ON THE MEAN VALUES OF DEDEKIND SUMS
AND HARDY SUMS

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Abstract. For a positive integer \( k \) and an arbitrary integer \( h \), the classical Dedekind sums \( s(h, k) \) is defined by

\[
s(h, k) = \sum_{j=1}^{k} \left( \left( \frac{j}{k} \right) \left( \frac{hj}{k} \right) \right),
\]

where

\[
\left( \left( x \right) \right) = \begin{cases} 
  x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \text{ is not an integer;} \\
  0 & \text{if } x \text{ is an integer}.
\end{cases}
\]

J. B. Conrey et al proved that

\[
\sum_{(h,k)=1}^{k} s^{2m}(h, k) = f_{m}(k) \left( \frac{k}{12} \right)^{2m} + O \left( \left( k^{\frac{2}{12}} + k^{2m-1} + k^{\frac{1}{12}} \right) \log^{3} k \right).
\]

For \( m \geq 2 \), C. Jia reduced the error terms to \( O \left( k^{2m-1} \right) \). While for \( m = 1 \), W. Zhang showed

\[
\sum_{(h,k)=1}^{k} s^{2}(h, k) = \frac{5}{144} k \phi(k) \prod_{p \mid k} \left[ \left( 1 + \frac{1}{p} \right)^{-2} - \frac{1}{p+1} \right]
\]

\[
+ O \left( k \exp \left( \frac{4 \log k}{\log \log k} \right) \right).
\]

In this paper we give some formulae on the mean value of the Dedekind sums and Hardy sums, and generalize the above results.

§ 1. Introduction

For a positive integer \( k \) and an arbitrary integer \( h \), the classical Dedekind sums \( s(h, k) \) is defined by

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\[ s(h, k) = \sum_{j=1}^{k} \left( \left( \frac{j}{k} \right) \left( \frac{hj}{k} \right) \right), \]

where

\[ ((x)) = \begin{cases} 
  x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\
  0, & \text{if } x \text{ is an integer.} 
\end{cases} \]

The sum \( s(h, k) \) plays an important role in the transformation theory of the Dedekind \( \eta \) function (see [17] and Chapter 3 of [1] for details).

Perhaps the most famous property of the Dedekind sums is the reciprocity formula

\[ s(h, k) + s(k, h) = h^2 + k^2 + 1 - \frac{1}{4}hk - \frac{1}{4}, \]

denoted by (1.1), for positive coprime integers \( h \) and \( k \). R. Dedekind [9], H. Rademacher [17], B. C. Berndt [2-4] and U. Dieter [10] gave different proofs for this famous reciprocity formula. Some three term versions of this formula were discovered by H. Rademacher [17], R. R. Hall [11], and J. Pommersheim [16].

Suppose that \( a, q, h, k > 0 \) with \( (a, q) = 1 \) and \( (h, k) = 1 \). Suppose further that \( z = qh - ak \) satisfies \( |z| \leq \frac{k}{q} \). J. B. Conrey, E. Fransen, R. Klein, and C. Scott [8] showed that

\[ s(h, k) = \frac{k}{12q^2} + O \left( |s(a, q)| + |z| + 1 \right). \]

Then they studied the \( 2m \)-th power mean of Dedekind sums, and proved the following proposition by using the circle method.

**Proposition 1.1.** Suppose that \( m \) is a given positive integer and \( k \) is any sufficiently large integer. Then

\[ \sum_{\substack{h=1 \\ (h, k)=1}}^{k} s^{2m}(h, k) = f_m(k) \left( \frac{k}{12} \right)^{2m} + O \left( \left( k^{2m-1} + \frac{1}{m+1} \right) \log^3 k \right), \]

where \( f_m(k) \) is defined by the Dirichlet series

\[ \sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2 \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \cdot \zeta(s), \]

and \( \zeta(s) \) is the Riemann zeta-function.

In [13], C. Jia improved the error terms in Proposition 1.1, and proved the following:

**Proposition 1.2.** For every given integer \( m \geq 2 \) and any sufficiently large integer \( k \), we have

\[ \sum_{\substack{h=1 \\ (h, k)=1}}^{k} s^{2m}(h, k) = f_m(k) \left( \frac{k}{12} \right)^{2m} + O \left( k^{2m-1} \right). \]
How to reduce the error terms in Proposition 1.1 for \( m = 1 \)? Recall that H. Walum [19] established a relation between Dedekind sums and Dirichlet \( L \)-functions as following:

\[
\sum_{\chi \mod p} \chi(-1)^{\chi(p)} = -\frac{1}{p^2} \sum_{h=1}^{p} |s(h, p)|^2,
\]

where \( p \) is a prime number, \( L(1, \chi) = \sum_{n=1}^{\infty} \chi(n) n \) is the Dirichlet \( L \)-functions, and \( \chi \) denotes a Dirichlet character modulo \( p \). In the spirit of [8] and [19], W. Zhang [20] showed that

\[
s(h, k) = \frac{1}{\pi^2 k} \sum_{d \mid k} \frac{d^2}{\phi(d)} \sum_{\chi \mod d} \chi(-1) \chi(h) |L(1, \chi)|^2,
\]

where \( \phi(d) \) is the Euler function, and finally in [21] he proved the following:

**Proposition 1.3.** For any sufficiently large integer \( k \), we have

\[
(1.3) \quad \sum_{h=1}^{k} s^2(h, k) = \frac{5}{144}k\phi(k) \prod_{p \mid k} \left( \frac{1 + \frac{1}{p}}{1 + \frac{1}{p} + \frac{1}{p^2}} \right) + \Omega \left( k \exp \left( \frac{4 \log k}{\log \log k} \right) \right),
\]

where \( \prod_{p \mid k} \) denotes the product over all prime divisors \( p \) of \( k \) with \( p^\alpha \mid k \) and \( p^{\alpha+1} \nmid k \), and \( \exp(y) = e^y \).

Taking \( m = 1 \) in Proposition 1.1, we get

\[
(1.4) \quad \sum_{h=1}^{k} s^2(h, k) = \frac{1}{144}f_1(k)k^2 + O \left( k^2 \log^3 k \right).
\]

Compare (1.3) and (1.4), we have

\[
f_1(k) = 5 \cdot \frac{\phi(k)}{k} \prod_{p \mid k} \left( \frac{1 + \frac{1}{p}}{1 + \frac{1}{p} + \frac{1}{p^2}} \right).
\]

Does \( f_m(k) \) have similar form for \( m > 1 \)? If yes, then we can get some interesting result from Propositions 1.1, 1.2, and 1.3. In Section 2 we shall prove the following:

