is continuous. Then the mapping $t \mapsto u_t$, from $(a, b)$ into $C([-q, 0]; X)$ is also continuous. The following result is a kind of variance of Lebesgue derivative type, which is useful in the sequel.

**Theorem 2.2.** Assume that $D$ is a nonempty subset of a separable Banach space $X$, $S(t)$ is a $C_0$-semigroup on $X$ and $f : (a, b) \times C([-q, 0]; X) \to X$ is a function which satisfies the conditions (A1), (A2) and (A3). Then there exists a negligible subset $Z$ of $(a, b)$ such that, for every $t \in (a, b) \setminus Z$, one has

$$
\lim_{h \downarrow 0} h^{-1} \int_t^{t+h} S(t + h - s)f(s, u_s)ds = f(t, u_t)
$$

for all continuous functions $u : (a, b) \to X$.

The proof of Theorem 2.2 is similar to that of [5] Theorem 2.3. So we omit it.

3. Main result

Now we are ready to state our main result of this paper.

**Theorem 3.1.** Let $D \subset X$ be a locally closed subset in a general Banach space, $f : (a, b) \times C([-q, 0]; X) \to X$ a function satisfying (A1)-(A3), and let $A : D(A) \to X$ be the infinitesimal generator of a compact $C_0$-semigroup $S(t) : X \to X, t \geq 0$. Then a necessary and sufficient condition in order that $D$ be a viable domain of (1.1) is the tangency condition (T).

*Proof of necessity.* Let $Z$ be given by Theorem 2.2, let $t_0 \in (a, b) \setminus Z$. Let $v \in C([-q, 0]; X)$ such that $v(0) \in D$. By hypothesis, there exists $T = T(t_0, v) > 0$ with $t_0 + T < b$ and a continuous function $u$ satisfying (1.3) with $\phi = v$. Since $u(t_0 + h) \in D$ for all $h \in [0, T]$, we have

$$
\begin{align*}
    h^{-1}d(S(h)v(0) + hf(t_0, v); D) &\leq h^{-1}\|S(h)v(0) + hf(t_0, v) - u(t_0 + h)\| \\
    &\leq \|f(t_0, v) - h^{-1}\int_{t_0}^{t_0+h} S(t_0 + h - s)f(s, u_s)ds\|.
\end{align*}
$$

(3.1)

Letting $h \downarrow 0$, one obtains the condition (T). \qed
In the proof of sufficiency, the following lemma is needed. We first note that, since $D$ is locally closed, there is a real number $r > 0$ such that $D \cap B(\phi(0), r)$ is closed. On the basis of the continuity of $\phi$ on $[-q, 0]$, there is a real number $T > 0$ such that

$$
\|\phi(\theta_1) - \phi(\theta_2)\| \leq \frac{1}{2} r, \quad \forall \theta_1, \theta_2 \in [-q, 0], |\theta_1 - \theta_2| \leq T.
$$

(3.2) Set $R = r + \|\phi(0)\|$ and

$$
M = \int_{t_0}^{t_0+T} m_R(t)dt,
$$

where $m_R$ is the function appeared in (A3). Moreover, we may choose $T$ small enough such that $t_0 + T < b$ and

$$
\max_{0 \leq t \leq T} \|S(t)\phi(0) - \phi(0)\| + N(M + T) \leq \frac{1}{2} r, \quad (N = Ce^{\alpha T}).
$$

Lemma 3.2. Suppose that the hypotheses of Theorem 3.1 hold. Suppose further that $f: (a, b) \times C([-q, 0]; X) \rightarrow X$ satisfies the tangency condition (T). Then for each $t_0 \in (a, b)$, $\phi \in C([-q, 0]; X)$ with $\phi(0) \in D$, each positive integer $n$, and each open subset $L_n \subset \mathbb{R}$ with $Z \subset L_n$ and $\lambda(L_n) < \frac{1}{n}$, there exist a $T \in [t_0, t_0 + T] \setminus Z$, an nondecreasing sequence $\{t^n_i\}_{i=1}^{\infty} \subset [t_0, t_0 + T]$, and an approximate solution $u^n$ on $[t_0, t_0 + T]$ in the following sense:

