CONSTRAINED JACOBI POLYNOMIAL AND
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Abstract. In this paper, we present the constrained Jacobi polynomial which is equal to the constrained Chebyshev polynomial up to constant multiplication. For degree \( n = 4, 5 \), we find the constrained Jacobi polynomial, and for \( n \geq 6 \), we present the normalized constrained Jacobi polynomial which is similar to the constrained Chebyshev polynomial.

1. Introduction

Degree reduction of Bézier curves is an old problem in CAGD (Computer Aided Geometric Design) or CAD/CAM. In general, degree reduction cannot be done exactly so that it invokes approximation problems. Thus many papers dealing with the problems have been published in the recent thirty years or so. Degree reduction has been developed in a variety of viewpoint, e.g., \( C^k \) constrained degree reduction \([1, 2, 3, 8, 11, 12, 20]\), degree reduction using control points \([6, 7, 9, 13, 15]\), degree reduction matrix \([14, 17, 18]\), degree reduction on simplex domain \([10, 16]\), etc.

The constrained Chebyshev polynomial is the error function of the best degree reduction with \( C^0 \) constraint at both end points. The author \([1]\) presented the constrained Jacobi polynomial as the error function of the near best degree reduction with \( C^k \) constraint at both end points. So, a question follows: Does the constrained Jacobi polynomial which is equal to the constrained Chebyshev polynomial up to constant multiplication exist? In this paper we find the constrained Jacobi polynomial for degree \( n = 4, 5 \). But unfortunately, we cannot find the constrained Jacobi polynomial for \( n \geq 6 \). Thus we extend the constrained Jacobi polynomial of degree \( n = 5 \) to all degree \( n \geq 6 \). We also find an interpolation of normalization factor at even integer \( 2n \), so that we finally present the normalized constrained Jacobi polynomial for all degree. By plotting the graph of the polynomial, we can see that they are almost similar to the constrained Chebyshev polynomial.
2. constrained Jacobi polynomial and constrained Chebyshev polynomial

The Jacobi polynomial defined \([4, 5, 19]\) by

\[
(2.1) \quad P_{\alpha,\beta}^n(x) = 2^{-n} \sum_{i=0}^{n} \binom{n + \alpha}{n - i} \binom{n + \beta}{i} (x - 1)^i (x + 1)^{n-i}, \quad x \in [-1, 1]
\]

for \(\alpha, \beta > -1\), is the well known orthogonal polynomial with respect to the weight \((1 - x)^\alpha (1 + x)^\beta\) such that

\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_{\alpha,\beta}^n(x) P_{\alpha,\beta}^m(x) \, dx = \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)n!} \delta_{m,n},
\]

where

\[
\delta_{m,n} = \begin{cases} 
1 & (m = n) \\
0 & (m \neq n)
\end{cases}
\]

and has the leading coefficient

\[
2^{-n} \sum_{i=0}^{n} \binom{n + \alpha}{n - i} \binom{n + \beta}{i} = 2^{-n} \binom{2n + \alpha + \beta}{n}.
\]

Note that

\[
L_n(x) = P_n^{(0,0)}(x), \\
T_n(x) = 2^{2n} \left(\frac{2n}{n}\right)^{-1} P_n^{-1/2,-1/2}(x), \\
U_n(x) = 2^{2n} \left(\frac{2n + 1}{n + 1}\right)^{-1} P_n^{1/2,1/2}(x)
\]

are called by Legendre polynomial, Chebyshev polynomial of first kind, and Chebyshev polynomial of second kind, which are the error functions of the best degree reduction with respect to the \(L_2\)-, \(L_\infty\)-, and \(L_1\)-norm, respectively. The Chebyshev polynomial

\[
T_n(x) = 2^{2n} \left(\frac{2n}{n}\right)^{-1} P_n^{-1/2,-1/2}(x) = \cos(n \cos^{-1} x)
\]

has the leading coefficient \(2^{n-1}\). It has zeros at \(x = \cos\left(\frac{2k+1}{2n}\pi\right), \ k = 0, 1, \ldots, \ n - 1\), and its largest zero is \(\cos(\pi/2n)\). The constrained Chebyshev polynomial is well known \([11]\]

\[
T_{n,1} = T_n(\cos(\pi/2n)x)
\]

which is the error function of the best degree reduction with \(C^0\)-continuity at both end points. We define the \(C^0\) constrained Jacobi polynomial \(J_{\alpha}^n(x)\) by

\[
(2.2) \quad J_{\alpha}^n(x) = (x - 1)(x + 1)P_{n-2}^{(\alpha, \alpha)}(x), \quad x \in [-1, 1]
\]
for \( n \geq 2 \).