**Theorem 1.1.** For any given integer \( m \geq 1 \) and any sufficiently large integer \( k \), we have the identity

\[
f_m(k) = \frac{2\zeta(2m)}{\zeta(4m)} \prod_{p^\alpha \mid k} \left[ \frac{1 - \frac{1}{p^{2m}}}{1 - \frac{1}{p^{2m+1}}} \left( \frac{1 - \frac{1}{p^{2m+1}}}{1 - \frac{1}{p^{2m}}} \right)^2 \right].
\]
Taking \( m = 1 \) in Theorem 1.1, we immediately get
\[
f_1(k) = 5 \prod_{p^s \mid k} \left[ \left( 1 - \frac{1}{p^2} \right)^2 - \frac{1}{p^{2n+1}} \left( 1 - \frac{1}{p} \right)^2 \right]
\]
\[
= 5 \cdot \phi(k) \prod_{p^s \mid k} \left[ \left( 1 - \frac{1}{p^2} \right)^2 - \frac{1}{p^{2n+1}} \left( 1 - \frac{1}{p} \right)^2 \right]
\]
\[
= 5 \cdot \phi(k) \prod_{p^s \mid k} \left[ \left( 1 + \frac{1}{p^2} \right)^2 - \frac{1}{p^{2n+1}} \frac{1}{1 + \frac{1}{p^2}} \right].
\]

Then from Propositions 1.1-1.3 and Theorem 1.1 we have

**Corollary 1.1.** Suppose that \( m \) is a given positive integer and \( k \) is any sufficiently large integer. Then
\[
\sum_{\substack{h=1 \atop (h,k)=1}}^{k} s^{2m}(h,k)
\]
\[
= \frac{2s^2(2m)}{\zeta(4m)} \left( \frac{k}{12} \right)^{2m} \prod_{p^s \mid k} \left[ \left( 1 - \frac{1}{p^{2m}} \right)^2 - \frac{1}{p^{4m-1}p^{2n+1}} \left( 1 - \frac{1}{p^{2m}} \right)^2 \right] + e_m(k),
\]
where
\[
e_m(k) \ll \begin{cases} k \exp \left( \frac{4 \log k}{\log \log k} \right), & \text{if } m = 1; \\ k^{2m-1}, & \text{if } m > 1. \end{cases}
\]

B. C. Berndt [5] defined the following Hardy sums:
\[
S(h, k) = \sum_{j=1}^{k-1} (-1)^j + \left\lfloor \frac{j}{h} \right\rfloor, \quad s_1(h, k) = \sum_{j=1}^{k} (-1)^j \left\lfloor \frac{j}{k} \right\rfloor \left( \left( \frac{j}{k} \right) \right),
\]
\[
s_2(h, k) = \sum_{j=1}^{k} (-1)^j \left( \left( \frac{j}{h} \right) \right) \left( \left( \frac{hj}{k} \right) \right), \quad s_3(h, k) = \sum_{j=1}^{k} (-1)^j \left( \left( \frac{hj}{k} \right) \right),
\]
\[
s_4(h, k) = \sum_{j=1}^{k} (-1)^j \left\lfloor \frac{j}{k} \right\rfloor, \quad s_5(h, k) = \sum_{j=1}^{k} (-1)^j + \left\lfloor \frac{j}{k} \right\rfloor \left( \left( \frac{j}{k} \right) \right),
\]
and studied their arithmetical properties in [6]. For \( (h, k) = 1 \), R. Sitaramachandrarao [18] and M. R. Pettet [15] expressed Hardy sums in terms of Dedekind sum \( s(h, k) \) as following:
\[
S(h, k) = \begin{cases} 8s(h, 2k) + 8s(2h, k) - 20s(h, k), & \text{if } h + k \text{ is odd}; \\ 0, & \text{if } h + k \text{ is even}, \end{cases}
\]
\[ s_1(h, k) = \begin{cases} 2s(h, k) - 4s(h, 2k), & \text{if } h \text{ is even;} \\ 0, & \text{if } h \text{ is odd}, \end{cases} \]

\[ s_2(h, k) = \begin{cases} -s(h, k) + 2s(2h, k), & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd}, \end{cases} \]

\[ s_3(h, k) = \begin{cases} 2s(h, k) - 4s(2h, k), & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even}, \end{cases} \]

\[ s_4(h, k) = \begin{cases} -4s(h, k) + 8s(h, 2k), & \text{if } h \text{ is odd;} \\ 0, & \text{if } h \text{ is even}, \end{cases} \]

\[ s_5(h, k) = \begin{cases} -10s(h, k) + 4s(2h, k) + 4s(h, 2k), & \text{if } h + k \text{ is even;} \\ 0, & \text{if } h + k \text{ is odd}. \end{cases} \]

For \( s_1(h, k) \), the author [14] proved the following:

**Proposition 1.4.** For any fixed integer \( m \geq 2 \) and any sufficiently large odd number \( k \), we have the asymptotic formula

\[ \sum_{h=1 \atop (h,k)=1 \atop 2|h}^{k} s_1^{2m}(h, k) = g_m(k) \left( \frac{k}{2} \right)^{2m} + O \left( k^{2m-1} \right), \]

where \( g_m(k) \) is defined by the Dirichlet series

\[ \sum_{k=1 \atop 2|k}^{+\infty} g_m(k) \frac{1}{k^s} = \left( \frac{s+4m-2}{2^s-1} \right) \frac{\zeta^2(2m)}{\zeta(4m)} \sum_{p \text{ prime}} \alpha^\parallel \frac{1}{p^{2m+1}} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s). \]

While X. Chen and W. Zhang [7] got the following:

**Proposition 1.5.** For any odd number \( k > 2 \), we have

\[ \sum_{h=1 \atop (h,k)=1 \atop 2|h}^{k} s_1^2(h, k) = \frac{1}{8} k\phi(k) \prod_{p \text{ prime}} \left[ \frac{(1 + \frac{1}{p})^2 - \frac{1}{p^{2m+1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} \right] + O \left( k \exp \left( \frac{4 \log k}{\log \log k} \right) \right). \]

In Section 3 we shall prove the following:

**Theorem 1.2.** For any given integer \( m \geq 2 \) and any sufficiently large odd number \( k \), we have the identity

\[ g_m(k) = \frac{\zeta^2(2m)}{2^{2m+1} \zeta(4m)} \prod_{p \text{ prime}} \left[ \frac{\left( 1 - \frac{1}{p^{2m}} \right)^2 - \frac{1}{p^{2m+1}} + \frac{1}{p^{2m+2}}}{\left( 1 - \frac{1}{p^{2m+1}} \right)^2} \right]. \]
Let $m = 1$, we immediately get
\[
g_1(k) = \frac{1}{2} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^2 - \frac{1}{p^{m+1}} \left(1 - \frac{1}{p}\right)^2 \frac{1}{1 - \frac{1}{p}}\phi(k) \prod_{p \mid k} \left(1 + \frac{1}{p} - \frac{1}{p^{m+1}}\right).
\]