(i) $t^n_0 = t_0$, $t^n_{i+1} - t^n_i = d^n_i \leq \frac{1}{n}, \lim_{i \rightarrow \infty} t^n_i = t_0 + T;
$

(ii) $u^n_i = \phi, u^n(t^n_i) = x^n_i \in D \cap B(\phi(0), r);
$

(iii) $h_n(s) = f(t^n_i, u^n_i)$ in case $t^n_i \not\in L_n$ while $h_n(s) = f(T, u^n_T)$ in case $t^n_i \in L_n$ for $s \in [t^n_i, t^n_{i+1}];
$

(iv) $u^n(t) = S(t - t^n_i)x^n_i + \int_{t^n_i}^{t} S(t - s)h_n(s)ds + (t - t^n_i)p^n_i$ for $t \in [t^n_i, t^n_{i+1}],$

where $x^n_i \in D$ and $p^n_i \in X$ with $\|p^n_i\| \leq \frac{1}{n}$. Moreover, $u^n_i \in B(\phi, r) \cap C([-q, 0]; X).

Proof. Let $t_0 \in (a, b)$, $\phi \in C([-q, 0]; X)$ and $n \in \mathbb{N}$ be given. We may assume that (2.2) and (2.3) hold for each $t \in [t_0, t_0 + T] \setminus L_n$. Fix $T \in [t_0, t_0 + T] \setminus L_n$. We shall construct $u^n$ and $t^n_i$ by induction. Set $t^n_0 = t_0, u^n(t^n_0) = \phi(0) = x^n_0, u^n_0 = \phi$. To simplify notation, we drop $n$ as a superscript for $t_i, x_i, u_i, p_i$ etc. Suppose that $u$ is constructed on $[t_0 - q, t_1]$. Then we define $t_{i+1}$ in the following manner. If $t_i = t_0 + T$, set $t_{i+1} = t_0 + T$, and if $t_i < t_0 + T$, then we define $t_{i+1}$ as the following two cases.

Case 1: $t_i \in L_n$. Set

$$
\delta_i = \sup \{h \in (0, \frac{1}{2}) : t_i + h \leq t_0 + T, [t_i, t_i + h] \subset L_n,
$$

$$
d(S(h)x_i + \int_{t_i}^{t_i+h} S(t_i + h - s)f(T, u_{t_i})ds; D) \leq \frac{1}{2^{i+1}}.
$$

(3.5) In view of (2.1) and the fact that

$$
\lim_{h \rightarrow 0} h^{-1} \int_{t_i}^{t_i+h} S(t_i + h - s)f(T, u_{t_i})ds = f(T, u_{t_i}),
$$
one can easily see that \( \delta_i > 0 \). Choose a number \( d_i \in (\frac{1}{2} \delta_i, \delta_i] \), such that

\[
d(S(d_i)x_i + \int_{t_i}^{t_i + d_i} S(t_i + d_i - s) f(\overline{t}, u_{t_i}) ds; D) \leq \frac{d_i}{2n}.
\]

Define \( t_{i+1} = t_i + d_i \). By (3.6), there is \( x_{i+1} \in D \) such that

\[
\| S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s) f(\overline{t}, u_{t_i}) ds - x_{i+1} \| \leq \frac{d_i}{n}.
\]

Consequently, \( x_{i+1} \) can be written as

\[
x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s) f(\overline{t}, u_{t_i}) ds + (t_{i+1} - t_i)p_i
\]

with \( \| p_i \| \leq \frac{1}{n} \). In this case we define \( u \) on \([t_i, t_{i+1}]\) as

\[
u(t) = S(t - t_i)x_i + \int_{t_i}^{t} S(t - s) f(\overline{t}, u_{t_i}) ds + (t - t_i)p_i.
\]

**Case 2**: \( t_i \not\in L_n \). In this case we set

\[
\delta_i = \sup \{ h \in (0, \frac{1}{n^2}] : t_i + h \leq t_0 + T, \quad
\overline{d}(S(h)x_i + \int_{t_i}^{t_i + h} S(t_i + h - s) f(t_i, u_{t_i}) ds; D) \leq \frac{h}{2n} \}.
\]

