COMMON FIXED POINT OF COMPATIBLE MAPS OF TYPE (γ) ON COMPLETE FUZZY METRIC SPACES

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ABSTRACT. In this paper, we establish a common fixed point theorem in complete fuzzy metric spaces which generalizes some results in [9].

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [16] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and ε(∞) theory which were given and studied by El Naschie [1, 2, 3, 4, 15]. Many authors [6, 10, 13] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous t-norm if it satisfies the following conditions

1. \( * \) is associative and commutative,
2. \( * \) is continuous,
3. \( a \ast 1 = a \) for all \( a \in [0, 1] \),
4. \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).

Two typical examples of continuous t-norm are \( a \ast b = ab \) and \( a \ast b = \min(a, b) \).

Definition 1.2. A 3-tuple \( (X, M, \ast) \) is called a fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( \ast \) is a continuous t-norm, and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \), satisfying the following conditions for each \( x, y, z \in X \) and \( t, s > 0 \),

1. \( M(x, y, t) > 0 \),
2. \( M(x, y, t) = 1 \) if and only if \( x = y \),
3. \( M(x, y, t) = M(y, x, t) \),
4. \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \),
(5) \(M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]\) is continuous.

Let \((X, M, \ast)\) be a fuzzy metric space. For \(t > 0\), the open ball \(B(x, r, t)\) with center \(x \in X\) and radius \(0 < r < 1\) is defined by

\[
B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.
\]

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\[
B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.
\]

Let \((X, M, \ast)\) be a fuzzy metric space. Let \(\tau\) be the set of all \(A \subset X\) with \(x \in A\) if and only if there exist \(t > 0\) and \(0 < r < 1\) such that \(B(x, r, t) \subset A\). Then \(\tau\) is a topology on \(X\) (induced by the fuzzy metric \(M\)). This topology is Hausdorff and first countable. A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if there exist \(t > 0\) and \(0 < r < 1\) such that \(B(x, r, t) \subset A\). Then \(\tau\) is a topology on \(X\) (induced by the fuzzy metric \(M\)). This topology is Hausdorff and first countable. A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if there exist \(t > 0\) and \(0 < r < 1\) such that \(B(x, r, t) \subset A\).

**Example 1.3.** Let \(X = \mathbb{R}\). Denote \(a \ast b = a \cdot b\) for all \(a, b \in [0, 1]\). For each \(t \in (0, \infty)\), define

\[
M(x, y, t) = \frac{t}{t + |x - y|}
\]

for all \(x, y \in X\).

**Lemma 1.4.** Let \((X, M, \ast)\) be a fuzzy metric space. Then \(M(x, y, t)\) is non-decreasing with respect to \(t\), for all \(x, y\) in \(X\).

**Definition 1.5.** Let \((X, M, \ast)\) be a fuzzy metric space. \(M\) is said to be continuous on \(X^2 \times (0, \infty)\) if

\[
\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t).
\]

Whenever a sequence \(\{(x_n, y_n, t_n)\}\) in \(X^2 \times (0, \infty)\) converges to a point \((x, y, t) \in X^2 \times (0, \infty)\), i.e.,

\[
\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).
\]

**Lemma 1.6.** Let \((X, M, \ast)\) be a fuzzy metric space. Then \(M\) is continuous function on \(X^2 \times (0, \infty)\).

**Proof.** See Proposition 1 of [11].

**Definition 1.7.** Let \(A\) and \(S\) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, \(Ax = Sx\) implies that \(ASx = SAx\).

**Definition 1.8.** Let \(A\) and \(S\) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be compatible if

\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Proposition 1.9 ([14]).** Self-mappings \( A \) and \( S \) of a fuzzy metric space \( (X, M, *) \) are compatible, then they are weak compatible.

The converse is not true as seen in following example.