Then from Propositions 1.4, 1.5, and Theorem 1.2 we have

**Corollary 1.2.** Suppose that $m$ is a given positive integer and $k$ is any sufficiently large odd number. Then
\[
\sum_{\substack{h=1 \\
(h,k)=1}}^{k} s_1^{2m}(h,k)
= \frac{\zeta(2m)}{\zeta(4m)} \left(\frac{k}{2}\right)^{2m} \prod_{p \mid k} \left(1 - \frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{m+1}} \left(1 - \frac{1}{p^{m-1}}\right)^2 \frac{1}{1 - \frac{1}{p^{m-1}}} + \epsilon_m'(k),
\]

where
\[
\epsilon_m'(k) \ll \begin{cases} 
    k \exp \left(\frac{4 \log k}{\log \log k}\right), & \text{if } m = 1; \\
    k^{2m-1}, & \text{if } m > 1.
\end{cases}
\]

In [22], W. Zhang obtained the following:

**Proposition 1.6.** Let $k = 2^\beta M$ be an integer with $\beta \geq 1$ and $2 \nmid M$. Then we have
\[
\sum_{\substack{h=1 \\
(h,k)=1}}^{k} s_2^{2m}(h,k) = \frac{5}{112} k \phi(k) \left(\frac{3}{8} - \frac{2}{2^{2m}}\right) \prod_{p \mid M} \left(1 + \frac{1}{p} - \frac{1}{p^{m+1}}\right) + O \left(k \exp \left(\frac{4 \log k}{\log \log k}\right)\right).
\]

We shall study the $2m$-th power mean of $s_2(h,k)$ in Section 4, and prove the following:

**Theorem 1.3.** For any fixed integer $m \geq 2$ and any sufficiently large even number $k$, we have the asymptotic formula
\[
\sum_{\substack{h=1 \\
(h,k)=1}}^{k} s_2^{2m}(h,k) = h_m(k) \left(\frac{k}{4}\right)^{2m} + O \left(k^{2m-1}\right),
\]

where
\[
h_m(k) = \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\
(a,q)=1, 2\mid h}}^{q} \sum_{\substack{h=-\infty \\
(h,k)=1}}^{+\infty} \frac{1}{(qh - ak)^{2m}}.
\]
In Section 5 we shall further show the following:

**Theorem 1.4.** Let \( k = 2^3 M \) be an integer with \( \beta \geq 1 \) and \( 2 \nmid M \). Then for any integer \( m \geq 2 \), we have

\[
\begin{align*}
\sum_{m=1 \atop (h,k)=1}^{k} s_2^{2m}(h,k) & = \frac{2^{2m} \zeta(2m) (2^{2m} - 1) \zeta(4m)}{(2^{2m} + 1) \zeta(4m)} \left( \sum_{m=1}^{2m} \frac{1}{p^m} \right) - \frac{2^{6m-2} + 1 - 2^{4m-1} - 2^{4m-2}}{(2^{4m-1} - 1) (2^{4m-1} - 1) (2^{4m-1} - 1) \beta + 2m - 1} \\
& \times \prod_{\rho^\omega \mid M} \left[ \left( 1 - \frac{1}{\rho^{2m}} \right)^2 - \frac{1}{\rho^{(4m-1)\beta + 2m - 1}} \right] + e''_m(k),
\end{align*}
\]

where

\[
e''_m(k) \ll \begin{cases} 
  k \exp \left( \frac{4 \log k}{\log \log k} \right), & \text{if } m = 1; \\
  \frac{k^2}{2m-1}, & \text{if } m > 1.
\end{cases}
\]

Then from Proposition 1.6, Theorem 1.3 and Theorem 1.4 we have

**Corollary 1.3.** Suppose that \( m \) is a given positive integer and \( k = 2^3 M \) is any sufficiently large even number with \( \beta \geq 1 \) and \( 2 \nmid M \). Then
On the other hand, from Proposition 1.4 and Theorem 1.2 we have
\[
\sum_{k=1}^{+\infty} \frac{1}{k^s} \prod_{p^m||k} \left[ \frac{(1 - \frac{1}{p^{2m}})^2 - \frac{1}{p^{2m-1}\mu + 1} \left(1 - \frac{1}{p^{2m-1}\mu + 1}\right)^2}{1 - \frac{1}{p^{2m-1}\mu + 1}} \right]
\]
\[
= \frac{(2^{s+4m} - 2)(2^s - 1)}{(2^{s+2m} - 1)^2} \cdot \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \cdot \zeta(s).
\]

Therefore
\[
\sum_{k=1}^{+\infty} \frac{h_m(k)}{k^s} = \sum_{\beta=1}^{+\infty} \sum_{k=1}^{2^\beta || k} \frac{h_m(k)}{k^s} = \sum_{\beta=1}^{+\infty} \sum_{M=1}^{2^\beta || M} \frac{h_m(2^\beta M)}{(2^\beta M)^s}
\]
\[
= \frac{2^{2m} \zeta^2(2m)}{(2^{2m} + 1) \zeta(4m)} \sum_{\beta=1}^{+\infty} \frac{1}{2^{3\beta s}} \left( \frac{2^{2m} - 1}{2^{2m}(2^{2m-1} - 1)} - \frac{2^{6m-2} + 1 - 2^{4m-1} - 2^{4m-2}}{(2^{4m-1} - 1)2^{4m-1}2^{4m-2}} \right)
\]
\[
\times \sum_{M=1}^{+\infty} \frac{1}{M^s} \prod_{p^m||M} \left[ \frac{(1 - \frac{1}{p^{2m}})^2 - \frac{1}{p^{2m-1}\mu + 1} \left(1 - \frac{1}{p^{2m-1}\mu + 1}\right)^2}{1 - \frac{1}{p^{2m-1}\mu + 1}} \right]
\]
\[
= \frac{2^{s+2} + 2^{2m+1} - 6}{(2^{s+2m} - 1)^2(2^{2m} + 1)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \cdot \zeta(s).
\]

Using the same methods, one can get similar interesting results for other Hardy sums.

§ 2. Proof of Theorem 1.1

First we prove the following lemma.

Lemma 2.1. For any positive integers $k$ and $m$, we have
\[
\Lambda_1 := \sum_{d|k} \mu(d) q^{d_2} \sum_{g|k} \frac{1}{g^{2m-1}} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \left( \frac{1 - \frac{1}{p^{2m}}}{1 - \frac{1}{p^{2m-1}\mu + 1}} \right)^2
\]
\[
= \zeta(2m) \prod_{p^m||k} \left[ \frac{(1 - \frac{1}{p^{2m}})^2 - \frac{1}{p^{2m-1}\mu + 1} \left(1 - \frac{1}{p^{2m-1}\mu + 1}\right)^2}{1 - \frac{1}{p^{2m-1}\mu + 1}} \right].
\]

Proof. By use of the Möbius relation
\[
\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}
\]
we have

\[
\sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{\left(\frac{q}{g}\right)\equiv 1 (q,g)=1} g^{2m} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{t|q} \mu(t) = \sum_{t|q} \frac{1}{g^{2m}} \sum_{t|q} \mu(t) \frac{\zeta(2m)}{t^{2m}} \sum_{t|q} \mu(t) \frac{1}{t^{2m}}.
\]

