THE ZERO-DIVISOR GRAPH UNDER A GROUP ACTION
IN A COMMUTATIVE RING

JUNCHEOL HAN

Abstract. Let $R$ be a commutative ring with identity, $X$ the set of all nonzero, nonunits of $R$ and $G$ the group of all units of $R$. We will investigate some ring theoretic properties of $R$ by considering $\Gamma(R)$, the zero-divisor graph of $R$, under the regular action on $X$ by $G$ as follows:
(1) If $R$ is a ring such that $X$ is a union of a finite number of orbits under the regular action on $X$ by $G$, then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if $R$ is a local ring or $R \cong \mathbb{Z}_2 \times F$ where $F$ is a field; (2) If $R$ is a local ring such that $X$ is a union of $n$ distinct orbits under the regular action of $G$ on $X$, then all ideals of $R$ consist of $\{0\}, J, J^2, \ldots, J^n, R$ where $J$ is the Jacobson radical of $R$; (3) If $R$ is a ring such that $X$ is a union of a finite number of orbits under the regular action on $X$ by $G$, then the number of all ideals is finite and is greater than equal to the number of orbits.

1. Introduction and basic definitions

The zero-divisor graph of a commutative ring has been studied extensively by Akbari, Anderson, Frazier, Lauve, Livinston and Maimani in [1, 2, 3] since its concept had been introduced by Beck in [4]. Recently, zero-divisor graph of a noncommutative ring (resp. a semigroup) has studied by Redmond and Wu (resp. F. DeMeyer and L. Demeyer) in [9, 10, 11] (resp [5]). Zero-divisor graph is very useful to find the algebraic structures and properties of rings. In this paper, the zero-divisor graph of a commutative ring is also studied by considering some group action.

Throughout this paper all rings are assumed to be rings with identity $1 \neq 0$. For a commutative ring $R$, let $Z(R)$ be the set of all zero-divisors of $R$, and $\Gamma(R)$ be the zero-divisor graph of $R$ consisting of all vertices in $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero zero-divisors of $R$, and edges $x \leftrightarrow y$, which means that $xy = 0$ for $x, y \in Z(R)^*$. In this paper, a loop (i.e., an edge from some vertex to
itself) can be considered an edge in a zero-divisor graph \( \Gamma(R) \). Recall that \( \Gamma(R) \) is connected if there is a path between any two distinct vertices. For vertices \( x \) and \( y \) of \( \Gamma(R) \), if there exists a path between \( x \) and \( y \), we will denote \( d(x, y) \) by the length of the shortest path between \( x \) and \( y \), otherwise, \( d(x, y) = \infty \).

The diameter of \( \Gamma(R) \) (denoted by \( \text{diam}(\Gamma(R)) \)) is defined by the supremum of \( d(x, y) \) for all distinct vertices \( x \) and \( y \) in \( \Gamma(R) \). In particular, if \( x = y \) and \( d(x, x) = k \geq 3 \), then the path is called the cycle of length \( k \). If \( \Gamma(R) \) contains a cycle, then the girth of \( \Gamma(R) \) (denoted by \( g(\Gamma(R)) \)) is defined by the length of the shortest cycle in \( \Gamma(R) \), otherwise, \( g(\Gamma(R)) = \infty \). In [6, Proposition 1.3.2], if \( \Gamma(R) \) contains a cycle, then \( 1 + 2\text{diam}(\Gamma(R)) \geq g(\Gamma(R)) \). We say that \( \Gamma(R) \) is complete if \( xy = 0 \) for any distinct vertices \( x, y \) in \( \Gamma(R) \). In [3], Anderson and Livingston have shown that for a commutative ring \( R \), (1) \( \Gamma(R) \) is connected and \( 3 \geq \text{diam}(\Gamma(R)) \); (2) there is a vertex of \( \Gamma(R) \) which is adjacent to every other vertex in \( \Gamma(R) \) if and only if \( R \cong \mathbb{Z}_2 \times A \) (\( A \) is an integral domain) or \( Z(R) \) is an annihilator ideal.

Let \( R \) be a ring, \( X(R) \) (simply, denoted by \( X \)) the set of all nonzero, nonunits of \( R \), \( G(R) \) (simply, denoted by \( G \)) the group of all units of \( R \) and \( J \), the Jacobson radical of \( R \). In this paper, we will consider a group action of \( G \) on \( X \) given by \( ((g,x) \rightarrow gx) \) from \( G \times X \) to \( X \), called the regular action. If \( \phi : G \times X \rightarrow X \) is the regular action, then for each \( x \in X \), we define the orbit of \( x \) by \( o(x) = \{\phi(g, x) : g \in G\} \). Recall that \( G \) is transitive on \( X \) (or \( G \) acts transitively on \( X \)) if there is an \( x \in X \) with \( o(x) = X \) and the group action on \( X \) by \( G \) is trivial if \( o(x) = \{x\} \) for all \( x \in X \). In [7], it has been shown that if \( X \) is a union of a finite \( n \) number of orbits under the regular action of \( G \) on \( X \), then (1) \( x^n \geq 0 \) for all \( x \in J \), and \( X \) is the set of all nonzero left zero-divisors of \( R \); (2) \( R \) is a local ring, \( J^n \neq (0) \) and \( J^{n+1} = (0) \) if and only if there exists \( x \in J \) such that \( x^n \neq (0) \) if and only if \( J > J^2 > \ldots > J^{n-1} > J^n \neq (0) \).

