ENERGY FINITE $p$-HARMONIC FUNCTIONS ON GRAPHS AND ROUGH ISOMETRIES

SEOK WOO KIM AND YONG HAH LEE
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Abstract. We prove that if a graph $G$ of bounded degree has finitely many $p$-hyperbolic ends $(1 < p < \infty)$ in which every bounded energy finite $p$-harmonic function is asymptotically constant for almost every path, then the set $\mathcal{HBD}_p(G)$ of all bounded energy finite $p$-harmonic functions on $G$ is in one to one corresponding to $\mathbb{R}^l$, where $l$ is the number of $p$-hyperbolic ends of $G$. Furthermore, we prove that if a graph $G'$ is roughly isometric to $G$, then $\mathcal{HBD}_p(G')$ is also in an one to one correspondence with $\mathbb{R}^l$.

1. Introduction

We say that a graph $G$ has the Liouville property if every bounded harmonic function on $G$ is constant. Thus the set of all bounded harmonic functions on $G$ having Liouville property is in one to one correspondence with the real line $\mathbb{R}$. With this viewpoint, given an operator $A$ on a graph, it seems natural to regard a class $\mathcal{S}$ of solutions of $A$ which is in one to one correspondence with the Euclidean space $\mathbb{R}^l$ for some positive integer $l$ as a generalized version of the Liouville property of the pair $(A, \mathcal{S})$. In this paper, we study case of the $p$-Laplacian operator $(1 < p < \infty)$ and the bounded $p$-harmonic functions on a graph $G$ of bounded degree. If $p = 2$, then we obtain harmonic functions on $G$ as a special case. (See [6] and [8].) In Section 3, we study a sort of an asymptotic behavior of $p$-harmonic functions which enables us to identify a subset of the set of the bounded $p$-harmonic functions on $G$. To be precise, if a graph $G$ has a finite number of $p$-hyperbolic ends and every bounded energy finite $p$-harmonic function on $G$ satisfies such an behavior, then we have the following theorem:

**Theorem 1.1.** Let $G$ be a graph with $l$ ($l \geq 1$) $p$-hyperbolic ends. Suppose that every $p$-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end, where $\mathcal{HBD}_p(G)$ denotes the set of all

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bounded energy finite $p$-harmonic functions on $G$. Then given any real numbers $a_1, a_2, \ldots, a_l \in \mathbb{R}$, there exists a unique $p$-harmonic function $v \in \mathcal{HBD}_p(G)$ such that

$$v(p) = a_i \text{ for } p\text{-almost every path } p \in P_{E_i}$$

for each $i = 1, 2, \ldots, l$, where $E_1, E_2, \ldots, E_l$ are $p$-hyperbolic ends of $G$, and $P_{E_i}$ denotes a family of paths lying in $E_i$ to be explained in Section 3.

In Section 4, we extend our result to graphs being roughly isometric to those satisfying the assumption of Theorem 1.1:

**Theorem 1.2.** Let $G$ be a graph with $l \ (l \geq 1)$ $p$-hyperbolic ends. Suppose that every $p$-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end. Let $G'$ be a graph being roughly isometric to $G$. Then given any real numbers $a_1, a_2, \ldots, a_l \in \mathbb{R}$, there exists a unique $p$-harmonic function $v \in \mathcal{HBD}_p(G')$ such that

$$v(p) = a_i \text{ for } p\text{-almost every path } p \in P_{E_i}$$

for each $i = 1, 2, \ldots, l$, where $E_1, E_2, \ldots, E_l$ are $p$-hyperbolic ends of $G'$.

2. Preliminaries

Let $G = (V_G, E_G)$ be a graph, where $V_G$ and $E_G$ denote the vertex set and the edge set, respectively, of $G$. If vertices $x$ and $y$ are the endpoints of the same edge, then we say that $x$ and $y$ are neighbors and write $y \in N_x$ and $x \in N_y$. The degree of $x$ is the number of all neighbors of $x$ and it is denoted by $\sharp N_x$. A graph $G$ is said to be of bounded degree if there exists a number $\nu < \infty$ such that $\sharp N_x \leq \nu$ for all $x \in V_G$. A sequence $x = (x_0, x_1, \ldots, x_r)$ of vertices in $V_G$ is called a path from $x_0$ to $x_r$ with the length $r$ if $x_k$ is an element of $N_{x_{k-1}}$ for each $k = 1, 2, \ldots, r$. We say that a graph $G$ is connected if any two points of $V_G$ can be joined by a path. Throughout this paper, $G$ is a connected infinite graph with no self-loops and is of bounded degree.