This gives

(2.2) \[ \Lambda_1 = \zeta(2m) \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{k}{d}} \frac{1}{g^{4m-1}} \sum_{t|\frac{k}{g}} \frac{\mu(t)}{t^{2m}}. \]

On the other hand, from the properties of multiplicative functions we have

(2.3)

\[
\sum_{d|k} \mu(d) d^{2m} \sum_{g|\frac{k}{d}} \frac{1}{g^{4m-1}} \sum_{t|\frac{k}{g}} \frac{\mu(t)}{t^{2m}} = \prod_{p^n \| k} \left[ \sum_{d|k} \mu(d) d^{2m} \sum_{g|\frac{k}{d}} \frac{1}{g^{4m-1}} \sum_{t|\frac{k}{g}} \frac{\mu(t)}{t^{2m}} \right]
\]

\[
= \prod_{p^n \| k} \left[ \sum_{d|k} \mu(d) d^{2m} \sum_{g|\frac{k}{d}} \frac{1}{g^{4m-1}} \sum_{t|\frac{k}{g}} \frac{\mu(t)}{t^{2m}} \right]
\]

\[
= \prod_{p^n \| k} \left[ \frac{1 - 1}{p^{4m-1}} - \frac{1}{p^{4m-1}} \left(1 - \frac{1}{p^{4m-1}}\right) \right]
\]

Then from (2.2) and (2.3) we have

\[
\Lambda_1 = \zeta(2m) \prod_{p^n \| k} \left[ \frac{1 - 1}{p^{4m-1}} - \frac{1}{p^{4m-1}} \left(1 - \frac{1}{p^{4m-1}}\right) \right].
\]

\[
\square
\]

Now we prove Theorem 1.1. From [8] or [13] we know that

\[
f_m(k) = \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{+\infty} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh-ak)^{2m}}.
\]

\[
a \equiv 1 (a,q) = 1 \begin{cases} h \equiv 1 (h,k) = 1 \end{cases} qh-ak \neq 0
\]
By (2.1) we get

\[
\begin{align*}
    f_m(k) &= \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{t|q \atop a\mid t} \mu(t) \sum_{h=-\infty \atop qh-ak \neq 0}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{d|h \atop d|k} \mu(d) \\
    &= \sum_{t=1}^{+\infty} \mu(t) \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty \atop qh-ak \neq 0}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{d|h \atop d|k} \mu(d) \\
    &= \frac{1}{\zeta(4m)} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty \atop qh-ak \neq 0}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{d|h \atop d|k} \mu(d).
\end{align*}
\]

(2.4)

Let \( g = (q, \frac{k}{d}) \). Then

\[
\begin{align*}
    &= \frac{1}{g^{2m}} \sum_{a=1}^{+\infty} \sum_{h=-\infty \atop qh-ak \neq 0}^{+\infty} \frac{1}{(qh-ak)^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{+\infty} \sum_{z=-\infty \atop z \neq 0}^{+\infty} \frac{1}{z^{2m}} = \frac{2\zeta(2m)}{g^{2m-1}}.
\end{align*}
\]

(2.5)

Now from (2.4), (2.5), and Lemma 2.1 we have

\[
\begin{align*}
    f_m(k) &= \frac{2\zeta(2m)}{\zeta(4m)} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{+\infty} \frac{1}{q^{2m-a}} \sum_{\substack{z=-\infty \atop z \neq 0}}^{+\infty} \frac{1}{z^{2m}} \\
    &= \frac{2\zeta^2(2m)}{\zeta(4m)} \prod_{p | k} \left[ \frac{1 - \frac{1}{p^{2m}}} {1 - \frac{1}{p^{2m-1}}} \right] \left[ \frac{1 - \frac{1}{p^{2m-a}}} {1 - \frac{1}{p^{2m-a+1}}} \right] \left[ \frac{1}{1 - \frac{1}{p^{2m-a+2}}} \right].
\end{align*}
\]

This proves Theorem 1.1.

§ 3. Proof of Theorem 1.2

First we have
Lemma 3.1. For any integer \( m \geq 1 \) and positive odd number \( k \), we have

\[
\Lambda_2 := \sum_{d \mid k} \frac{\mu(d)}{d^{2m}} \sum_{g \mid k} \frac{1}{g^{4m-1}} \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \\
= \frac{(2^{2m} - 1)}{2^{2m}} \cdot \zeta(2m) \prod_{p \mid k} \left[ \left( 1 - \frac{1}{p^{2m}} \right)^2 - \frac{1}{p^{4m-1} - 1} \left( 1 - \frac{1}{p^{2m-1}} \right)^2 \right].
\]

Proof. By (2.1) we get

\[
\sum_{q=1}^{\infty} \frac{1}{q^{2m}} = \frac{1}{g^{2m}} \sum_{q=1}^{\infty} \frac{1}{q^{2m}} = \frac{1}{g^{2m}} \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{t \mid q} \mu(t) \\
= \frac{1}{g^{2m}} \sum_{t \mid 2^{2m}} \sum_{q=1}^{\infty} \frac{1}{q^{2m}} = \frac{(2^{2m} - 1)}{2^{2m}} \cdot \zeta(2m) \cdot \frac{1}{g^{2m}} \sum_{t \mid 2^{2m}} \mu(t).
\]

Then from (2.3) we have

\[
\Lambda_2 = \frac{(2^{2m} - 1)}{2^{2m}} \cdot \zeta(2m) \cdot \sum_{d \mid k} \frac{\mu(d)}{d^{2m}} \sum_{g \mid k} \frac{1}{g^{4m-1}} \sum_{t \mid g^{2m}} \mu(t) \\
= \frac{(2^{2m} - 1)}{2^{2m}} \cdot \zeta(2m) \prod_{p \mid k} \left[ \left( 1 - \frac{1}{p^{2m}} \right)^2 - \frac{1}{p^{4m-1} - 1} \left( 1 - \frac{1}{p^{2m-1}} \right)^2 \right].
\]

Now we prove Theorem 1.2. From [14] we know that

\[
g_m(k) = \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{h_{a}=-\infty} \frac{1}{(gh-ak)^{2m}} + \sum_{h=1}^{\infty} \frac{1}{h^{2m}} \\
\quad := g'_m(k) + g''_m(k).
\]
Noting that $k$ is odd. It is easy to show that

$$g''_m(k) = \sum_{h=1}^{+\infty} \frac{1}{h^{2m}} = \sum_{h=1}^{+\infty} \frac{1}{h^{2m}} \sum_{d|h} \mu(d)$$

(3.2)

$$= \sum_{d|k} \mu(d) \sum_{h=1}^{+\infty} \frac{1}{h^{2m}} \sum_{d|h} \mu(d) = \zeta(2m) \sum_{d|k} \mu(d).$$

On the other hand, by (2.1) we get

$$g'_m(k) = \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{(a,q)=1} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh-ak)^{2m}}$$

$$= \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{t|q} \frac{1}{t^{2m}} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{d|h} \mu(d)$$