For a subset \( S \) of \( Z(R)^* \), we will denote the induced subgraph of \( \Gamma(R) \) with vertices in \( S \) by \( \Gamma_S(R) \), that is, \( x, y \in S \) are adjacent in \( \Gamma_S(R) \) if and only if \( x \) and \( y \) are adjacent in \( \Gamma(R) \). In particular, if \( R \) is a commutative ring such that \( X \) is a union of a finite number of orbits under the regular action of \( G \) on \( X \), then \( X \) is the set of all nonzero zero-divisors of \( R \), i.e., \( X = Z(R)^* \), and so \( \Gamma(R) = \Gamma_X(R) \). In Section 2, for a commutative ring \( R \) such that \( X \) is a union of a \( n \) orbits under the regular action on \( X \) by \( G \), we will investigate some ring theoretic properties of \( R \) by considering \( \Gamma(R) \), the zero-divisor graph of \( R \), as follows: (1) if \( n = 1 \), then \( \Gamma(R) \) is complete; (2) there is an element \( x \in X \) such that \( x \) is adjacent to every other vertex in \( \Gamma(R) \) if and only if \( R \) is a local ring or \( R \cong \mathbb{Z}_2 \times F \) (\( F \) is a field); (3) if \( R \) is a local ring, then every ideal of \( R \) is an annihilator of some element \( x \in X \) (denoted by \( \text{ann}(x) \)); (3) the number of all ideals in \( R \) is equal to the number of all annihilators in \( R \) and is greater than or equal to \( n \), the number of orbits.

Recall that a ring \( R \) is called von Neumann regular (simply, regular) (resp. unit-regular) if for every \( x \in R \) there exists \( y \in R \) (resp. \( g \in G \)) such that \( xyx = x \) (resp. \( xyx = x \)). Note that for a commutative ring \( R \), \( R \) is regular if
and only if $R$ is unit-regular. In Section 3, we will investigate some properties of a commutative regular ring $R$ as follows: (1) $\Gamma_X(R)$ is complete if and only if the set of all idempotents in $R$ is orthogonal and the regular action of $G$ on $X$ is trivial; (2) if $2 = 2 \cdot 1$ is a unit in $R$, then there exists a cycle of length 4 in $\Gamma(R)$.

2. Zero-divisor graph under the regular action

For each $x \in X$, we will denote the set of every element which is adjacent to $x$ by $av(x)$. In fact, $av(x) = ann(x)^* = ann(x) \setminus \{0\}$.

**Proposition 2.1.** Let $R$ be a commutative ring. If the regular action of $G$ on $X$ is transitive, then $\Gamma_X(R)$ is complete.

*Proof.* It follows from [7, Theorem 2.2].

**Example 1** (See Example 2.1 in [3]). Let $R_1 = \mathbb{Z}_9$ and $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Even though $R_1$ is not isomorphic to $R_2$, $\Gamma(R_1) = \Gamma(R_2)$. On the other hand, we can note that (1) all the vertices of $\Gamma(R_1)$ are nilpotent but all the vertices of $\Gamma(R_2)$ are not nilpotent; (2) the adjacency matrix of $R_1$ is not also equal to the one of $R_2$; (3) the regular action in $R_1$ is transitive but the regular action in $R_2$ is trivial.

**Example 2** (See Example 2.1 in [3]). Let $R_1 = \mathbb{Z}_2[x, y]/(x^2, xy, y^2)$ and $R_2 = F_4[x]/(x^2)$. Even though $R_1$ is not isomorphic to $R_2$, but $\Gamma(R_1) = \Gamma(R_2)$. On the other hand, we can note that (1) all the vertices of $\Gamma(R_1)$ (resp. $\Gamma(R_2)$) are nilpotent; (2) the adjacency matrix of $R_1$ is equal to the one of $R_2$; (3) the regular action in $R_1$ is trivial but the regular action in $R_2$ is transitive.

**Proposition 2.2.** Let $R$ be a commutative ring such that $X$ is a union of 2 orbits $o(x)$ and $o(y)$ under the regular action on $X$ by $G$. If $\Gamma_{o(x)}(R)$ and $\Gamma_{o(y)}(R)$ are complete, then $\Gamma_X(R)$ is complete.