For any vertices $x$ and $y$, we define $d(x, y)$ to be the length of the shortest path joining $x$ to $y$. Then $d$ defines a metric on $V_G$. For this metric $d$ and $r \in \mathbb{N}$, define an $r$-neighborhood $N_r(x) = \{y \in V_G : d(x, y) \leq r\}$ for each $x \in V_G$. Given any subset $S \subset V_G$, the outer boundary $\partial S$ and the inner boundary $\delta S$ of $S$ are defined by

$$\partial S = \{x \in V_G : d(x, S) = 1\} \text{ and } \delta S = \{x \in V_G : d(x, V_G \setminus S) = 1\},$$

respectively.

For each real valued function $u$ on $S \cup \partial S$, define the norm of $p$-gradient, the $p$-Dirichlet sum, and the $p$-Laplacian of $u$ at a point $x \in S$, where $1 < p < \infty,$
in such a way that
\[
|Du|(x) = \left( \sum_{y \in N_x} |u(y) - u(x)|^p \right)^{1/p},
\]
\[
I_p(u, S) = \sum_{x \in S} |Du|^p(x),
\]
\[
\Delta_p u(x) = \sum_{y \in N_x} \text{sign}(u(y) - u(x))|u(y) - u(x)|^{p-1}
\]
\[
= \sum_{y \in N_x} |u(y) - u(x)|^{p-2}(u(y) - u(x)),
\]
respectively.

We say that \( u \) is \( p \)-harmonic on \( S \) if \( \Delta_p u(x) = 0 \) for all \( x \in S \). We introduce some useful properties of \( p \)-harmonic functions on graphs in \cite{1}. If a subset \( S \subset V_G \) is finite, then the following conditions are equivalent:

(i) A function \( u \) is \( p \)-harmonic on \( S \).

(ii) A function \( u \) satisfies \( p \)-Laplacian equation in a weak form. That is,
\[
\sum_{x \in S} \sum_{y \in N_x} |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) = 0
\]
for any real valued function \( w \) on \( S \cup \partial S \) such that \( w = 0 \) on \( \partial S \).

(iii) A function \( u \) is a minimizer of \( p \)-Dirichlet sum \( I_p(\cdot, S) \) among functions on \( S \cup \partial S \) with the same values on \( \partial S \). That is,
\[
\sum_{x \in S} |Du|^p(x) \leq \sum_{x \in S} |Dv|^p(x)
\]
for every function \( v \) on \( S \cup \partial S \) such that \( v = u \) on \( \partial S \).

Let us set \( T(u, v; x, y) = |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) \) whenever functions \( u \) and \( w \) are defined at \( x \) and \( y \). Then it is easy to check that
\[
T(v, v - u; x, y) \geq T(u, v - u; x, y)
\]
if \( u \) and \( v \) are defined at \( x \) and \( y \). The equality occurs only if \( v(x) - u(x) = v(y) - u(y) \). By (2), the following comparison principle holds on \( S \): Suppose there exist \( p \)-harmonic functions \( u \) and \( v \) on a finite set \( S \subset V_G \) such that \( u \geq v \) on \( \partial S \). Then \( u \geq v \) on \( S \).

Let \( S \) be a finite subset of \( V_G \). Suppose that \( \{u_i\} \) is a sequence of functions on \( S \cup \partial S \) converging to a function \( u \) pointwisely. Then for each point \( x \in S \),
\[
|Du_i|^p(x) \to |Du|^p(x) \quad \text{and} \quad \Delta_p u_i(x) \to \Delta_p u(x)
\]
and
\[
I_p(u_i, S) \to I_p(u, S)
\]
as \( i \to \infty \). By these facts together with the comparison principle, the following existence and uniqueness result holds: Let \( S \) be a finite subset of \( V_G \). For any
function \( v \) on \( \partial S \), there exists a unique function on \( S \cup \partial S \) which is \( p \)-harmonic on \( S \) and equal to \( v \) on \( \partial S \).

Let \( \{S_i\} \) be an increasing sequence of finite connected subsets of \( V_G \) and \( S = \bigcup S_i \). Let \( \{u_i\} \) be a sequence of functions on \( S \cup \partial S \) such that each \( u_i \) is \( p \)-harmonic on \( S_i \) and \( u_i(x) \to u(x) < \infty \) as \( i \to \infty \) for all \( x \in S \cup \partial S \). Then the limit function \( u \) is \( p \)-harmonic on \( S \).