By (2.2) we see that \( \delta_i > 0 \). Choose \( d_i \in (\frac{1}{2} \delta_i, \delta_i] \), such that

\[
d(S(d_i)x_i + \int_{t_i}^{t_i + d_i} S(t_i + d_i - s) f(t_i, u_{t_i}) ds; D) \leq \frac{d_i}{2n}.
\]

Define \( t_{i+1} = t_i + d_i \). By (3.10), there is \( x_{i+1} \in D \) such that

\[
\| S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s) f(t_i, u_{t_i}) ds - x_{i+1} \| \leq \frac{d_i}{n}.
\]

Consequently, \( x_{i+1} \) can be written as

\[
x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s) f(t_i, u_{t_i}) ds + (t_{i+1} - t_i)p_i
\]

with \( \| p_i \| \leq \frac{1}{n} \). In this case we define \( u \) on \([t_i, t_{i+1}]\) as

\[
u(t) = S(t - t_i)x_i + \int_{t_i}^{t} S(t - s) f(t_i, u_{t_i}) ds + (t - t_i)p_i.
\]

Setting \( h(s) = f(\overline{t}, u_{t_i}) \) in case \( t_i \in L_n \) and \( h(s) = f(t_i, u_{t_i}) \) in case \( t_i \not\in L_n \) for \( s \in [t_i, t_{i+1}] \). Let us define the step functions \( \alpha_n \) and \( \beta_n \) as \( \alpha_n(s) = t_i \) in case \( t_i \not\in L_n \), \( \alpha_n(s) = \overline{t} \) in case \( t_i \in L_n \) and \( \beta_n(s) = t_i \) for \( s \in [t_i, t_{i+1}] \). Then
Thus, properties (ii), (iii) and (iv) are verified.

Let us check that \( u_{i+1} \in B(\phi, r) \). To do this, we have to estimate \( \| u_{i+1}(\theta) - \phi(\theta) \| \) for each \( \theta \in [-q, 0] \). If \(-q \leq \theta \leq t_0 - t_{i+1} \), then by (3.2),

\[
\| u_{i+1}(\theta) - \phi(\theta) \| = \| u(0) + (t_{i+1} + \theta - t_0) \| = \| \phi(t_{i+1} + \theta - t_0) - \phi(0) \| \leq \frac{1}{2}r < r
\]

since \( t_{i+1} - t_0 \leq T \). If \( t_0 - t_{i+1} \leq \theta \leq 0 \), then \( t_{i+1} + \theta \geq t_0 \), so by (3.13), (3.2) and (3.4), we have

\[
\| u_{i+1}(\theta) - \phi(\theta) \| \\
\leq \| u(t_{i+1} + \theta) - \phi(0) \| + \| \phi(0) - \phi(\theta) \| \\
\leq \| S(t_{i+1} + \theta - t_0)\phi(0) - \phi(0) \| + N \sum_{m=0}^{i} \int_{t_m}^{t_{m+1}} \| h(s) \| ds \\
+ \sum_{m=0}^{i} (t_{m+1} - t_m)N \| p_m \| + \| \phi(0) - \phi(\theta) \| \\
\leq \| S(t_{i+1} + \theta - t_0)\phi(0) - \phi(0) \| + N(M + T) + \| \phi(0) - \phi(\theta) \| \\
\leq \frac{1}{2}r + \frac{1}{2}r = r
\]

and hence \( u_{i+1} \in B(\phi, r) \). Using again (3.13), we derive

\[
\| u(t) - \phi(0) \| \leq \| S(t - t_0)\phi(0) - \phi(0) \| + N(M + T) \leq \frac{1}{2}r < r
\]

for all \( t \in [t_0, t_{i+1}] \), i.e., \( u(t) \in B(\phi(0), r) \) for \( t \in [t_0, t_{i+1}] \). This remark, along with the fact that \( \phi \in C([-q, 0]; X) \), implies that \( u_{i+1} \in B(\phi, r) \cap C([-q, 0]; X) \).

Thus, properties (ii), (iii) and (iv) are verified.