**Example 1.10.** Let \( (X, M, *) \) be a fuzzy metric space, where \( X = [0, 2] \), with t-norm defined \( a * b = \min\{a, b\} \) for all \( a, b \in [0, 1] \) and \( M(x, y, t) = \frac{t}{t + d(x, y)} \) for all \( t > 0 \) and \( x, y \in X \). Define self-maps \( A \) and \( S \) on \( X \) as follows:
\[
Ax = \begin{cases} 
2 & \text{if } 0 \leq x \leq 1, \\
\frac{x}{2} & \text{if } 1 < x \leq 2,
\end{cases}
\]
and \( x_n = 2 - \frac{1}{2n} \). Then we have \( S1 = A1 = 2 \) and \( S2 = A2 = 1 \). Also \( SA1 = AS1 = 1 \) and \( SA2 = AS2 = 2 \). Thus \( (A, S) \) is weak compatible. Again,
\[
Ax_n = 1 - \frac{1}{4n}, \quad Sx_n = 1 - \frac{1}{10n}.
\]
Thus,
\[
Ax_n \to 1, \quad Sx_n \to 1.
\]
Further,
\[
SAx_n = \frac{4}{5} - \frac{1}{20n}, \quad ASx_n = 2.
\]
Now,
\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = \lim_{n \to \infty} M\left(2, \frac{4}{5} - \frac{1}{20n}, t\right) = \frac{t}{t + \frac{4}{5}} < 1, \quad \forall \ t > 0.
\]
Hence \( (A, S) \) is not compatible.

**2. Compatible maps of type \((\gamma)\)**

In this section, we give the concept of compatible maps of type \((\gamma)\) in fuzzy metric spaces and some properties of these maps.

**Definition 2.1.** Let \( A \) and \( S \) be mappings from a fuzzy metric space \( (X, M, *) \) into itself. Then the mappings are said to be compatible maps of type \((\gamma)\) if satisfying:
(i) \( A \) and \( S \) are compatible, that is
\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X,
\]
(ii) they are continuous at \(x\).

On the other hand we have,
\[
A(x) = A\left( \lim_{n \to \infty} Ax_n \right) = A\left( \lim_{n \to \infty} Sx_n \right) = \lim_{n \to \infty} ASx_n = S\left( \lim_{n \to \infty} Ax_n \right) = S(x).
\]

**Definition 2.2.** Let \(A\) and \(S\) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. The maps \(A\) and \(S\) are said to be weak compatible of type \((\gamma)\) if
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x
\]
for some \(x \in X\) implies that \(Ax = Sx\).

Clearly if self-mappings \(A\) and \(S\) of a fuzzy metric space \((X, M, \ast)\) are compatible of type \((\gamma)\), then they are weak compatible of type \((\gamma)\). But the converse is not true as seen in following example.

**Example 2.3.** Let \((X, M, \ast)\) be a fuzzy metric space, where \(X = [0, 2]\), with t-norm defined \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\) and 
\[
M(x, y, t) = \frac{t}{t + d(x, y)}
\]
for all \(t > 0\) and \(x, y \in X\). Define self-maps \(A\) and \(S\) on \(X\) as follows:
\[
Ax = \begin{cases} 1 & \text{if } x \in Q, \\ \frac{1}{2} & \text{otherwise}, \end{cases} \quad Sx = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{otherwise}, \end{cases}
\]
and \(x_n = 2 - \frac{1}{2^n}\). Then we have
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = 1,
\]
and \(S1 = A1 = 1\). That is \((A, S)\) is weak compatible of type \((\gamma)\) also \((A, S)\) is compatible, for \(Ax_n = A(2 - \frac{1}{2^n}) = 1\) and \(Sx_n = S(2 - \frac{1}{2^n}) = 1\) hence
\[
\lim_{n \to \infty} ASx_n = \lim_{n \to \infty} SAx_n = 1,
\]
but \(A, S\) are not continuous at 1 and hence \((A, S)\) is not compatible of type \((\gamma)\).

**Lemma 2.4.** Let \((X, M, \ast)\) be a fuzzy metric space.

(i) If we define \(E_{\lambda, M} : X^2 \to [0, \infty] \cup \{0\}\) by
\[
E_{\lambda, M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\}
\]
for each \(\mu \in (0, 1)\) there exists \(\lambda \in (0, 1)\) such that
\[
E_{\lambda, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n)
\]
for any \(x_1, x_2, \ldots, x_n \in X\).