*Proof.* Note that the set of all the vertices of $\Gamma(R)$ is $X = o(x) \cup o(y)$. Since $\Gamma_X(R)$ is connected by [3, Theorem 2.3], there exists $a \in o(x)$ and $b \in o(y)$ such that $ab = 0$. Let $x_1 \in o(x)$ (resp. $y_1 \in o(y)$) be arbitrary. Then $x_1 = ga$ and $y_1 = hb$ for some $g, h \in G$, and then $x_1 y_1 = (gh)(ab) = 0$. Hence $X$ is complete.

**Lemma 2.3.** Let $R$ be a commutative ring such that $X$ is a union of finite number of orbits under the regular action on $X$ by $G$. Then for each $x \in X$, $av(x)$ is a union of finite number of orbits.

*Proof.* It follows from the observation that $av(x) = \bigcup_{g \in av(x)} o(y)$ for each $x \in X$.

**Theorem 2.4.** Let $R$ be a commutative ring such that $X$ is a union of a finite number of orbits under the regular action on $X$ by $G$. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if $R$ is a local ring or $R \simeq \mathbb{Z}_2 \times F$ where $F$ is a field.
Proof. (⇒) Suppose that there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. Then $R \cong \mathbb{Z}_2 \times A$ (A is an integral domain) or $Z(R)$ is an annihilator ideal by [1, Theorem 2.5]. Let $X$ be a union of $n$ distinct orbits under the regular action on $X$ by $G$. Then $Z(R)* = X$ by [7, Lemma 2.1]. Since $Z(R)* = X$, in case that $R \cong \mathbb{Z}_2 \times A$, $A$ must be a field; in case that $Z(R)$ is an annihilator ideal, $Z(R) = X \cup \{0\}$ is an ideal, which means that $R$ is a local ring.

(⇐) Suppose that $R$ is a local ring. Then there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$ and $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ by [7, Lemma 2.3]. Thus $av(x^n) = X$ and so there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. Suppose that $R \cong \mathbb{Z}_2 \times F$ where $F$ is a field. Without loss of generality, we can let $R = \mathbb{Z}_2 \times F$. Then there exists $(1,0) \in R$ such that $av((1,0)) = X$, and so there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. □

Corollary 2.5. Let $R$ be a finite commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$ if and only if $R$ is a local ring or $R \cong \mathbb{Z}_2 \times F$ where $F$ is a finite field.

Proof. Since $R$ is a finite commutative ring, clearly $X$ is a union of finite number of orbits under the regular action of $G$ on $X$. Hence it follows from Theorem 2.4. □

Proposition 2.6. Let $R$ be a commutative ring with $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ under the regular action on $X$ by $G$ for some positive integer $n$. If $n = 1$ and $|X| \geq 3$, or $n = 2$ and $o(x^2) \neq \{x^2\}$, or $n = 3$ and $o(x^i) \neq \{x^i\}$ for some $i = 2$ or $3$, or $n \geq 4$, then there exists a cycle of length 3 in $\Gamma(R)$.

Proof. If $n = 1$, i.e., the regular action is transitive, then $\Gamma(R)$ is complete by Proposition 2.1. Since $|X| \geq 3$, there exists a cycle of length 3 in $\Gamma(R)$. If $n = 2$ and $o(x^2) \neq \{x^2\}$, then there exists $g \in G$ such that $gx^2 \neq x^2$. Since $X = o(x) \cup o(x^2)$ and $x^2g \in X$, $gx^2 = hx$ or $hx^2$ for some $h \in G$. Thus $x^2 \rightarrow x \rightarrow gx^2 \rightarrow x^2$ is a cycle of length 3. If $n = 3$ and $o(x^i) \neq \{x^i\}$ for some $i = 2$ or $3$, then there exists $g \in G$ such that $gx^i \neq x^i$. Since $X = o(x) \cup o(x^2) \cup o(x^3)$ and $gx^i \in X$, $gx^i = hx$ or $hx^i$ or $hx^3$ for some $h \in G$. Thus $x^2 \rightarrow x \rightarrow gx^i \rightarrow x^3$ is a cycle of length 3. Finally, if $n \geq 4$, then clearly $x^{n-2} \rightarrow x^{n-1} \rightarrow x^n \rightarrow x^{n-2}$ is a cycle of length 3. □

Theorem 2.7. Let $R$ be a local commutative ring such that $X$ is a union of $n$ distinct orbits under the regular action of $G$ on $X$. Then the set of all the distinct nonzero proper ideals of $R$ consists of $\{\{0\}, J, J^2, \ldots, J^n, R\}$.

Proof. Since $R$ is a local ring with identity such that $X$ is a union of $n$ distinct orbits under the regular action of $G$ on $X$, there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$, $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$ by [7, Lemma 2.3] and also $J^i \neq \{0\} = J^{n+1}$ by [7, Lemma 2.9]. Thus $J \supset J^2 \supset \cdots \supset J^n$ and $J^i \neq J^j$ for all $i, j = 1, \ldots, n$ ($i \neq j$). Consider $av(x), av(x^2), \ldots, av(x^n)$. 