We say that a real valued function \( u \) is energy finite if it has finite \( p \)-Dirichlet sum on the whole set \( V_G \), i.e., \( I_p(u, V_G) < \infty \). Let \( BD_p(G) \) denote the set of all bounded energy finite functions on \( V_G \). Then, \( BD_p(G) \) is a Banach space with the norm
\[
||u||_p = \sup_{V_G} |u| + I_p(u, V_G)^{1/p}.
\]
We denote by \( BD_{p,0}(G) \) the closure of the set of all finitely supported functions on \( V_G \) in \( BD_p(G) \) with respect to the norm \( || \cdot ||_p \). The subset of all bounded \( p \)-harmonic functions in \( BD_p(G) \) is denoted by \( HBD_p(G) \).

The subgraph \( \Gamma \) induced by a set \( S \subset V_G \) is the graph \( \Gamma = (S, E_\Gamma) \), where \( E_\Gamma \) is the set of all edges in \( E_G \) with both ends points in \( S \). In particular, that a subset \( S \subset V_G \) is connected means that the subgraph \( \Gamma = (S, E_\Gamma) \) induced by \( S \) is connected. A connected subset \( S \subset V_G \) with \( \partial S \neq \emptyset \) is called \( D_p \)-massive if there exists a nonnegative \( p \)-harmonic function \( u \) on \( S \) such that \( u = 0 \) on \( \partial S \), \( \sup_S u = 1 \) and \( I_p(u, S) < \infty \). We say that a connected infinite set \( S \subset V_G \) is \( p \)-hyperbolic if there exists a nonempty finite set \( A \subset S \) such that
\[
\text{Cap}_p(A, \infty, S) = \inf_u I_p(u, S) > 0,
\]
where the infimum is taken over all finitely supported function \( u \) on \( S \cup \partial S \) such that \( u = 1 \) on \( A \). Otherwise, \( S \) is called \( p \)-parabolic.

We now introduce the \( p \)-Royden decomposition: (See [9].)

**Proposition 2.1.** If a graph \( G \) is \( p \)-hyperbolic, then for each function \( u \in BD_p(G) \), there exist unique functions \( h \in HBD_p(G) \) and \( g \in BD_{p,0}(G) \) such that \( u = h + g \).

For each nonnegative real valued function \( w \) on \( E_G \), define
\[
E_p(w) = \sum_{e \in E_G} w^p(e).
\]
Let \( P \) be a family of infinite paths in \( G \). The \( p \)-extremal length \( \lambda_p(P) \) of \( P \) is defined by
\[
\lambda_p(P) = \left( \inf_w E_p(w) \right)^{-1},
\]
where the infimum is taken over the set of all nonnegative functions \( w \) on \( E_G \) such that \( E_p(w) < \infty \) and \( \sum_{e \in E_x} w(e) \geq 1 \) for each path \( x \in P \), where \( E_x \) denotes the edge set of \( x \). The following proposition gives some fundamental properties of the extremal length. (See [4].)

**Proposition 2.2.** Let \( P_n, n = 1, 2, \ldots, \) be families of paths in a graph \( G \).
(i) If $P_1 \subset P_2$, then $\lambda_p(P_1) \geq \lambda_p(P_2)$.
(ii) $\sum_{n=1}^{\infty} \lambda_p(P_n)^{-1} \geq \lambda_p(\cup_{n=1}^{\infty} P_n)^{-1}$.

On the other hand, the $p$-extremal length is closely related to the $p$-capacity: Let $S \subset \mathbb{V}_G$ be a connected infinite subset. For a nonempty finite subset $A \subset S$, let $P_{S,A}$ be the set of all non-self-intersecting infinite paths in $S$ starting from a vertex in $A$. Then we have

$$\lambda_p(P_{S,A}) = \text{Cap}_p(A, \infty, S)^{-1}. \quad (3)$$

(See [9] and [7].) Furthermore, if $S \subset \mathbb{V}_G$ is $p$-hyperbolic, then by (3),

$$\lambda_p(P_{S,A}) = \text{Cap}_p(A, \infty, S)^{-1} < \infty. \quad (4)$$

We say that a property holds for $p$-almost every path in $P$ if the subset of all paths for which the property is not true has $p$-extremal length $\infty$.