(ii) The sequence \(\{x_n\}_{n \in \mathbb{N}}\) is convergent in fuzzy metric space \((X, M, \ast)\) if and only if \(E_{\lambda, M}(x_n, x) \to 0\). Also the sequence \(\{x_n\}_{n \in \mathbb{N}}\) is Cauchy sequence if and only if it is Cauchy with \(E_{\lambda, M}\).
Proof. (i) For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that
\[
(1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda) \geq 1 - \mu
\]
by triangular inequality we have
\[
M(x_1, x_n, E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta)
\geq M(x_1, x_2, E_{\lambda,M}(x_1, x_2) + \delta) \ast \cdots \ast M(x_{n-1}, x_n, E_{\lambda,M}(x_{n-1}, x_n) + \delta)
\geq (1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda) \geq 1 - \mu
\]
for very $\delta > 0$, which implies that
\[
E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n).
\]
Since $\delta > 0$ is arbitrary, we have
\[
E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n).
\]
(ii) Note that since $M$ is continuous in its third place and
\[
E_{\lambda,M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\}.
\]
Hence, we have
\[
M(x_n, x, \eta) > 1 - \lambda \iff E_{\lambda,M}(x_n, x) < \eta
\]
for every $\eta > 0$. $\square$

Lemma 2.5. Let $(X, M, \ast)$ be a fuzzy metric space. If
\[
M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)
\]
for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in (0, 1)$ and $x_n, x_{n+1} \in X$, we have
\[
E_{\lambda,M}(x_{n+1}, x_n) = \inf\{t > 0 : M(x_{n+1}, x_n, t) > 1 - \lambda\}
\leq \inf\{t > 0 : M(x_0, x_1, k^n t) > 1 - \lambda\}
= \inf\{\frac{t}{k^n} : M(x_0, x_1, t) > 1 - \lambda\}
= \frac{1}{k^n} \inf\{t > 0 : M(x_0, x_1, t) > 1 - \lambda\}
= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1).
By Lemma 2.4, for every $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that
\[
E_{\mu,M}(x_n, x_m) \leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m)
\]
\[
\leq \frac{1}{k^n} E_{\lambda,M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,M}(x_0, x_1)
\]
\[
= E_{\lambda,M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \to 0.
\]
Hence sequence $\{x_n\}$ is Cauchy sequence. $\Box$

3. The main results

**Lemma 3.1.** Let $P$ and $Q$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying:

(i) there exists a constant $k \in (0, 1)$ such that
\[
M^2(Px, Qy, kt) \ast [M(x, Px, kt)M(y, Qy, kt)]
\]
\[
\ast M^2(y, Qy, kt) + aM(y, Qy, kt)M(x, Qy, 2kt)
\]
\[
\geq [pM(x, Px, t) + qM(x, y, t)]M(x, Qy, 2kt)
\]
for every $x, y$ in $X$ and $t > 0$, where $0 < p, q < 1, 0 \leq a < 1$ such that $p + q - a = 1$. Then $P$ and $Q$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$ be an arbitrary point, there exist $x_1 \in X$ such that $Px_0 = x_1$, $Qx_1 = x_2$. Inductively, construct sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = Px_{2n}$, $x_{2n+2} = Qx_{2n+1}$ for $n = 0, 1, 2, \ldots$.