Let \( x^{n-i+1}, x^{n-i+2}, \ldots, x^n \in av(x^i) \) for all \( i = 1, 2, \ldots, n \), we can have that \( av(x^i) = o(x^{n-i+1}) \cup o(x^{n-i+2}) \cup \cdots \cup o(x^n) \). Also we can note that \( av(x^i) \neq av(x^k) \) and \( av(x^i) \supseteq av(x^k) \) for all \( j, k \) \( (n \geq j > k \geq 1) \). Next, we will show that \( J^k = \text{ann}(x^{n-k+1})(av(x^{n-k+1}) \cup \{0\}) \) for all \( k = 1, \ldots, n \) by using induction on \( n \). When \( n = 1 \), \( J = \text{ann}(x) \) since \( x = av(x) \). Assume that \( J^k = \text{ann}(x^{n-k+1}) \) holds. To show that \( J^{k+1} = \text{ann}(x^{n-k}) \), let \( y(\neq 0) \in J^{k+1} \) be arbitrary. Then \( y = y_1y_2 \) for some \( y_1 \in J^k, y_2 \in J \). By assumption, \( y_1 \in \text{ann}(x^{n-k+1}) \) and \( y_2 \in \text{ann}(x^n) \). Since \( \text{ann}(x^{n-k+1}) \setminus \{0\} = av(x^{n-k+1}) \cup o(x^{n-k+1}) \cup \cdots \cup o(x^n) \) and \( \text{ann}(x^n) \setminus \{0\} = av(x^n) = o(x) \cup o(x^2) \cup \cdots \cup o(x^n) \), \( y_1 = ax^k, y_2 = bx \) for some \( a, b \in R \setminus \{0\} \). Thus \( y^{n-k} = y_1y_2x^{n-k} = abx^{k+1}x^{n-k} = abx^{n+1} = 0 \), which implies \( y \in av(x^{n-k}) \).

Hence \( J^{k+1} \subseteq \text{ann}(x^{n-k}) \). To show the converse inclusion holds, let \( z \in \text{ann}(x^{n-k}) \) be arbitrary. Then \( x^{n-k} = 0 \). Since \( \text{ann}(x^{n-k}) \setminus \{0\} = av(x^{n-k}) = o(x^{n-k+1}) \cup o(x^{n-k+2}) \cup \cdots \cup o(x^n) \), \( z \in o(x^i) \) for some \( i (n \geq i \geq k+1) \), and so \( z = gx^i \) for some \( g \in G \). Thus \( z = gx^i = (gx)(x^{i-1}) \in J^{k+1} \) since \( gx \in J \) and \( x^{i-1} \in J \). Thus \( J^{k+1} \supseteq \text{ann}(x^{n-k}) \). Hence we have \( J^{k+1} = \text{ann}(x^{n-k}) \).

Let \( A = \{J, J^2, \ldots, J^n\} \). Therefore, \( J^k = \text{ann}(x^{n-k+1})(av(x^{n-k+1}) \cup \{0\}) \) for all \( k = 1, \ldots, n \). Finally, we will show that for any nonzero proper ideal \( I \) of \( R \), \( I \subseteq A \). Since \( I \) is a nonzero ideal of \( R \), there exists \( y \in X \). Since \( x = o(x) \cup o(x^2) \cup \cdots \cup o(x^n) \), \( y \in o(x^i) \) for some \( i \), and then \( o(x^i) \subseteq I \). Since \( x^i \in I \) and \( I \) is an ideal of \( R, x^i, x^{i+1}, \ldots, x^n \in I \), and so \( o(x^i), o(x^{i+1}), \ldots, o(x^n) \subseteq I \), which implies that \( J^i = o(x^i) \cup \cdots \cup o(x^n) \cup \{0\} \subseteq I \). If \( I \neq J^i \), then there exists \( z \in I \setminus J^i \). Then \( z \in o(x^i) \) for some \( j (i > j \geq 1) \). By the same argument given as above, \( J^j \subseteq I \) \( (i > j) \). If \( I \neq J^1 \), then we will continue in this way. Since \( A = \{J, J^2, \ldots, J^n\} \) is a finite set of ideals in \( R \), it must be \( J^k \) for some \( k \) \( (n \geq k \geq 1) \). Hence the set of all ideals of \( R \) consists of \( \{0\}, J, J^2, \ldots, J^n, R \). \( \square \)

For any set \( S \), we denote the cardinality of \( S \) by \( |S| \).

**Corollary 2.8.** Let \( R \) be a local commutative ring such that \( X \) is a union of \( n \) orbits under the regular action of \( G \) on \( X \). If \( S = \{av(a) : \forall a \in X\} \), then \( S = \{J^i \setminus \{0\} : i = 1, \ldots, n\} \), and so \( |S| = n \).