(3.3)

$$= \sum_{t=1}^{+\infty} \frac{1}{t^{4m}} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{(a,q)=1} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{d|h} \mu(d)$$

$$= \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{(a,q)=1} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh-ak)^{2m}}.$$

Let $g = (q, \frac{k}{2})$. Then

$$\sum_{a=1}^{q} \sum_{h=-\infty}^{+\infty} \frac{1}{(qh - ak)^{2m}} = \sum_{a=1}^{q} \sum_{h=-\infty}^{+\infty} \frac{1}{(g - \frac{ak}{g})^{2m}}$$

(3.4)

$$= \frac{1}{2^{2m} g^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{+\infty} \frac{1}{(g - \frac{ak}{g})^{2m}}.$$
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\[= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} \sum_{z \equiv -\frac{a}{g} \pmod{\frac{q}{g}}} \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{z=1}^{-1} \frac{1}{z^{2m}} \]

\[= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{q-1} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} \sum_{z \equiv -\frac{a}{g} \pmod{\frac{q}{g}}} \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{+\infty} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} \]

\[= \frac{1}{2^{2m}g^{2m-1}} \sum_{z=1}^{+\infty} \frac{1}{2^{2m}g^{2m}} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} = \frac{\zeta(2m)}{2^{2m}g^{2m-1}} - \frac{\zeta(2m)}{2^{2m}q^{2m}}.\]

Now from (3.3) and (3.4) we have

\[g'_{m}(k) = \frac{2^{2m}}{(2^{4m} - 1)} \cdot \frac{\zeta(2m)}{(4m)} \sum_{d|k} \mu(d) \sum_{g|x} \frac{1}{g^{2m-1}} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \]

\[= \frac{2^{2m}}{(2^{4m} - 1)} \cdot \frac{\zeta(2m)}{(4m)} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{4m}}.\]

Noting that

\[= \frac{2^{2m}}{(2^{4m} - 1)} \cdot \frac{\zeta(2m)}{(4m)} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} = \frac{\zeta(2m)}{2^{2m}} \sum_{d|k} \mu(d) \sum_{q=1}^{+\infty} \frac{1}{q^{4m}}.\]

then combining (3.1), (3.2), (3.5), (3.6), and Lemma 3.1 we have

\[g_{m}(k) = \frac{\zeta^2(2m)}{(2^{2m} + 1) \zeta(4m)} \prod_{p^r|k} \left[ \left( 1 - \frac{1}{p^{2m}} \right)^2 - \frac{1}{p^{4m+1}} \left( 1 - \frac{1}{p^{4m+1}} \right)^2 \right].\]

This completes the proof of Theorem 1.2.
§ 4. Proof of Theorem 1.3

We need the following lemmas.

Lemma 4.1. For any given positive integer $k$ and any integer $h$ with $(h, k) = 1$ and any $P > 1$, there exist a positive integer $q \leq P$ and an integer $a$ with $(a, q) = 1$ such that

$$\left| \frac{h}{k} - \frac{a}{q} \right| < \frac{1}{qP}.$$ 

Proof. This is a well-known result; See Theorem 36 of [12]. □

Lemma 4.2. Let $a, b, c, d, h$ and $k$ be positive integers with $ad - bc = 1$ and $(h, k) = 1$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Then we have

$$s(a, c) + s(h, k) - s(x, y) = \frac{c^2 + k^2 + y^2}{12cky} - \frac{1}{4}.$$ 

Proof. This is equation (26) of [11]. □

Lemma 4.3. Let $k$ be a positive even number, $h$ be an odd number with $(k, h) = 1$. Then

$$s_2(h, k) = 2s\left(\frac{h}{2}, \frac{k}{2}\right) - s(h, k).$$ 

Proof. See [18] or [15]. □

Lemma 4.4. For any positive integer $q$, we have

$$\sum_{\substack{a=1 \\ (a, q)=1}}^{q} |s(a, q)| \ll q \log^2 q.$$ 

Proof. This is Lemma 6 of [8]. □

Lemma 4.5. Let $k$ be a positive even number, $a$, $q$ and $h$ be positive integers with $(h, k) = (a, q) = 1$, $z = qh - ak$. If $1 \leq |z| \leq \frac{k}{q}$, then

$$s_2(h, k) = \begin{cases} \frac{k}{4q^2} + O\left(\left|s\left(a, \frac{q}{2}\right)\right| + |s(a, q)| + |z|\right), & \text{if } q \text{ is an even number;} \\ O\left(q + |z|\right), & \text{if } q \text{ is an odd number.} \end{cases}$$ 

Proof. Suppose that $q$ is even. Since $(a, q) = 1$, $a$ must be odd number.

First we consider the case that $z < 0$. Since $(a, q) = 1$, there exist positive integers $b$ and $d$ such that

$$ad - bq = 1, \quad 1 \leq d < q.$$
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Since \( q \) is even, \( d \) must be odd. Let \( f = dh - bk \). Then we have
\[
\begin{pmatrix}
d & -b \\
-q & a
\end{pmatrix}
\begin{pmatrix}
h \\
k
\end{pmatrix}
= \begin{pmatrix}
f \\
-z
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
a & 2b \\
\frac{q}{2} & d
\end{pmatrix}
\begin{pmatrix}
f \\
-\frac{z}{2}
\end{pmatrix}
= \begin{pmatrix}
h \\
\frac{k}{2}
\end{pmatrix}
.
\]
The fact that \( d < q \) and \( z \geq -\frac{k}{q} \) yields
\[
f = kd \left( \frac{h}{k} - \frac{b}{d} \right) = kd \left( \frac{h}{k} - \frac{a}{q} + \frac{1}{qd} \right) = kd \left( \frac{z}{qk} + \frac{1}{qd} \right) = \frac{kd}{q} \left( \frac{z}{k} + \frac{1}{d} \right) > 0.
\]
On the other hand, since \((h, k) = 1\) and \( f \) is odd, we get \((f, -z) = 1\). Then by Lemma 4.2,
\[
l(h, \frac{k}{2}) = \frac{k}{6qz} + O \left( \left| s \left( a, \frac{q}{2} \right) \right| + |z| \right).
\]
That is,
\[
l(h, \frac{k}{2}) = \frac{k}{12qz} + O (|s(a, q)| + |z|).
\]
From Lemma 8 of [8] we also have
\[
s(h, k) = \frac{k}{12qz} + O (|s(a, q)| + |z|).
\]
Therefore by Lemma 4.3 we immediately get
\[
s_2(h, k) = 2s \left( h, k, \frac{k}{2} \right) - s(h, k) = \frac{k}{4qz} + O \left( \left| s \left( a, \frac{q}{2} \right) \right| + |s(a, q)| + |z| \right) \text{ if } z < 0.
\]
For \( z > 0 \), we can find positive integers \( b \) and \( d \) satisfying
\[
ad - bq = -1, \quad 1 \leq d < q.
\]
Since \( q \) is even, \( d \) must be odd. Let \( f = bk - dh \). Then we have
\[
\begin{pmatrix}
b & -d \\
-a & q
\end{pmatrix}
\begin{pmatrix}
h \\
k
\end{pmatrix}
= \begin{pmatrix}
f \\
-k
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\frac{q}{2} & d \\
a & 2b
\end{pmatrix}
\begin{pmatrix}
f \\
\frac{z}{2}
\end{pmatrix}
= \begin{pmatrix}
k \\
\frac{h}{2}
\end{pmatrix}
.
\]
Similarly we can get \((f, z) = 1\) and
\[
f = kd \left( \frac{b}{d} - \frac{h}{k} \right) = kd \left( \frac{1}{q} + \frac{a}{q} - \frac{h}{k} \right) = kd \left( \frac{1}{q} - \frac{z}{qk} \right) = \frac{kd}{q} \left( \frac{1}{d} - \frac{z}{k} \right) > 0.
\]
Then by Lemma 4.2,
\[
l \left( \frac{q}{2}, a \right) + s \left( f, \frac{z}{2} \right) - s \left( \frac{k}{2}, h \right) = \frac{a^2 + \frac{z^2}{4} + h^2}{6azh} - \frac{1}{4}.
\]
Noting that (see (1.1))
\[ s\left(\frac{q}{2}, a\right) + s\left(a, \frac{q}{2}\right) = \frac{q^2}{4} + a^2 + 1 - \frac{1}{4} \]
and
\[ s\left(h, \frac{k}{2}\right) + s\left(h, \frac{k}{2}\right) = \frac{k^2}{4} + h^2 + 1 - \frac{1}{4} \]
we have
\[ s\left(h, \frac{k}{2}\right) = \frac{k}{6q} + O\left(\left|s\left(a, \frac{q}{2}\right)\right| + |z|\right). \]
So from Lemma 8 of [8] and Lemma 4.3 we immediately get
\[ s_2(h, k) = 2s\left(h, \frac{k}{2}\right) - s(h, k) = \frac{k}{4q} + O\left(\left|s\left(a, \frac{q}{2}\right)\right| + |s(a, q)| + |z|\right) \]
for \( z > 0 \).
This proves that
\[ s_2(h, k) \approx q + |z|, \] if \( q \) is an even number.