Now, we prove $\{x_n\}$ is a Cauchy sequence. For $x = x_{2n}$ and $y = x_{2n+1}$ by (i) we have
\[
M^2(Px_{2n}, Qx_{2n+1}, kt) \ast [M(x_{2n}, Px_{2n}, kt)M(x_{2n+1}, Qx_{2n+1}, kt)]
\]
\[
\ast M^2(x_{2n+1}, Qx_{2n+1}, kt) + aM(x_{2n+1}, Qx_{2n+1}, kt)M(x_{2n+1}, Qx_{2n+1}, 2kt)
\]
\[
\geq [pM(x_{2n}, Px_{2n}, t) + qM(x_{2n}, x_{2n+1}, t)]M(x_{2n}, Qx_{2n+1}, 2kt)
\]
and
\[
M^2(x_{2n+1}, x_{2n+2}, kt) \ast [M(x_{2n+1}, x_{2n+1}, kt)M(x_{2n+1}, x_{2n+2}, kt)]
\]
\[
\ast M^2(x_{2n+1}, x_{2n+2}, kt) + aM(x_{2n+1}, x_{2n+2}, kt)M(x_{2n+1}, x_{2n+2}, 2kt)
\]
\[
\geq [pM(x_{2n}, x_{2n+1}, t) + qM(x_{2n}, x_{2n+1}, t)]M(x_{2n}, x_{2n+2}, 2kt)
\]
then
\[ M^2(x_{2n+1}, x_{2n+2}, kt) \geq [M(x_{2n}, x_{2n+1}, kt)] M(x_{2n+1}, x_{2n+2}, kt) \]
\[ + aM(x_{2n+1}, x_{2n+2}, kt) M(x_{2n}, x_{2n+2}, 2kt) \]
\[ \geq (p + q)M(x_{2n}, x_{2n+1}, t) M(x_{2n}, x_{2n+2}, 2kt). \]
Hence we have
\[ M(x_{2n+1}, x_{2n+2}, kt) M(x_{2n}, x_{2n+2}, 2kt) \]
\[ + aM(x_{2n+1}, x_{2n+2}, kt) M(x_{2n}, x_{2n+2}, 2kt) \]
\[ \geq (p + q)M(x_{2n}, x_{2n+1}, t) M(x_{2n}, x_{2n+2}, 2kt) \]
so
\[ M(x_{2n+1}, x_{2n+2}, kt) + aM(x_{2n+1}, x_{2n+2}, kt) \geq (p + q)M(x_{2n}, x_{2n+1}, t) \]
and
\[ M(x_{2n+1}, x_{2n+2}, kt) \geq M(x_{2n}, x_{2n+1}, t) \cdot \frac{p + q}{1 + a} = M(x_{2n}, x_{2n+1}, t). \]
Hence we have
\[ M(x_{2n+1}, x_{2n+2}, kt) \geq M(x_{2n}, x_{2n+1}, t). \]
Similarly, we also have
\[ M(x_{2n+2}, x_{2n+3}, kt) \geq M(x_{2n+1}, x_{2n+2}, t). \]
For \( k \in (0, 1) \) if set \( k_1 = \frac{1}{k} > 1 \) and set \( t = kt_1 \), then we have
\[ M(x_n, x_{n+1}, t_1) \geq M(x_{n-1}, x_{n}, k_1 t_1) \geq \cdots \geq M(x_0, x_1, k^n t_1). \]
Hence by Lemma 2.5 \( \{x_n\} \) is Cauchy and the completeness of \( X \), \( \{x_n\} \) converges to \( z \) in \( X \). That is, \( \lim_{n \to \infty} x_n = z \). Hence
\[ \lim_{n \to \infty} P x_{2n} = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} Q x_{2n+1} = z. \]
Now, taking \( x = z \) and \( y = x_{2n+1} \) in (i), we have
\[ M^2(Pz, Q x_{2n+1}, kt) \geq [M(z, Pz, kt)] M(x_{2n+1}, Q x_{2n+1}, kt) \]
\[ + aM(x_{2n+1}, Q x_{2n+1}, kt) M(z, Q x_{2n+1}, 2kt) \]
\[ \geq [pM(z, Pz, t) + qM(z, x_{2n+1}, t)] M(z, Q x_{2n+1}, 2kt) \]
as \( n \to \infty \)
\[ M^2(Pz, z, kt) \geq [M(z, Pz, kt)] M(z, z, kt) \]
\[ + aM(z, z, kt) M(z, z, 2kt) \]
\[ \geq [pM(z, Pz, t) + qM(z, z, t)] M(z, z, 2kt) \]
then, it follows that
\[ M(Pz, z, t) + a \geq pM(Pz, z, t) + q \]
thus
\[ M(Pz, z, t) \geq \frac{q - a}{1 - p} = 1 \]
for all \( t > 0 \), so \( Pz = z \). Taking \( x = x_{2n} \) and \( y = z \) in (i), we have
\[ M^2(Px_{2n}, Qz, kt) * [M(x_{2n}, Px_{2n}, kt)M(z, Qz, kt)] \]
\[ \times M^2(z, Qz, kt) + aM(z, Qz, kt)M(x_{2n}, Qz, 2kt) \]
\[ \geq [pM(x_{2n}, Px_{2n}, t) + qM(x_{2n}, z, t)]M(x_{2n}, Qz, 2kt) \]
as \( n \to \infty \)
\[ M^2(z, Qz, kt) * [M(z, z, kt)M(z, Qz, kt)] \]
\[ \times M^2(z, Qz, kt) + aM(z, Qz, kt)M(z, Qz, 2kt) \]
then
\[ M(z, Qz, kt)M(z, Qz, 2kt) + aM(z, Qz, 2kt) \geq [p + q]M(z, Qz, 2kt). \]
\[ M(z, Qz, t) + a \geq p + q \]
and
\[ M(z, Qz, t) \geq p + q - a = 1 \]
for all \( t > 0 \), so \( Qz = z \).