**Proof.** Let \( I^*_a = av(a) \) for all \( a \in X \). Then \( I^*_a \) is a union of some orbits by Lemma 2.3. Since \( I_a = I^*_a \cup \{0\} = \text{ann}(a) \) is an ideal of \( R \), \( I_a = J^k \) for some \( k \) \( (n \geq k \geq 1) \) by Theorem 2.7. In the proof in Theorem 2.7, \( J^k = \text{ann}(x^{n-k+1}) \).

Hence we have the result from Theorem 2.7. \( \square \)

**Corollary 2.9.** Let \( R \) be a finite local commutative ring such that \( X \) is a union of \( n \) orbits under the regular action of \( G \) on \( X \) and let \( m \) be the number of all ideals of \( R \). Then

\[
m - 2 = n = \frac{1}{|G|} \sum_{g \in G} |X_g|,
\]

where \( X_g = \{ x \in X : gx = x \} \).
Proof. It follows from the Theorem 2.7 and the Burnside’s formula. 

Lemma 2.10. Let \( R = R_1 \times R_2 \times \cdots \times R_t \) be the direct product of commutative rings \( R_1, R_2, \ldots, R_t \) and let \( B = \{ \text{ann}(x) : \forall x \in X \} \cup \{ \{0\}, R \} \) and \( B_i = \{ \text{ann}(x_i) : x_i \in X \} \cup \{ \{0\}, R_i \} \) for all \( i = 1, \ldots, t \) where each \( X_i \) is the set of all nonzero, nonunits of \( R_i \) and \( 0 \) is the additive identity of \( R_i \). Then \( B_1 \times B_2 \times \cdots \times B_t \subseteq B \).

Proof. Let \( b_1 \times b_2 \times \cdots \times b_t \in B_1 \times B_2 \times \cdots \times B_t \) be arbitrary.

Case 1. \( b_i \neq \{0\}, R_i \) for all \( i \), i.e., \( b_i = \text{ann}(x_i) \) for some \( x_i \in X_i \).

Thus \( b_1 \times b_2 \times \cdots \times b_t = \text{ann}(x_1) \times \text{ann}(x_2) \times \cdots \times \text{ann}(x_t) \). Then clearly, \( \text{ann}(x_1) \times \text{ann}(x_2) \times \cdots \times \text{ann}(x_t) \subseteq \text{ann}(x_1, x_2, \ldots, x_t) \in B \).

Case 2. \( b_i = \{0\} \) for some \( i \).

Thus \( b_1 \times b_2 \times \cdots \times b_t = \text{ann}(x_1) \times \cdots \times \{0\} \times \cdots \times \text{ann}(x_t) \). Then \( \text{ann}(x_1) \times \cdots \times \{0\} \times \cdots \times \text{ann}(x_t) \subseteq \text{ann}(x_1, x_2, \ldots, x_t) \in B \), where \( 1 \) is the unity of \( R_i \).

Case 3. \( b_i = R_i \) for some \( i \).

Thus \( b_1 \times b_2 \times \cdots \times b_t = \text{ann}(x_1) \times \cdots \times R_i \times \cdots \times \text{ann}(x_t) \). Then \( \text{ann}(x_1) \times \cdots \times R_i \times \cdots \times \text{ann}(x_t) \subseteq \text{ann}(x_1, \ldots, 0_i, \ldots, x_t) \in B \).

Case 4. \( b_i = \{0\} \) for some \( i \) and \( b_j = R_j \) for some \( j \) (\( i \neq j \)).

Thus by Case 2 and Case 3, \( b_1 \times \cdots \times b_i \times \cdots \times b_j = \text{ann}(x_1) \times \cdots \times \{0\} \times \cdots \times \text{ann}(x_t) \). Then \( \text{ann}(x_1) \times \cdots \times \{0\} \times \cdots \times R_j \times \cdots \times \text{ann}(x_t) \subseteq \text{ann}(x_1, \ldots, 0_j, \ldots, x_t) \in B \).

Case 5. \( b_i = \{0\} \) or \( b_i = R_i \) for all \( i \).

Thus \( b_1 \times \cdots \times b_i \times \cdots \times b_t = \text{ann}((a_1, \ldots, a_i, \ldots, a_t)) \in B \), where \( a_i = 1 \) or \( a_i = 0 \), for all \( i \). 

Lemma 2.11. Let \( R = R_1 \times R_2 \times \cdots \times R_t \) be the direct product of commutative rings \( R_1, R_2, \ldots, R_t \) and let \( C = \{ o(x) : \forall x \in X \} \cup \{ \{0\}, R \} \) and \( C_i = \{ o(x_i) : \forall x_i \in X \} \cup \{ \{0\}, R_i \} \) for all \( i = 1, \ldots, t \) where each \( X_i \) is the set of all nonzero, nonunits of \( R_i \) and \( 0_i \) is the additive identity of \( R_i \). Then \( C \subseteq C_1 \times C_2 \times \cdots \times C_t \).