The following proposition gives some $p$-almost every path properties of energy finite functions: (See [4] and [9].)

**Proposition 2.3.** Let $P_o$ be the family of all non-self-intersecting infinite paths from a fixed point $o \in \mathbb{V}_G$.

(i) If $u \in BD_p(G)$, then $u(x)$ exists and is finite for $p$-almost every path $x \in P_o$, where $u(x) = \lim u(x)$ as $x \to \infty$ along the vertices of $x$.
(ii) $u \in BD_{p,0}(G)$ if and only if $u(x) = 0$ for $p$-almost every path $x \in P_o$.

3. Asymptotically constant for $p$-almost every path on ends

We now define ends of a graph $G$ with its vertex set $\mathbb{V}_G$: Fix a point $o \in \mathbb{V}_G$. For each $r \in \mathbb{N}$, we denote by $\sharp(r)$ the number of infinite connected components of $\mathbb{V}_G \setminus N_r(o)$. Let $\lim_{r \to \infty} \sharp(r) = l$, where $l$ may be infinity, then we say that the number of ends of $G$ is $l$. If $l$ is finite, then we can choose $r_0 \in \mathbb{N}$ such that $\sharp(r) = l$ for all $r \geq r_0$.

Using the $p$-hyperbolicity, we can divide ends of $G$ into two classes as follows: An end $E$ of $G$ is called $p$-hyperbolic if

$$\text{Cap}_p(\partial E, \infty, E) = \inf_u I_p(u, E) > 0,$$

where the infimum is taken over all finitely supported function $u$ on $E \cup \partial E$ such that $u = 1$ on $\partial E$. Otherwise, the end is called $p$-parabolic.

From the definition of a $p$-hyperbolic end, we have the following lemma:

**Lemma 3.1.** If $E$ is a $p$-hyperbolic end, then there exists a $p$-harmonic function $u_E$ on $E$, called a $p$-harmonic measure of $E$, with the following properties:

(i) $0 \leq u_E \leq 1$ on $E$;
(ii) $u_E = 0$ on $\partial E$;
(iii) $\limsup_{x \in E} u_E(x) = 1$;
(iv) $u_E$ has finite $p$-Dirichlet sum over $E$. 

Suppose the lemma is not true. Then by assumption, there exists a convergent subsequence, and its limit function.

Proof of Theorem 1.1.

Clearly, \( \Omega \) is a nonconstant for \( p \)-almost every path in \( E \).

For each end \( E \) of \( G \), let us denote \( P_E \subset P_G \) to be the family of all paths lying in \( E \setminus N_{r_1}(o) \) starting from a vertex in \( \delta N_{r_1}(o) \) for some large \( r_1 \in \mathbb{N} \). We say that a real valued function \( u \) on \( V_G \) is asymptotically constant for \( p \)-almost every path in \( E \) if there exists a constant \( c \) such that

\[
 u(x) = c \quad \text{for} \quad p \text{-almost every path} \ x \in P_E,
\]

where \( u(x) = \lim u(x) \) as \( x \) goes to \( \infty \) along vertices on \( x \).

Lemma 3.2. Let \( E \) be a \( p \)-hyperbolic end of a graph \( G \) and \( u \) be a nonconstant function in \( HBD_p(G) \) such that \( 0 \leq u \leq 1 \). Suppose that \( u \) is asymptotically constant for \( p \)-almost every path in \( E \). If \( \limsup_{x \to \infty} u = 1 \), then \( u(x) = 1 \) for \( p \)-almost every path \( x \in P_E \).

Proof. Suppose the lemma is not true. Then by assumption, there exists a constant \( c \) such that \( u(x) = c \) for \( p \)-almost every path \( x \in P_E \) and \( 0 \leq c < 1 \). Since \( u \) is nonconstant, there exists a proper subset \( \Omega \) of \( E \) such that \( \Omega = \{ x \in E : u(x) > 1 - \epsilon \} \), where \( \epsilon \) is a positive constant so small that \( 1 - \epsilon > c \). Clearly, \( \Omega \) is a \( D \)-massive subset. By (4), there exists a subfamily \( P_\Omega \) of \( P_E \) such that \( \lambda_p(P_\Omega) < \infty \). But from the definition of \( \Omega \), one can conclude that \( u(x) > c \) for all paths \( x \in P_\Omega \). This contradicts the fact that \( u(x) = c \) for \( p \)-almost every path \( x \in P_E \). This completes the proof.