On the other hand, if \( q \) is an odd number, using similar methods we can get
\[ s_2(h, k) \ll q + |z|. \]
□

Now we prove Theorem 1.3. We suppose that \( m \geq 2, \) a sufficiently large even number \( k \) are given and we set
\[ Q = \left\lfloor k^{1/2} \right\rfloor, \quad P = 2Q. \]
Let
\[ I(a, q) = \left(\frac{a}{q} - \frac{1}{qP}, \frac{a}{q} + \frac{1}{qP}\right). \]
When \( \frac{a_1}{q_1} \neq \frac{a_2}{q_2} \) and \( q_1, q_2 \leq Q \), one has
\[ \left|\frac{a_1}{q_1} - \frac{a_2}{q_2}\right| \geq \frac{1}{q_1q_2} \geq \left(\frac{1}{q_1P} + \frac{1}{q_2P}\right). \]
Thus the intervals \( I(a, q) \) are pairwise disjoint.

If \( 1 \leq h \leq k \) and \( (h, k) = 1 \), then by Lemma 4.1, \( \frac{h}{k} \) falls into an interval \( I(a, q) \) with \( 1 \leq q \leq P, 0 \leq a \leq q \) and \( (a, q) = 1 \).
Let \( z = qh - ak \). It is easy to see that \( z \neq 0 \) and
\[ |z| = qk \left|\frac{h}{k} - \frac{a}{q}\right| \leq \frac{k}{P} \leq \frac{k}{q}. \]
If \( \frac{h}{k} \) falls into an interval \( I(a, q) \) with \( 1 \leq q \leq P, q \) is an odd number, \( 0 \leq a \leq q \) and \( (a, q) = 1 \), by Lemma 4.5, we have
\[ s_2(h, k) = O(q + |z|) \ll P + \frac{k}{P} \ll k^{1/2}. \]
Thus,
\[ \sum^* s_2^{2m}(h, k) \ll k^{m+1} \ll k^{2m-1}, \]
where the asterisk indicates summation over those integers \( h, 1 \leq h \leq k, \) \((h, k) = 1\) for which \( \frac{h}{k} \) falls into an interval \( I(a, q)\) with \( 1 \leq q \leq P, \) \( q \) is an odd number, \( 0 \leq a \leq q \) and \((a, q) = 1.\)

If \( \frac{h}{k} \) falls into an interval \( I(a, q)\) with \( Q \leq q \leq P, \) \( q \) is an even number, \( 1 \leq a \leq q \) and \((a, q) = 1,\) by Lemma 4.5, we have
\[
s_2(h, k) = \frac{k}{4qz} + O \left( \left| s \left( a, \frac{q}{2} \right) \right| + |s(a, q)| + |z| \right) \ll \frac{k}{q} + \frac{k}{P} \ll \frac{k}{Q} + \frac{k}{P} \ll k^2.
\]

Thus,
\[ \sum^* s_2^{2m}(h, k) \ll k^{m+1} \ll k^{2m-1}, \]
where the asterisk indicates summation over those integers \( h, 1 \leq h \leq k, \) \((h, k) = 1\) for which \( \frac{h}{k} \) falls into an interval \( I(a, q)\) with \( Q \leq q \leq P, \) \( q \) is an even number, \( 1 \leq a \leq q \) and \((a, q) = 1.\)

Therefore
\[ \sum_{h=1}^{k} s_2^{2m}(h, k) = \sum_{q=1}^{Q} \sum_{a=1}^{q \atop 2|q \atop (a, q) = 1} \sum^* s_2^{2m}(h, k) + O \left( k^{2m-1} \right), \]
where the asterisk means that \( 1 \leq h \leq k, \) \((h, k) = 1.\)

Lemma 4.5 produces
\[ s_2(h, k) = \frac{k}{4qz} + O \left( \left| s \left( a, \frac{q}{2} \right) \right| + |s(a, q)| + |z| \right). \]

Using the estimate
\[(A + B + C)^{2m} = A^{2m} + O \left( |A|^{2m-1} (|B| + |C|) \right) + O \left( B^{2m} + C^{2m} \right),\]
we obtain
\[ s_2^{2m}(h, k) = \left( \frac{k}{4qz} \right)^{2m} + O \left( \left( \frac{k}{q |z|} \right)^{2m-1} \left( \left| s \left( a, \frac{q}{2} \right) \right| + |s(a, q)| + |z| \right) \right)
+ O \left( \left( \left| s \left( a, \frac{q}{2} \right) \right| + |s(a, q)| \right)^{2m} + z^{2m} \right). \]