Proof. Let \( c \in C \) be arbitrary

Case 1. \( c = \{0\} \) or \( c = R \).

Then clearly, \( c \in C_1 \times C_2 \times \cdots \times C_t \).

Case 2. \( c = o(x) \) for some \( x = (x_1, \ldots, x_t) \in X \).

Subcase 1. \( x_i \in X_i \) for all \( i \).

Subcase 2. \( x_i = 0 \), for some \( i \).

Then \( c = o(x) = o((x_1, \ldots, 0_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times \{0\} \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_t \).

Subcase 3. \( x_i = 1 \), for some \( i \).

Then \( c = o(x) = o((x_1, \ldots, 1_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times R_i \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_t \).

Subcase 4. \( x_i = 0 \), for some \( i \) and \( x_j = 1 \), for some \( j \) (\( i \neq j \)).

Thus by Subcase 2 and Subcase 3, \( c = o(x) = o((x_1, \ldots, 0_i, \ldots, x_t)) \subseteq o(x_1) \times \cdots \times \{0\} \times \cdots \times R_j \times \cdots \times o(x_t) \in C_1 \times \cdots \times C_i \times \cdots \times C_j \times \cdots \times C_t \).
Remark 1. Let $R$ be a commutative ring such that $X$ is a union of a finite number of orbits under the regular action of $G$ on $X$. Then $R$ is an Artinian ring since $I \setminus \{0\}$ is a union of some orbits for every ideal $I$ of $R$ by Lemma 2.3. Therefore, $R$ is a finite direct product of Artinian local rings, say $R = R_1 \times R_2 \times \cdots \times R_t$ with each $R_i$ Artinian local ring ($i = 1, \ldots, n$).

Theorem 2.12. Let $R$ be a commutative ring such that $X$ is a union of a finite number of orbits under the regular action of $G$ on $X$ and let $R = R_1 \times R_2 \times \cdots \times R_t$ where each $R_i$ is Artinian local ring ($i = 1, \ldots, n$) as mentioned in Remark 1. Then

$(1)$ for all ideal $I$ of $R$, $I = I_1 \times I_2 \times \cdots \times I_t$ where $I_i \in \{\{0\}, J_i, J_i^2, \ldots, J_i^{n_i}, \ R_i\}$ $(\{0\}$ is the zero ideal of $R_i$ and $J_i$ is the Jacobson radical of $R_i$ with $J_i^{n_i} \neq \{0\}$) for all $i = 1, \ldots, t$.

$(2)$ the number of all nonzero proper ideals of $R$ is $(n_1 + 2) \cdots (n_t + 2) - 2$, the number of orbits under the regular action of $G$ on $X$ and is equal to $|\{\text{av}(x) : \forall x \in X\}|$ and is less than or equal to $|\{o(x) : \forall x \in X\}|$.

Proof. $(1)$ Note that any ideal $I$ of $R$ is of the form $I_1 \times I_2 \times \cdots \times I_t$ where $I_i$ is an ideal of $R_i$ for all $i = 1, \ldots, n$. Since $R_i$ is a local commutative ring for all $i = 1, \ldots, n$, $I_i \in \{\{0\}, J_i, J_i^2, \ldots, J_i^{n_i}, R_i\}$ by Theorem 2.7 and so we have the result.

$(2)$ Let $A$ (resp. $A_i$) be the set of all ideals of $R$ (resp. the set of all ideals of $R_i$) for all $i = 1, \ldots, t$. Then $B = \{\text{ann}(x) : \forall x \in X\} \cup \{\{0\}, R\}$ and $C = \{o(x) : \forall x \in X\}$ by $(1)$. $A = A_1 \times \cdots \times A_t$ and so $|A| = \prod_{i=1}^{t} |A_i| = (n_1 + 2) \cdots (n_t + 2)$. In the proof of Theorem 2.7, we have that

$(\ast)$

$$A_i = \{\{0\}, J_i, \ldots, J_i^{n_i}, R_i\}$$

with $J_i^{n_i+1} = \{0\}$ and $J_i^{k_i} = \text{ann}(x_i^{n_i-k_i+1})$ for some $x_i \in X_i$, the set of all nonzero, nonunits of $R_i$ for all $i = 1, \ldots, t$ where $n_i \geq k_i \geq 1$. Since for all $x \in X$, $\text{ann}(x)$ is a nonzero proper ideal of $R$, $B \subseteq A$, and so $(|A| - 2) \geq |\{\text{ann}(x) : \forall x \in X\}|$. Let $B_i = \{\text{ann}(x_i) : \forall x_i \in X\} \cup \{\{0\}, R_i\}$ for all $i = 1, \ldots, t$. Clearly, $A_i \subseteq B_i$ for all $i = 1, \ldots, t$. By above $(\ast)$, we have $B_i \subseteq A_i$ for all $i = 1, \ldots, t$. By Lemma 2.10, we have $B_i \times \cdots \times B_t \subseteq B$. Hence $B \subseteq A = A_1 \times \cdots \times A_t = B_1 \times \cdots \times B_t = B$, and so $A = B$. Therefore, $|A| - 2 = |\{\text{ann}(x) : \forall x \in X\}| = |\{o(x) : \forall x \in X\}|$.