Proof of Theorem 1.1. For each \( i = 1, 2, \ldots, l \), extend \( u_{E_i} \) to be zero outside \( E_i \) and then construct a sequence of real valued functions \( \{u_{r,i}\} \) on \( V_G \) such that

\[
\begin{cases}
\Delta_p u_{r,i} = 0 & \text{on } N_r(o); \\
u_{r,i} = u_{E_i} & \text{on } V_G \setminus N_r(o),
\end{cases}
\]

where \( u_{E_i} \) is a \( p \)-harmonic measure of \( E_i \) constructed in Lemma 3.1 for each \( i \). By the comparison principle, \( u_{E_i} \leq u_{r,i} \leq 1 \) on \( N_r(o) \) for each \( i \). Thus there exists a convergent subsequence, and its limit function \( u_i \) satisfies that

\[
\begin{cases}
\Delta_p u_i = 0 & \text{on } V_G; \\
0 \leq u_i \leq 1; \\
\limsup_{x \to \infty, x \in E_i} u_i = 1.
\end{cases}
\]

By the minimizing property of \( p \)-harmonic functions, \( u_i \) is energy finite for each \( i \).

Without loss of generality, we may assume that \( 0 < a_1 \leq a_2 \leq \cdots \leq a_l \leq 2a_1 \). Let us construct a sequence of real valued functions \( \{v_r\} \) on \( V_G \) such that

\[
\begin{cases}
\Delta_p v_r = 0 & \text{on } N_r(o); \\
v_r = a_i & \text{on } E_i \setminus N_r(o); \\
v_r = 0 & \text{on } V_G \setminus (\bigcup_{k=1}^l E_k \cup N_r(o)),
\end{cases}
\]
where \( i = 1, 2, \ldots, l \). Then
\[
a_i u_i \leq v_r \leq a_i (2 - u_i) \quad \text{on} \quad (\delta N_r(o) \cup \partial N_r(o)) \cap E_i,
\]
where \( u_i \) is the \( p \)-harmonic function constructed above. Hence by the comparison principle, we conclude that
\[
a_i u_i \leq v_r \leq a_i (2 - u_i) \quad \text{on} \quad N_r(o) \cap E_i.
\]
There exists a subsequence, denoted by \( \{ u_{i_n} \} \), converging to a \( p \)-harmonic function \( v \) on \( V_G \). By Lemma 3.2, \( u_i(x) = 1 \) for \( p \)-almost every path \( x \in P_{E_i} \) for each \( i \). Hence \( v \) satisfies (1). By the minimizing property of \( p \)-harmonic function, \( v \) has finite \( p \)-Dirichlet sum.

Suppose that there exists a \( p \)-harmonic function \( w \in \mathcal{HBD}_p(G) \) satisfying (1). Put \( P_{E_i} = P_{i,w,1} \cup P_{i,w,2} \) for each \( i \), where
\[
P_{i,w,1} = \{ x \in P_{E_i} : w(x) = a_i \} \quad \text{and} \quad P_{i,w,2} = \{ x \in P_{E_i} : w(x) \neq a_i \}.
\]
Then we have \( \lambda_p(P_{E_i}) < \infty \) and \( \lambda_p(P_{i,w,2}) = \infty \) for each \( i \). Similarly, let us set \( P_{E_i} = P_{i,v,1} \cup P_{i,v,2} \) for each \( i \), where
\[
P_{i,v,1} = \{ x \in P_{E_i} : v(x) = a_i \} \quad \text{and} \quad P_{i,v,2} = \{ x \in P_{E_i} : v(x) \neq a_i \}.
\]
Then we have \( \lambda_p(P_{i,v,1}) < \infty \) and \( \lambda_p(P_{i,v,2}) = \infty \) for each \( i \). From Proposition 2.2 and Proposition 2.3, we conclude that
\[
\lambda_p(P_{E_i} \setminus (P_{i,w,1} \cap P_{i,v,1})) = \lambda_p((P_{E_i} \setminus P_{i,w,1}) \cup (P_{E_i} \setminus P_{i,v,1})) \geq 1/(\lambda_p(P_{E_i} \setminus P_{i,w,1})^{-1} + \lambda_p(P_{E_i} \setminus P_{i,v,1})^{-1}) = \infty
\]
for each \( i \). This implies that
\[
(v - w)(x) = 0 \quad \text{for} \quad p \text{-almost every} \quad x \in P_{E_i}
\]
for each \( i = 1, 2, \ldots, l \). On the other hand, since \( \lambda_p(P_G \setminus \cup_{i=1}^{l} P_{E_i}) = \infty \), we have
\[
(v - w)(x) = 0 \quad \text{for} \quad p \text{-almost every} \quad x \in P_G.
\]
Consequently, by Proposition 2.3, we conclude that \( v - w \in \mathcal{BD}_{p,0}(G) \). Thus there exists a sequence of finitely supported functions converging to \( v - w \) in \( \mathcal{BD}_p(G) \). By this fact together with the Hölder inequality, since \( v \) and \( w \) are \( p \)-harmonic functions on \( V_G \), it is easy to see that
\[
\sum_{x \in V_G} \sum_{y \in N_x} |v(y) - v(x)|^{p-2}(v(y) - v(x))((v - w)(y) - (v - w)(x)) = 0
\]
and
\[
\sum_{x \in V_G} \sum_{y \in N_x} |w(y) - w(x)|^{p-2}(w(y) - w(x))((v - w)(y) - (v - w)(x)) = 0.
\]
Thus by (2), we conclude that \( v - w \) is constant function on \( N_x \) for all points \( x \in V_G \). Since \( V_G \) is connected, by (5), we conclude that \( v \equiv w \) on \( V_G \). \( \square \)
4. Asymptotically constant for \( p \)-almost every path and rough isometries