To prove property (i), we first note that \( \lim_{i \to \infty} t_i \) exists, since \( \{t_i\}_{i=1}^{\infty} \) is increasing and \( t_i \leq t_0 + T \) for all \( i = 1, 2, \ldots \). Suppose that \( \lim_{i \to \infty} t_i = t^* \), then \( t^* \leq t_0 + T \). We have to prove \( t^* = t_0 + T \). To do this, we first show that \( \lim_{i \to \infty} x_i \) also exists. In fact, let \( j \geq i \). Using (3.13) for \( t = t_i \) and \( t = t_j \), we
derive
\[ \|x_j - x_i\| \leq \|S(t_i - t_0)(S(t_j - t_i)\phi(0) - \phi(0))\| \\
+ \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_i - s)(S(t_j - t_i)h(s) - h(s))\|ds \\
+ \sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_i - t_{m+1})(S(t_j - t_i)p_m - p_m)\| \\
+ \sum_{m=1}^{j-1} \|f_{t_{m+1}}^t S(t_j - s)h(s)ds\| \\
+ \sum_{m=1}^{j-1} (t_{m+1} - t_m)\|S(t_j - t_{m+1})p_m\| \\
\leq N\|S(t_j - t_i)\phi(0) - \phi(0)\| \\
+ N \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\|ds \\
+ N \sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_j - t_i)p_m - p_m\| \\
+ N \int_{t_i}^{t_j} m_R(s)ds + N(t_j - t_i)\frac{1}{n} \tag{3.14} \]

Now given \( \varepsilon > 0 \). Since \( m_R \in L(a, b, X) \), there is \( \eta > 0 \) such that \( \int_{t'}^{t''} m_R(s)ds \leq \varepsilon/(5N) \) for \( t', t'' \in (a, b) \) with \( |t' - t''| < \eta \). By the existence of \( \lim_{t \to \infty} t_i = t' \), there is a positive integer \( k_0 \) such that

\[ t_j - t_i < \min \left\{ \frac{\varepsilon}{10N(N+1)M}, \frac{\varepsilon}{10(N+1)}, \eta \right\} \tag{3.15} \]

for all \( j > i \geq k_0 \). Choose \( k_1 > k_0 \) with the properties: for \( j > i \geq k_1 \),

- \( \|S(t_j - t_i)\phi(0) - \phi(0)\| \leq \varepsilon/(5N) \);
- \( \|S(t_j - t_i)p_m - p_m\| \leq \varepsilon/(10NT) \), \( 1 \leq m \leq k_0 - 1 \);
- \( \|S(t_j - t_i)f(t_m, u_{t_m}) - f(t_m, u_{t_m})\| \leq \varepsilon/(10NT) \), \( 1 \leq m \leq k_0 - 1 \) with \( t_m \in L_n \);
- \( \|S(t_j - t_i)f(t, u_{t_m}) - f(t, u_{t_m})\| \leq \varepsilon/(10NT) \), \( 1 \leq m \leq k_0 - 1 \) with \( t_m \in L_n \).

Then we have

\[ N\|S(t_j - t_i)\phi(0) - \phi(0)\| \leq N \frac{\varepsilon}{5N} = \frac{\varepsilon}{5} \tag{3.16} \]

\[ N \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\|ds \]

\[ \leq N \sum_{m=0}^{k_0-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\|ds \\
+ \sum_{m=k_0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\|ds \\
\leq N(t_{k_0} - t_0)\frac{\varepsilon}{10NT} + N(t_i - t_{k_0})(N + 1)M \leq \frac{\varepsilon}{5} \tag{3.17} \]
(3.18) \[ N \int_{t_i}^{t_j} m_R(s) \, ds \leq N \frac{\varepsilon}{5N} = \frac{\varepsilon}{5}; \]

\[ N \sum_{m=0}^{i-1} (t_{m+1} - t_m) \| S(t_j - t_i) p_m - p_m \| \]
\[ \leq N \sum_{m=0}^{i-1} (t_{m+1} - t_m) \| S(t_j - t_i) p_m - p_m \| \]
\[ + N \sum_{m=k_0}^{i-1} (t_{m+1} - t_m) \| S(t_j - t_i) p_m - p_m \| \]
\[ \leq N(t_{k_0} - t_0) \frac{\varepsilon}{10N} + (t_i - t_{k_0}) N(N + 1) \]
\[ \leq \varepsilon \frac{4}{5}; \]