On the other hand, let $C_i = \{o(y_i) : \forall y_i \in X\} \cup \{\{0\}, R_i\}$ for all $i = 1, \ldots, t$. By Lemma 2.11, we also have $C \subseteq C_1 \times \cdots \times C_t$, and so $|C| \leq |C_1| \times \cdots \times |C_t|$. Since $|B_i| = |\{\text{av}(y_i) : \forall y_i \in X\} \cup \{\{0\}, R_i\}| = |C_i|$ for all $i = 1, \ldots, t$ by Corollary 2.8, $|A| = |B| = |B_1| \times \cdots \times |B_t| = |C_1| \times \cdots \times |C_t| \geq |C|$.

We can have the following question:

Question 1. Let $R$ be a commutative ring with identity such that $X$ is a union of $n$ orbits under the regular action of $G$ on $X$. Is $|\{\text{av}(x) : \forall x \in X\}| = |\{o(x) : \forall x \in X\}|$?
Example 3. Let \( R = \mathbb{Z}_{36} \). Then \( R \) has 7 nonzero proper ideals. We can compute that \( av(x) \) and \( o(x) \) for all \( x \in X \) as follows: \( av(2) = 18R = \{18\}, av(3) = 12R = \{12, 24\}, av(4) = 9R = \{9, 18, 27\}, av(6) = 6R = \{6, 12, 18, 24, 30\} \), \( av(9) = 4R = \{4, 8, \ldots, 32\} \), \( av(12) = 3R = \{3, 6, \ldots, 33\} \), \( av(2) = 2R = \{2, 4, \ldots, 34\} \); \( o(18) = \{18\} \), \( o(6) = \{6, 30\} \), \( o(9) = \{9, 27\} \), \( o(12) = \{12, 24\} \), \( o(3) = \{3, 15, 21, 33\} \), \( o(2) = \{2, 10, 14, 22, 26, 34\} \) and \( o(4) = \{4, 8, 16, 20, 28, 32\} \). Note that the number of \( av(x) \)'s is 7 and is equal to the number of \( o(x) \)'s.

Example 4. Let \( R = \mathbb{Z}_3[x]/(x^3) \) and for simple notation, denote \( f(x) = f(x) + (x^3) \in R \) for all \( f(x) \in \mathbb{Z}_3[x] \). Then \( X = \{x, 2x, x^2, 2x^2, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\} \) and \( R \) has 2 nonzero proper ideals \( xR \) and \( x^2R \). We can also compute that \( av(y) \) and \( o(y) \) for all \( y \in X \) as follows: \( av(x) = \{x, 2x^2\}, av(x^2) = \{x, 2x, x^2, 2x^2, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\} \); \( o(x^2) = \{x^2, 2x^2\} \), \( o(x) = \{x, 2x, x + x^2, 2x + x^2, x + 2x^2, 2x + 2x^2\} \). Note that the number of \( av(y) \)'s is 2 and is also equal to the number of \( o(y) \)'s.

3. Zero-divisor graph of regular rings

In [4], it has been shown that if \( R \) is a unit-regular ring, then for every orbit \( o(x) (x \in X) \) under the regular action of \( G \) on \( X \), there exists some idempotent \( e \in X \) such that \( o(x) = o(e) \). Note that for a commutative ring \( R \) with identity, \( R \) is regular if and only if \( R \) is unit-regular.

Proposition 3.1. Let \( R \) be a commutative regular ring. Then \( \Gamma_X(R) \) is complete if and only if the set of all idempotents in \( R \) is orthogonal and the regular action of \( G \) on \( X \) is trivial, i.e., \( o(x) = \{x\} \) for all \( x \in X \).

Proof. \((\Rightarrow)\) Suppose that \( \Gamma_X(R) \) is complete. Clearly, the set of all idempotents in \( R \) is orthogonal. Assume that the regular action of \( G \) on \( X \) is not trivial. Then there exists \( y \in X \) such that \( o(y) \neq \{y\} \). By [8, Lemma 2.3], there exists idempotent \( e (= y) \in X \) such that \( y = ge \) for some \( g \in G \). Since \( \Gamma_X(R) \) is complete and \( y, e \in X \), \( 0 = ge = (ge)e = ge = y \), a contradiction. Hence the regular action of \( G \) on \( X \) is trivial.