**Proposition 2.1.** \( J_4^\alpha (x) \equiv T_{4,1}(x) \) up to constant multiplication if and only if \( \alpha = \sqrt{2} \).

**Proof.** Since \( \cos(\pi/8) = (3 + 2\sqrt{2})/8 \), we have

\[
T_{4,1}(x) = 2^3 \cos^4\left(\frac{\pi}{8}\right)(x-1)(x+1)(x-\frac{\cos \frac{3}{8}\pi}{\cos \frac{5}{8}\pi})(x+\frac{\cos \frac{3}{8}\pi}{\cos \frac{5}{8}\pi})
= (3 + 2\sqrt{2})(x^2 - 1)(x^2 - (\sqrt{2} - 1)^2)
\]
and by Equations (2.1)-(2.2),

\[
J_4^\alpha (x) = \frac{(2\alpha + 2)(2\alpha + 3)}{2^2 \cdot 2!} (x^2 - 1)(x^2 - \frac{1}{2\alpha + 3}).
\]

Thus we have \( 1/(2\alpha + 3) = (\sqrt{2} - 1)^2 \), and \( \alpha = \sqrt{2} \) is the solution of \( J_4^\alpha (x) \equiv T_{4,1}(x) \) up to constant multiplication. \( \square \)

**Proposition 2.2.** \( J_5^\alpha (x) \equiv T_{5,1}(x) \) up to constant multiplication if and only if \( \alpha = \frac{3\sqrt{5} - 1}{4} \).

**Proof.** Since \( \cos(\pi/10) = (\sqrt{10} + 2\sqrt{5})/4 \), we have

\[
T_{5,1}(x) = 2^4 \cos^5\left(\frac{\pi}{10}\right)x(x^2 - 1)(x^2 - \left(\frac{3}{\cos \frac{3}{10}\pi}\right)^2)
= \frac{5\sqrt{50} + 22\sqrt{5}}{4} x(x^2 - 1)(x^2 - \frac{3 - \sqrt{5}}{2})
\]
and

\[
J_5^\alpha (x) = \frac{(2\alpha + 4)(2\alpha + 5)(2\alpha + 6)}{2^1 \cdot 3!} x(x^2 - 1)(x^2 - \frac{3}{2\alpha + 5}).
\]

Thus we have \( \frac{3 - \sqrt{5}}{2\alpha + 5} = \frac{3}{2\alpha + 5} \), and \( \alpha = (3\sqrt{5} - 1)/4 \) is the solution of \( J_5^\alpha (x) \equiv T_{5,1}(x) \) up to constant multiplication. \( \square \)

We call the polynomial \( J_{2n}^\alpha (x)/|J_{2n}^\alpha (0)| \) by normalized constrained Jacobi polynomial (NCJP). But \( J_{2n-1}^\alpha (0) = 0 \). Thus it is required to find interpolation function of \( |J_{2n}^\alpha (0)| \) at even integer \( 2n \).

**Proposition 2.3.** \( \delta^\alpha_x = \frac{\Gamma(x+\alpha+1)}{\Gamma(2(n+x+\alpha+1))} \), \( x > 0 \), is an interpolation of \( |J_{2n}^{(\alpha,\alpha)}(0)| \) at even integer \( x = 2n \).

**Proof.** Since

\[
P_{2n}^{(\alpha,\alpha)}(0) = 2^{-2n} \sum_{i=0}^{2n} (-1)^i \binom{2n+\alpha}{2n-i} \binom{2n+\alpha}{i}
= \frac{(-1)^n (n+1+\alpha) \cdots (2n+\alpha)}{n!},
\]
we have

$$\delta^\alpha_{2n} = \frac{\Gamma(2n + \alpha + 1)}{2^{2n}\Gamma(n + 1)\Gamma(n + \alpha + 1)}$$

$$= \frac{1}{2^{2n}} \frac{(n + 1 + \alpha) \cdots (2n + \alpha)}{n!} = |P_{2n}^{(\alpha,\alpha)}(0)|.$$
Thus the assertion follows.

We present the polynomial $NCJP_{\alpha}^{n}(x)/\delta_n^{\alpha}$ for all degree $n \geq 5$ as an error function of degree reduction with $C^0$ continuity at both end-points. For $\alpha = \frac{3\sqrt{5}-1}{4}$ and $5 \leq n \leq 10$, we can see that they are almost similar to the constrained Chebyshev polynomial, as shown in Figures 1-2.

References

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