We begin with introducing rough isometries between metric spaces. A map \( \varphi : X \to Y \) is called a rough isometry between metric spaces \( X \) and \( Y \) if it satisfies the following condition:

\[
(R) \quad \text{for some constant } \tau > 0, \text{ the } \tau \text{-neighborhood of the image } \varphi(X) \text{ covers } Y; \\
 \text{there exist constants } a \geq 1 \text{ and } b \geq 0 \text{ such that } \\
\quad a^{-1}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b \\
\text{for all points } x_1, x_2 \in X, \text{ where } d \text{ denotes the distances of } X \text{ and } Y \induced from their metrics, respectively.}
\]

If such a map exists, then \( X \) is said to be roughly isometric to \( Y \). Being roughly isometric is an equivalent relation. (See [2].) In particular, if \( \varphi : X \to Y \) is a rough isometry satisfying \((R)\), then for any point \( y \in Y \), there exists at least one point \( x \in X \) such that \( d(\varphi(x), y) < \tau \). If we set \( \varphi^{-1}(y) = x \), then \( \varphi^{-1} \) satisfies \((R)\) with constants \( \tau', a' \) and \( b' \), where \( \tau' = a(b + \tau), a' = a \) and \( b' = a(b + 2\tau) \).

On the other hand, since the vertex set of each graph is a metric space, we can define rough isometries between the vertex sets of graphs similarly as above. Let \( G = (V_G, E_G) \) and \( G' = (V_{G'}, E_{G'}) \) be graphs, and \( \varphi : V_{G'} \to V_G \) be a rough isometry. For convenience' sake, we prefer to write the rough isometry \( \varphi : G' \to G \) rather than \( \varphi : V_{G'} \to V_G \).

Slightly modifying the proof of [5, 3], the number of ends of a graph is a rough isometric invariant. In fact, the rough isometry between graphs gives a one to one correspondence between ends of the graphs and, furthermore, it induces the rough isometry between each end and its corresponding end. On the other hand, the \( p \)-parabolicity of ends is preserved under rough isometries between ends. Also, we can prove that the property of asymptotically constant for \( p \)-almost every path is invariant under rough isometries between ends as follows:

**Theorem 4.1.** Let \( G \) and \( G' \) be graphs with finitely many ends and roughly isometric to each other. Suppose that every \( p \)-harmonic function in \( \mathcal{HBD}_p(G) \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \). Then every \( p \)-harmonic function in \( \mathcal{HBD}_p(G') \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G' \).

To prove Theorem 4.1, we need the following lemmas:

**Lemma 4.2.** Let \( G \) and \( G' \) be graphs with finitely many ends, and \( \varphi : G' \to G \) be a rough isometry. Suppose that every \( p \)-harmonic function in \( \mathcal{HBD}_p(G) \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \). Then for each \( u \in \mathcal{HBD}_p(G') \), \( u \circ \varphi^{-1} \) is asymptotically constant for \( p \)-almost every path in each \( p \)-hyperbolic end of \( G \).
For each $u \in \mathcal{HBD}_p(G')$, it is easy to check that $u \circ \varphi^- \in \mathcal{BD}_p(G)$. So, by Proposition 2.1, there exist unique $h \in \mathcal{HBD}_p(G)$ and $g \in \mathcal{D}_{x,0}(G)$ such that

$$u \circ \varphi^- = h + g.$$  

By the assumption, $h$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. On the other hand, by Proposition 2.3, $g$ is asymptotically constant 0 for $p$-almost every path in each $p$-hyperbolic end of $G$.