Therefore, \( z \) is a common fixed of \( P \) and \( Q \).

Uniqueness, let \( v \) be second common fixed point of \( P \) and \( Q \). Then using inequality (i), we have
\[ M^2(Pz, Qv, kt) * [M(z, Pz, kt)M(v, Qv, kt)] \]
\[ \times M^2(v, Qv, kt) + aM(v, Qv, kt)M(z, Qv, 2kt) \]
\[ \geq [pM(z, Pz, t) + qM(z, v, t)]M(z, Qv, 2kt) \]
so
\[ M^2(z, v, kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt) \]
and
\[ M(z, v, t)M(z, v, 2kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt). \]
Thus, it follows that
\[ M(z, v, t) \geq \frac{p - a}{1 - q} = 1 \]
for all \( t > 0 \), so \( z = v \). Hence \( P \) and \( Q \) have unique common fixed point. \( \square \)
Theorem 3.2. Let $P, S, T$ and $Q$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying:
(i) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$,
(ii) there exists a constant $k \in (0, 1)$ such that
\[ M^2(Px, Qy, kt) \ast [M(Sx, Px, kt)M(Ty, Qy, kt)] \]
\[ \ast M^2(Ty, Qy, kt) + aM(Ty, Qy, kt)M(Sx, Qy, 2kt) \]
\[ \geq [pM(Sx, Px, t) + qM(Sx, Ty, t)]M(Sx, Qy, 2kt) \]
for every $x, y \in X$ and $t > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,
(iii) the pairs $(P, S)$ and $(Q, T)$ are weak compatible of type $(\gamma)$.
Then $P, S, Q$ and $T$ have a unique common fixed point in $X$.