Therefore
\[ \sum_{q=1}^{Q} \sum_{a=1}^{q \atop 2|q \atop (a, q) = 1} \sum^* s_2^{2m}(h, k) = \Omega_1 + O(\Omega_2) + O(\Omega_3), \]
where

\[(4.5) \quad \Omega_1 = \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \left( \frac{k}{4q^2} \right)^{2m}, \]

\[(4.6) \quad \Omega_2 = \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \left( \frac{k}{q^2} \right)^{2m-1} \left( \left| s\left( a, \frac{q}{2} \right) \right| + \left| s(a, q) \right| + |z| \right), \]

\[(4.7) \quad \Omega_3 = \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \left( \left( \left| s\left( a, \frac{q}{2} \right) \right| + \left| s(a, q) \right| \right)^2 + z^{2m} \right). \]

Note that for the fixed \(a, q, k\) and \(z\), the equation \(z = qh - ak\) has at most one solution \(h\). By Lemma 4.4, we have

\[(4.8) \quad \Omega_2 \ll k^{2m-1} \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \frac{1}{q^{2m-1}} \cdot \frac{1}{z^{2m-2}} \left( \left| s\left( a, \frac{q}{2} \right) \right| + \left| s(a, q) \right| + 1 \right) \]

\[\ll k^{2m-1} \sum_{q=1}^{Q} \frac{1}{q^{2m-1}} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \frac{1}{z^{2m-2}} \left| s\left( a, \frac{q}{2} \right) \right| + \left| s(a, q) \right| + 1 \sum_{z \neq 0}^{\ast} \frac{1}{z^2} \]

\[\ll k^{2m-1} \sum_{q=1}^{Q} \frac{1}{q^{2m-1}} \cdot q \cdot \log^2(q + 1) \ll k^{2m-1} \sum_{q=1}^{Q} \frac{\log^2(q + 1)}{q^2} \ll k^{2m-1}. \]

Moreover,

\[(4.9) \quad \Omega_3 \ll \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \left( q^{2m} + \left( \frac{k}{q} \right)^{2m} \right) \ll k^{m} \sum_{h=1}^{k} \frac{1}{(h,k)} \ll k^{m+1} \ll k^{2m-1}. \]

Combining (4.3)-(4.9), we obtain

\[(4.10) \quad \sum_{h=1}^{k} s_2^{2m}(h, k) = \Omega_1 + O(k^{2m-1}), \]

where

\[\Omega_1 = \left( \frac{k}{4} \right)^{2m} \sum_{q=1}^{Q} \sum_{a=1}^{q} \sum_{h/k \in I(a,q)}^{\ast} \frac{1}{z^{2m}}. \]
It remains to obtain an asymptotic formula for $\Omega_1$. Note that if $1 \leq h \leq k$, then $\frac{h}{k} \notin I(a, q)$ if and only if $|z| \geq \frac{k}{P}$. Hence

$$
\left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h/k \in I(a, q)} \frac{1}{z^{2m}} \leq \left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{q_1^{2m}} \sum_{a=1}^{q} \sum_{|z| \geq k/P} \frac{1}{z^{2m}} \ll k^{2m} \left(\frac{P}{k}\right)^{2m-1} \sum_{q=1}^{Q} \frac{1}{q^{2m-1}} \ll k^{2m-1}.
$$

Thus

$$
\Omega_1 = \left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=1}^{k} \frac{1}{(qh - ak)^{2m}} + O(k^{2m-1}).
$$

Using the estimate

$$
\sum_{h \geq k+1} \frac{1}{(qh - ak)^{2m}} \leq \int_{k}^{+\infty} \frac{dx}{(qx - ak)^{2m}} = \int_{(q-\alpha)k}^{+\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{qk^{2m-1}},
$$

we get

$$
\left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h \geq k+1} \frac{1}{(qh - ak)^{2m}} \ll k^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{qk^{2m-1}} \ll k.
$$

Since

$$
\sum_{h \leq 0} \frac{1}{(qh - ak)^{2m}} \leq \frac{1}{k^{2m}} + \sum_{r \geq 1} \frac{1}{(qr + ak)^{2m}} \leq \frac{1}{k^{2m}} + \int_{0}^{+\infty} \frac{dx}{(qx + ak)^{2m}} = \frac{1}{k^{2m}} + \int_{ak}^{+\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{k^{2m}} + \frac{1}{qk^{2m-1}} \ll \frac{1}{k^{2m-1}},
$$

we have

$$
\left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h \leq 0} \frac{1}{(qh - ak)^{2m}} \ll k^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{k^{2m-1}} \ll k.
$$

Therefore

$$
\Omega_1 = \left(\frac{k}{4}\right)^{2m} \sum_{q=1}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=0}^{+\infty} \frac{1}{(qh - ak)^{2m}} + O(k^{2m-1}).
$$
Since
\[
\left( \frac{k}{4} \right)^{2m} \sum_{q > Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{\infty} \frac{1}{(qh-ak)^{2m}} \ll k^{2m} \sum_{q > Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{z=-\infty}^{\infty} \frac{1}{z^{2m}} \ll k^{2m} \sum_{q > Q} \frac{1}{q^{2m-1}} \ll k^{m+1} \ll k^{2m-1},
\]
we have
\[
\Omega_1 = \left( \frac{k}{4} \right)^{2m} \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{\infty} \frac{1}{(h,k)=1} \frac{1}{(qh-ak)^{2m}} + O(k^{2m-1}).
\]

Therefore by (4.10) and (4.11)
\[
\sum_{h=1}^{k} s^{2m}_2(h,k) = h_m(k) \left( \frac{k}{4} \right)^{2m} + O(k^{2m-1}),
\]
where
\[
h_m(k) = \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{\infty} \frac{1}{(h,k)=1} \frac{1}{(qh-ak)^{2m}}
\]
\[
= \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{\infty} \frac{1}{(h,k)=1} \frac{1}{(2h, qh-ak)^{2m}}.
\]

This proves Theorem 1.3.

§ 5. Proof of Theorem 1.4

Let \( k = 2^\beta \) be an integer with \( \beta \geq 1 \) and \( 2 \nmid M \). By (2.1) we get
\[
h_m(k) = \sum_{q=1}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{q} \sum_{h=-\infty}^{\infty} \frac{1}{(h,k)=1} \frac{1}{(2h, qh-ak)^{2m}}
\]
Then we have (5.1)

\[ h_m(k) = \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\beta)^{2m}} \]

\[ \sum_{d|h} \mu(d) \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\beta)^{2m}} \]

\[ + \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\frac{\beta}{q})^{2m}} \]

\[ + \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\frac{\beta}{q})^{2m}} \]

\[ = \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\beta)^{2m}} \]

\[ + \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(qh - a\frac{\beta}{q})^{2m}} \]

\[ + \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{\gamma \in \mathbb{M}} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{\beta+\infty} \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{1}^{\infty} \sum_{h=-\infty}^{h=\infty} \frac{1}{(2^q qh - a\frac{\beta}{q})^{2m}} \]

\[ := \Psi_1 + \Psi_2 + \Psi_3. \]
First we consider $\Psi_1$. Let $g = (q, M/d)$. Then

$$
\sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{(qh - ak)^{2m}} = \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{g^{2m}} \sum_{a=1}^{\infty} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{(qh - ak)^{2m}}
$$

$$
= \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}}
$$

$$
= \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}}
$$

$$
= \frac{2^\gamma - 1}{2^{2m-1}} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}} = \frac{(2^2m - 1)}{2^{2m-1}} \cdot \frac{2^\gamma}{g^{2m-1}} \cdot \zeta(2m).
$$