Let $E_1, E_2, \ldots, E_l$ be $p$-hyperbolic ends of $G$. Then there exist constants $c_1, c_2, \ldots, c_l$ such that

$$h(y) = c_i$$

for each $i = 1, 2, \ldots, l$. Put $P_i = P_{i, h, 1} \cup P_{i, h, 2}$ for each $i$, where

$$P_{i, h, 1} = \{ y \in P_i : h(y) = c_i \} \quad \text{and} \quad P_{i, h, 2} = \{ y \in P_i : h(y) \neq c_i \}.$$  

Then we have $\lambda_p(P_{i, h, 1}) < \infty$ and $\lambda_p(P_{i, h, 2}) = \infty$ for each $i$. Similarly, let us set $P_i = P_{i, g, 1} \cup P_{i, g, 2}$ for each $i$, where

$$P_{i, g, 1} = \{ y \in P_i : g(y) = 0 \} \quad \text{and} \quad P_{i, g, 2} = \{ y \in P_i : g(y) \neq 0 \}.$$  

Then, by our claim, we have $\lambda_p(P_{i, g, 1}) < \infty$ and $\lambda_p(P_{i, g, 2}) = \infty$ for each $i$.

Arguing similarly as in the proof of Theorem 1.1, we have

$$\lambda_p(P_i \setminus (P_{i, h, 1} \cap P_{i, g, 1})) = \infty$$

for each $i$. Hence $u \circ \varphi^-$ is asymptotically constant $c_i$ at infinity of $E_i$ for $p$-almost every path $y \in P_i$ for each $i$. This completes the proof. □

**Lemma 4.3.** Let $G$ and $G'$ be graphs with finitely many ends and $\varphi : G' \to G$ be a rough isometry. Let $u \in \mathcal{HBD}_p(G')$. Suppose that $u \circ \varphi^-$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. Then $u$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G'$.

**Proof.** Let $E$ be a $p$-hyperbolic end of $G$ and $E'$ be the corresponding end of $G'$ under $\varphi$. Since $u \in \mathcal{HBD}_p(G')$, by Proposition 2.3,

$$u(x) \text{ exists and finite for } p\text{-almost every path } x \in P_a.$$  

Put $P_{E'} = P_1 \cup P_2 \cup P_3$, where $P_1 = \{ x \in P_{E'} : u(x) = c \}$, $P_2 = \{ x \in P_{E'} : u(x) \neq c \}$ and $P_3 = \{ x \in P_{E'} : u(x) \text{ does not exists.} \}$. Since $\lambda_p(P_3) = \infty$, we have only to show that $\lambda_p(P_2) = \infty$.

For each path $x \in P_2$, we will assign a suitable path $y \in P_{2, \varphi^-}$, where $P_{2, \varphi^-} = \{ y \in P_G : (u \circ \varphi^-)(y) \neq c \}$. Let us choose any path $x \in P_2$. We may assume that $x = (a, x_1, x_2, \ldots, x_n, \ldots)$. By definition of the inverse rough isometry $\varphi^-$, there exists a point $y_n \in E$ such that $d(x_n, \varphi^-(y_n)) < a(b + \tau)$ for each positive integer $n$. Let us choose a positive constant $\rho$ such that $\lambda_p(P_3) = \infty$. Hence

$$d(y_n, y_{n+1}) \leq \rho.$$  

For each positive integer $n$, we can choose a minimal path $(z_0^n, z_1^n, \ldots, z_{m_n}^n)$ in such a way that $z_0^n = y_n$, $z_{m_n} = y_{n+1}$, and $m_n \leq \rho$. It follows that there exists an infinite path $y = (o', l_1, l_2, \ldots, l_j, \ldots) \in P_E$ and a nondecreasing sequence of
subscripts \( j(n) \to \infty \) as \( n \to \infty \) such that \( t_{j(n)} = y_n \) and \( j(n+1) - j(n) \leq \rho \).