(3.20) \[ N(t_j - t_i) < \frac{\varepsilon}{5}. \]

From (3.14) to (3.20), we obtain that

(3.21) \[ \| x_j - x_i \| \leq \varepsilon \]

for all \( j > i \geq k_1 \), i.e., \( \{ x_i \} \) is a Cauchy sequence. Therefore \( \lim_{i \to \infty} x_i = x^* \) exists, and \( x^* \in B(\phi(0), r) \cap D \) since \( B(\phi(0), r) \cap D \) is closed. We define \( u(t^*) = x^* \). By (iv) we have

\[ \| u(t) - x_i \| \leq \| S(t - t_i)x_i - x_i \| + (t_i - t)(M + 1) \]

and therefore \( \lim_{t \to t^*} u(t) = x^* = u(t^*) \). Accordingly, \( u(t) \) is continuous on \( [t - q, t^*] \), and hence \( \lim_{i \to \infty} u_{t_i} = u_{t^*} \in C([-q, 0]; X) \cap B(\phi, r) \).

We assert that \( t^* \notin L_n \) for sufficiently large \( n \). Indeed, if \( t^* \in L_n \), then there are only finite many \( t_i \notin L_n \) since \( [t_0, t^*] \ \setminus \ L_n \) is closed. Therefore there is a positive integer \( i_0 \) such that \( t_i \in L_n \) for all \( i \geq i_0 \). But then \( [t_{i_0}, t^*] \subset L_n \) by (3.5), which contradicts the fact that \( \lambda(L_n) < \frac{1}{n} \) for sufficiently large \( n \).

We now assume by contradiction that \( t^* < t_0 + T \). We choose \( h^* \in (0, \frac{1}{n}] \) such that

(3.22) \[ d(S(h^*)x^* + \int_{t_i}^{t_i + h^*} S(t_i + h^* - s)f(t^*, u_{t^*}) \, ds; D) \leq \frac{h^*}{4n}. \]

Since \( \frac{1}{n} \delta_i < d_i = t_{i+1} - t_i \to 0 \) as \( i \to \infty \), there is a positive integer \( i_0 \) such that \( \delta_i < h^* \) for all \( i > i_0 \). On the basis of (3.9), we have

(3.23) \[ d(S(h^*)x^* + \int_{t_i}^{t_i + h^*} S(t_i + h^* - s)f(t^*, u_{t^*}) \, ds; D) > \frac{h^*}{2n} \]

for \( i > i_0 \) and \( t_i \notin L_n \). Letting \( i \to \infty \) in (3.23), one obtains an inequality which contradicts (3.22). Hence \( t^* = t_0 + T \), which concludes the proof. \( \square \)