Therefore

$$
(5.2) \quad \Psi_1 = \frac{2^{2m}}{(2^2m + 1)} \cdot \zeta(2m) \sum_{\gamma=1}^{3-1} \frac{1}{2^{(2m-1)}} \sum_{d|M} \mu(d) \sum_{d|M} \mu(d) \sum_{\substack{d | \frac{d}{d}}}^{+\infty} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2q | g}}^{+\infty} \frac{1}{q^{2m}}.
$$

Now we consider $\Psi_2$. Let $g = (q, M/d)$. Then

$$
\sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{(qh - ak)^{2m}} = \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{g^{2m}} \sum_{a=1}^{\infty} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{(qh - ak)^{2m}}
$$

$$
= \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2h | \gamma h - ak \neq 0}}^{+\infty} \frac{1}{z^{2m}}
$$
\[ \Psi_2 = \frac{2^{2m}}{(2^{2m+1}-1)\zeta(4m)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \frac{1}{2^{2m-1}} \sum_{d|N} \frac{\mu(d)}{d} \sum_{\gamma=1}^{+\infty} \frac{1}{q_{2m}^{\gamma}} \sum_{a=1}^{+\infty} \frac{1}{q_{2m}^{\gamma(a)}}, \quad 2|q, 2|h \neq \phi \neq 0 \]

Therefore

(5.3) \quad \Psi_2 = \frac{2^{2m}}{(2^{2m+1}-1)\zeta(4m)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \frac{1}{2^{2m-1}} \sum_{d|N} \frac{\mu(d)}{d} \sum_{\gamma=1}^{+\infty} \frac{1}{q_{2m}^{\gamma}} \sum_{a=1}^{+\infty} \frac{1}{q_{2m}^{\gamma(a)}}, \quad 2|q, 2|h \neq \phi \neq 0 \]

For \( \Psi_3 \), we have

\[ \Psi_3 = \frac{2^{2m}}{(2^{2m+1}-1)\zeta(4m)} \sum_{d|N} \mu(d) d^{2m} \sum_{\gamma=1}^{+\infty} \frac{1}{q_{2m}^{\gamma}} \sum_{a=1}^{+\infty} \frac{1}{q_{2m}^{\gamma(a)}}, \quad 2|q, 2|h \neq \phi \neq 0 \]
\[
- \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|M} \mu(d) \sum_{\gamma=\beta+1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
+ \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|M} \mu(d) \sum_{\gamma=\beta+1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
= \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|M} \mu(d) \sum_{\gamma=\beta+1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
- \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|M} \mu(d) \sum_{\gamma=\beta+1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
- \frac{2^{4m}}{(2^{4m} - 1) \zeta(4m)} \sum_{d|M} \mu(d) \sum_{\gamma=\beta+1}^{+\infty} \sum_{q=1}^{+\infty} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
= \Omega_1 - \Omega_2 - \Omega_3 + \Omega_4.
\]

Let \( g = (q, \frac{M}{z}) \). Then

\[
\sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{h=-\infty}^{+\infty} \frac{1}{(2^\gamma qh - \frac{a}{2^g})^{2m}}
\]

\[
= \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{z=-\infty}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{z=0}^{+\infty} \frac{1}{z^{2m}}
\]

\[
= \frac{2^\beta}{g^{2m-1}} \sum_{z=0}^{+\infty} \frac{1}{z^{2m}} = \frac{2^\beta + 1}{2^m - 1}.
\]
Similarly, we have
\[
\sum_{a=1}^{2^r q} \sum_{h=\infty}^{2\gamma/\beta+1} \frac{1}{(2\gamma/\beta+1 q h - \frac{ak}{2\gamma d})^{2m}} = \frac{2^\beta \zeta(2m)}{g^{2m-1}},
\]
\[
\sum_{a=1}^{2^r q} \sum_{h=-\infty}^{2\gamma/\beta-1} \frac{1}{(2\gamma/\beta-1 q h - \frac{ak}{2\gamma d})^{2m}} = \frac{2^\beta+1 \zeta(2m)}{g^{2m-1}},
\]
\[
\sum_{a=1}^{2^r q} \sum_{h=-\infty}^{2\gamma/\beta-1} \frac{1}{(2\gamma/\beta q h - \frac{ak}{2\gamma d})^{2m}} = \frac{2^\beta \zeta(2m)}{g^{2m-1}}.
\]
Therefore
\[\Psi_3 = \frac{2^{4m} \zeta(2m)}{(2^{4m} - 1) \zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta+1} g^{2m-1}} \sum_{d|M} \sum_{q|\nu} \sum_{q} \sum_{m=1}^{\infty} \frac{1}{g^{2m-1}} \frac{1}{q^{2m}} (q, \frac{\nu}{d}) = g \]
\[
- \frac{2^{4m} \zeta(2m)}{(2^{4m} - 1) \zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta+1} g^{2m-1}} \sum_{d|M} \sum_{q|\nu} \sum_{q} \sum_{m=1}^{\infty} \frac{1}{g^{2m-1}} \frac{1}{q^{2m}} (q, \frac{\nu}{d}) = g \]
\[
- \frac{2^{4m} \zeta(2m)}{(2^{4m} - 1) \zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta+2m} g^{2m-1}} \sum_{d|M} \sum_{q|\nu} \sum_{q} \sum_{m=1}^{\infty} \frac{1}{g^{2m-1}} \frac{1}{q^{2m}} (q, \frac{\nu}{d}) = g \]
\[
+ \frac{2^{4m} \zeta(2m)}{(2^{4m} - 1) \zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta+2m} g^{2m-1}} \sum_{d|M} \sum_{q|\nu} \sum_{q} \sum_{m=1}^{\infty} \frac{1}{g^{2m-1}} \frac{1}{q^{2m}} (q, \frac{\nu}{d}) = g \]
Now from (5.1)-(5.4) and Lemma 3.1 we have
\[
h_m(k) = \frac{2^{4m} \zeta(2m)}{(2^{4m} - 1) \zeta(4m)} \times \left[ \frac{(2^{2m} - 1)}{2^{2m}} \sum_{\gamma=1}^{2m} \frac{1}{2^{(2m-1)\beta+2m}} + \frac{1}{(2^{2m} - 1)\beta+2m} \right]
\]
\[
- \frac{1}{2^{(2m-1)\beta+2m}} \sum_{\gamma=1}^{2m} \frac{1}{2^{(2m-1)\beta+1}} \sum_{\gamma=1}^{2m} \frac{1}{2^{(2m-1)\beta+2m}} + \frac{1}{(2^{2m} - 1)\beta+2m} \right]
\]
This completes the proof of Theorem 1.4.

References


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