One can choose a minimal path \((s^n_0, v^n_1, \ldots, v^n_p)\) in such a way that \( s^n_0 = x_n, \ s^n_j = \varphi^-(t_{j(n)}) \) and \( l_n \leq a(b + \tau) \). Let us observe that

\[
|u(x_n) - u(\varphi^-(t_{j(n)}))| \leq a(b + \tau) \sum_{i=1}^{l_n} |u(s^n_i) - u(s^n_{i-1})| \leq C \sum_{x' \in N_{a(b + \tau)}(x_n)} |Du|(x').
\]

Since \( u \in \mathcal{BD}_p(E') \), we conclude that

\[
|u(x_n) - u(\varphi^-(t_{j(n)}))|^p \leq C \sum_{x' \in N_{a(b + \tau)}(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.
\]

This implies that \((u \circ \varphi^-)(t_{j(n)}) \to u(y) \neq c \text{ as } n \to \infty\). On the other hand, we have

\[
|u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))| \leq \rho \sum_{i=1}^{m_n} |u(\varphi^-(s^n_i)) - u(\varphi^-(s^n_{i-1}))| \leq C \sum_{x' \in N_{u(x_n)}(x_n)} |Du|(x')
\]

for each subscript \( j \in [j(n), j(n+1)] \). Hence we have

\[
|u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))|^p \leq C \sum_{x' \in N_{u(x_n)}(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.
\]

Thus \((u \circ \varphi^-)(t_j) \to u(x) \neq c \text{ as } j \to \infty\). Hence \( y \) belongs to \( P_{2, \varphi^-} \).

Since \( \lambda_p(P_{2, \varphi^-}) = \infty \), by the equivalent condition for a family of paths to have infinite \( p \)-extremal length \( [4] \), there exists a nonnegative function \( w \) on the edge set \( E_\xi \) of \( E \) such that \( \sum_{\tilde{e} \in E_\xi} w^p(\tilde{e}) = \mathcal{E}_\xi(w) < \infty \) and \( \sum_{\tilde{e} \in E_\xi} w(\tilde{e}) = \infty \) for all paths \( y \in P_{2, \varphi^-} \). For each positive integer \( \zeta \) and each edge \( e = [z_1, z_2] \in E_\xi \), let us define a set \( U(\zeta, \xi) = \{ e = [a_1, a_2] \in E_\xi : d(z_1, \varphi^-(a_2)) \leq \zeta \text{ for some } i, j = 1, 2 \} \). Let us define a function \( w^* \) on \( E_\xi \) in the following way: \( w^*(e) = \sup_{\tilde{e} \in U(\zeta, \xi)} w(\tilde{e}) \) for all edges \( e \in E_\xi \). Since \( w^*(e) \leq \sum_{\tilde{e} \in U(\zeta, \xi)} w^*(\tilde{e}) \) for each edge \( e \in E_{\xi} \), we have

\[
\mathcal{E}_\xi(w^*) \leq C \sum_{\tilde{e} \in E_{\xi}} w^p(\tilde{e}) < \infty, \quad \rho \in E_\xi
\]

where \( C \) is a positive constant depending on \( \zeta \). Let us fix a positive integer \( \kappa \) such that \([t_{j-1}, t_j] \in U([x_n, x_{n+1}], \kappa)\) for all \( j(n) \leq j \leq j(n+1) \), where \( y = (y', t_1, t_2, \ldots, t_j, \ldots) \) is a path in \( P_{2, \varphi^-} \) and \( x = (a, x_1, x_2, \ldots, x_n, \ldots) \) is a path in \( P_2 \) which are given above. Then for each path \( x \in P_2 \),

\[
\sum_{e \in E(x)} w^*(e) \geq \frac{1}{\rho} \sum_{\tilde{e} \in E(y)} w(\tilde{e}) = \infty.
\]
Therefore, we have $\lambda_p(P_2) = \infty$. This completes the proof. \hfill \square

We are now ready to prove Theorem 4.1:

*Proof of Theorem 4.1.* Let $u$ be a $p$-harmonic function in $\mathcal{HBD}_p(G')$. By Lemma 4.2, the function $u \circ \varphi^{-}$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G$. Then, by Lemma 4.3, the function $u$ is asymptotically constant for $p$-almost every path in each $p$-hyperbolic end of $G'$. This completes the proof. \hfill \square

Combining Theorem 1.1 and Theorem 4.1, we get Theorem 1.2.

*References*


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