Proof of sufficiency. Let \( \{ L_n \} \) be a sequence of open subsets of \( \mathbb{R} \) such that \( Z \subset L_n \) and \( \lambda(L_n) < \frac{1}{n} \) for all \( n \in \mathbb{N} \). Take \( L = \cap_{n \geq 1} L_n \) and a sequence of
n-approximate solutions \( \{u^n\} \) and \( \{t^n_i\} \) obtained in Lemma 3.2. Let us define
\[
g_n(t) = \sum_{m=0}^{i-1} (t^n_{m+1} - t^n_m) S(t - t^n_{m+1}) p^n_m + (t - t^n_i) p^n_i
\]
for \( t \in [t_i, t_{i+1}] \). Then \( \|g_n(t)\| \leq \frac{NT}{n} \) for all \( t \in [t_0, t_0 + T] \) and \( u^n \) can be written in the form
\[
u^n(t) = S(t - t_0)\phi(0) + \int_{t_0}^t S(t - s)h_n(s)ds + g_n(t)
\]
for all \( t \in [t_0, t_0 + T] \), \( u^n_{t_0} = \phi \). Set
\[
y^n(t) = \int_{t_0}^t S(t - s)h_n(s)ds, \quad t \in [t_0, t_0 + T].
\]
Since the semigroup \( S(t) : X \to X, t \leq 0 \), is compact and \( \{h_n\} \) is uniformly integrable on \([t_0, t_0 + T]\), by a standard argument involving a compactness result, it follows that there is a \( y \in C([t_0, t_0 + T]; X) \) such that at least on a subsequence we have
\[
\lim_{n \to \infty} y^n(t) = y(t)
\]
uniformly in \( t \in [t_0, t_0 + T] \). Since \( \|g_n(t)\| \leq \frac{NT}{n} \) for all \( t \in [t_0, t_0 + T] \), it follows that
\[
\lim_{n \to \infty} u^n(t) = S(t - t_0)\phi(0) + y(t) \equiv u(t)
\]
uniformly in \( t \in [t_0, t_0 + T] \). Let us observe that if \( s \not\in L \), then \( s \not\in L_n \) for sufficiently large \( n \), and then we have \( \alpha_n(s) \to s \) as \( n \to \infty \). Also we have \( \beta_n(s) \to s \) as \( n \to \infty \) for all \( s \in [t_0, t_0 + T] \). Therefore \( h_n(s) \to f(s, u_0) \) as \( n \to \infty \) for a.e. \( s \in [t_0, t_0 + T] \). Moreover, \( u^n(\alpha_n(s)) \in D \cap B(\phi(0), r) \) (which is closed). Finally, passing to limit in (3.24), one obtains (1.3), which completes the proof.

Concerning the continuation of the solution to (1.1) satisfying (1.2). Recall that a solution \( v : [t_0, t_0 + T_1] \to X \) of (1.1), with \( T_1 \geq T \) is said to be a continuation to the right of the solution \( u : [t_0, t_0 + T] \to X \) of (1.1), if \( v(t) = u(t) \) for all \( t \in [t_0, t_0 + T] \). A solution \( u \) is said to be noncontinuable if it has no proper continuation. Using a standard argument based on Zorn’s Lemma, one can easily verify that, if the hypotheses of Theorem 3.1 hold, and \( u : [t_0, b_0] \to X \) is a noncontinuable mild solution to (1.1) satisfying (1.2), then either \( b_0 = b \) or \( \lim_{t \to b_0} \|u(t)\| = +\infty \). Moreover, the tangency condition (T) is also necessary. Precisely, we have

**Theorem 3.3.** Under the hypotheses of Theorem 3.1, a necessary and sufficient condition in order that for each \( t_0 \in (a, b) \), and each \( \phi \in C([-q, 0]; X) \) with \( \phi(0) \in D \), there is a noncontinuable mild solution \( u(t) \in D \) to (1.1) satisfying (1.2) is the tangency condition (T).
Remark 3.4. If, in addition to the hypotheses of Theorem 3.1, we suppose that \( \phi(\theta) \in D \) for all \( \theta \in [-q,0] \), then there exists a solution to (1.1) and (1.2) with \( u(t) \in D \) for all \( t \in [t_0 - q, t_0 + T] \).

Remark 3.5. If \( D \) is open, then the tangency condition (T) is automatically satisfied. In this case, by Theorem 3.1, one obtains the locally existence result of problem (1.1) and (1.2), which extends the well-known result of J. K. Hale [8], who considered the case in which \( X \) is finite dimensional (i.e., \( X = \mathbb{R}^n \)) and \( A = 0 \).

Theorem 3.6. Let \( X \) be a real Banach space, \( f : (a,b) \times C([-q,0];X) \to X \) a function satisfying (A1)-(A3), and let \( A \) be the infinitesimal generator of a compact \( C_0 \)-semigroup \( S(t) : t \geq 0 \). Then for each \( t_0 \in (a,b) \), and each \( \phi \in C([-q,0];X) \) with \( \phi(0) \in D \), the problem (1.1) and (1.2) has a locally mild solution, for some \( T = T(t_0, \phi) > 0 \), with \( T < b - t_0 \).

References


QIXIANG DONG
School of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China
E-mail address: qxdongyz@yahoo.com.cn

GANG LI
School of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China
E-mail address: gli@yzu.edu.cn