\((\Leftarrow)\) Suppose that the set of all idempotents in \( R \) is orthogonal and the regular action of \( G \) on \( X \) is trivial. Let \( x, y (x \neq y) \in X \) be arbitrary. By [8, Lemma 2.3], there exist idempotents \( e_1, e_2 \in X \) such that \( o(x) = o(e_1) \) and \( o(y) = o(e_2) \). Since the regular action of \( G \) on \( X \) is trivial, \( \{x\} = o(x) = o(e_1) = \{e_1\} \) and \( \{y\} = o(y) = o(e_2) = \{e_2\} \), and so \( x = e_1, y = e_2 \). Since \( x \neq y, e_1 \neq e_2 \) and so \( xy = e_1e_2 = 0 \) by assumption. Thus \( \Gamma_X(R) \) is complete. \(\square\)

Lemma 3.2. Let \( R \) be a commutative regular ring. Then the following are equivalent:

1. \( x^2 = x \) for all \( x \in X \);
2. the regular action of \( G \) on \( X \) is trivial,
3. \( G = \{1\} \).
Proof. (1) ⇒ (2). Suppose that \( x^2 = x \) for all \( x \in X \). Let \( y \in o(x) \) be arbitrary. Then \( y = gx \) for some \( g \in G \). Since \( y \in X \), \( y^2 = y \) by assumption, and then \( y^2 = (gx)^2 = g^2x = y = gx \), which implies \( y = gx = x \), and so \( o(x) = \{x\} \). Thus the regular action of \( G \) on \( X \) is trivial.

(2) ⇒ (3). Suppose that the regular action of \( G \) on \( X \) is trivial and let \( e \in X \) be an idempotent. Then \( o(e) = \{e\} \) and \( o(1 - e) = \{1 - e\} \), and so \( ge = e \) and \( g(1 - e) = 1 - e \) for all \( g \in G \). Thus \( g - e = g(1 - e) = 1 - e \), which implies \( g = 1 \). Thus \( G = \{1\} \).

(3) ⇒ (1). Suppose that \( G = \{1\} \). Let \( x \in X \) be arbitrary. Since \( G = \{1\} \), \( o(x) = \{x\} \), and so \( o(x) = \{x\} = e \) for some idempotent \( e \in X \) by [8, Lemma 2.3]. Hence \( x^2 = x \) for all \( x \in X \).

\[ \square \]

Corollary 3.3. Let \( R \) be a commutative regular ring. Then \( \Gamma_X(R) \) is complete if and only if the set of all idempotents in \( R \) is orthogonal and one of the statements in Lemma 3.2 is satisfied.

\[ \square \]

Proof. It follows from Proposition 3.1 and Lemma 3.2.

Remark 2. Let \( R \) be a ring. If the regular action of \( G \) on \( X \) is transitive, then there exists no idempotent in \( X \). Indeed, assume that there exists an idempotent \( e \in X \). Since the regular action of \( G \) on \( X \) is transitive, \( X = o(1 - e) \), and then \( e = g(1 - e) \) for some \( g \in G \). Thus \( 0 = e(1 - e) = g(1 - e)^2 = g(1 - e) \), and so \( 1 = e \), a contradiction. Therefore for a unit-regular (commutative regular) ring \( R \) with identity, there is no transitive regular action of \( G \) on \( X \) by the above argument and [8, Lemma 2.3].

Proposition 3.4. Let \( R \) be a commutative regular ring with \( X \neq \emptyset \). Then for each \( x \in X \), there exists an idempotent \( e \in X \) such that \( av(x) = av(e) \).

Proof. By [8, Lemma 2.3], for each \( x \in X \) there exists an idempotent \( e \in X \) such that \( o(x) = 0(e) \). Thus \( e = gx \) for some \( g \in G \), and then \( av(e) = av(x) \).

\[ \square \]

Proposition 3.5. Let \( R \) be a commutative regular ring such that \( 2 = 2 \cdot 1 \) is a unit in \( R \). Then there exists a cycle of length \( 4 \) in \( \Gamma(R) \).

Proof. Let \( e \in X \) be an idempotent. Since \( 2 = 2 \cdot 1 \in G \), \( e \neq 1 - e, -e \). Thus \( e \leftrightarrow -e \leftrightarrow e - 1 \leftrightarrow e \) is a cycle of length \( 4 \) in \( \Gamma(R) \).

\[ \square \]

We note that for any idempotent \( e(\neq 0,1) \) in a commutative regular ring \( R \), under the regular action of \( G \) on \( X \), \( o(1 - e) \subseteq av(e) \). In particular, if \( R = F_1 \times F_2 \) (\( F_1, F_2 : \text{fields} \)), then \( o(1 - e) = av(e) \) for all idempotent \( e(\neq 0,1) \in R \).

We raise the following question:

Question 2. For any idempotent \( e(\neq 0,1) \) in a commutative regular ring \( R \) with identity, when is \( o(1 - e) = av(e) \)?
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References


Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: jchan@pusan.ac.kr