Proof. Let $x_0 \in X$ be an arbitrary point as $P(X) \subseteq T(X), Q(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Px_0 = Tx_1 = y_1$, $Qx_1 = Sx_2 = y_2$. Inductively, construct sequence \{${y_n}$\} and \{${x_n}$\} in $X$ such that $y_{n+1} = Px_n = Tx_{n+1}$, $y_{2n+2} = Qx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Now, we prove \{${y_n}$\} is a Cauchy sequence. For $x = x_{2n}$ and $y = x_{2n+1}$ by (ii) we have
\[ M^2(Px_{2n}, Qx_{2n+1}, kt) \ast [M(Sx_{2n}, Px_{2n}, kt)M(Tx_{2n+1}, Qx_{2n+1}, kt)] \]
\[ \ast M^2(Tx_{2n+1}, Qx_{2n+1}, kt) + aM(Tx_{2n+1}, Qx_{2n+1}, kt)M(Sx_{2n}, Qx_{2n+1}, 2kt) \]
\[ \geq [pM(Sx_{2n}, Px_{2n}, t) + qM(Sx_{2n}, Tx_{2n+1}, t)]M(Sx_{2n}, Qx_{2n+1}, 2kt) \]
and
\[ M^2(y_{n+1}, y_{n+2}, kt) \ast [M(y_{2n}, y_{2n+1}, kt)M(y_{2n+2}, y_{2n+2}, kt)] \]
\[ \ast M^2(y_{2n+1}, y_{2n+2}, kt) + aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt) \]
\[ \geq [pM(y_{2n}, y_{2n+1}, t) + qM(y_{2n}, y_{2n+1}, t)]M(y_{2n}, y_{2n+2}, 2kt) \]
then
\[ M^2(y_{2n+1}, y_{2n+2}, kt) \ast [M(y_{2n}, y_{2n+1}, kt)M(y_{2n+1}, y_{2n+2}, kt)] \]
\[ + aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt) \]
\[ \geq (p + q)M(y_{2n}, y_{2n+1}, t)M(y_{2n}, y_{2n+2}, 2kt). \]
Hence we have
\[ M(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt) \]
\[ + aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt) \]
\[ \geq (p + q)M(y_{2n}, y_{2n+1}, t) \]
so
\[ M(y_{2n+1}, y_{2n+2}, kt) + aM(y_{2n+1}, y_{2n+2}, kt) \geq (p + q)M(y_{2n}, y_{2n+1}, t) \]
and
\[ M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \cdot \frac{p + q}{1 + a} = M(y_{2n}, y_{2n+1}, t). \]
Hence we have
\[ M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t). \]
Similarly, we also have
\[ M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t). \]
For \( k \in (0, 1) \) if set \( k_1 = \frac{1}{k} > 1 \) and set \( t = k_1t_1 \), then we have
\[ M(y_n, y_{n+1}, t_1) \geq M(y_{n-1}, y_n, k_1t_1) \geq \cdots \geq M(y_0, y_1, k_1^n t_1). \]
Hence by Lemma 2.5 \( \{y_n\} \) is Cauchy and the completeness of \( X \), \( \{y_n\} \) converges to \( z \) in \( X \). That is, \( \lim_{n \to \infty} y_n = z \). Hence
\[
\lim_{n \to \infty} Px_{2n} = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} Qx_{2n+1} = \lim_{n \to \infty} Sz_{2n+2} = \lim_{n \to \infty} Sz_{2n} = z.
\]
Since \( P, S \) are weak compatible of type (\( \gamma \)) we get \( Pz = Sz \). Now, taking \( x = z \) and \( y = x_{2n+1} \) in (ii), we have
\[
M^2(Pz, Qx_{2n+1}, kt) \ast [M(Sz, Pz, kt)M(Tx_{2n+1}, Qx_{2n+1}, kt)]
\]
\[
\ast M^2(Tx_{2n+1}, Qx_{2n+1}, kt) + aM(Tx_{2n+1}, Qx_{2n+1}, kt)M(Sz, Qx_{2n+1}, 2kt)
\]
\[
\geq [pM(Sz, Pz, t) + qM(Sz, Tx_{2n+1}, t)]M(Sz, Qx_{2n+1}, 2kt)
\]
as \( n \to \infty \)
\[
M^2(Pz, z, kt) \ast [M(Sz, Pz, kt)M(z, z, kt)]
\]
\[
\ast M^2(z, z, kt) + aM(z, z, kt)M(Sz, z, 2kt)
\]
\[
\geq [pM(Sz, Pz, t) + qM(Sz, z, t)]M(Sz, z, 2kt)
\]
then, it follows that
\[
M^2(Pz, z, kt) + aM(Pz, z, 2kt) \geq [p + qM(Pz, z, t)]M(Pz, z, 2kt)
\]
and since \( M(x, y, \cdot) \) is non-decreasing for all \( x, y \) in \( X \), we have
\[
M(Pz, z, 2kt)M(Pz, z, t) + aM(Pz, z, 2kt) \geq [p + qM(Pz, z, t)]M(Pz, z, 2kt)
\]
thus
\[
M(Pz, z, t) + a \geq p + qM(Pz, z, t)
\]
and
\[
M(Pz, z, t) \geq \frac{p - a}{1 - q} = 1
\]
for all \( t > 0 \) so \( Pz = z \). Therefore \( Pz = Sz = z \).
Similarly since pair $Q, T$ are weak compatible of type ($\gamma$) hence we get $Qz = Tz$. Now, we show that $Qz = z$. For this taking $x = x_{2n}$ and $y = z$ in (ii), we have

\[
M^2(Px_{2n}, Qz, kt) \ast [M(Sx_{2n}, Px_{2n}, kt)M(Tz, Qz, kt)]
\ast M^2(Tz, Qz, kt) + aM(Tz, Qz, kt)M(Sx_{2n}, Qz, 2kt)
\geq \left[pM(Sx_{2n}, Px_{2n}, t) + qM(Sx_{2n}, Tz, t)\right]M(Sx_{2n}, Qz, 2kt)
\]

as $n \rightarrow \infty$

\[
M^2(z, Qz, kt) \ast [M(z, z, kt)M(Tz, Qz, kt)]
\ast M^2(Tz, Qz, kt) + aM(Tz, Qz, kt)M(z, Qz, 2kt)
\geq \left[pM(z, z, t) + qM(z, Tz, t)\right]M(z, Qz, 2kt)
\]

then

\[
M^2(z, Qz, kt) + aM(z, Qz, 2kt) \geq \left[p + qM(z, Tz, t)\right]M(z, Qz, 2kt).
\]

Since $M(x, y, \cdot)$ is non-decreasing for all $x, y$ in $X$, we have

\[
M(z, Qz, kt)M(z, Qz, 2kt) + aM(z, Qz, 2kt) \geq \left[p + qM(z, Tz, t)\right]M(z, Qz, 2kt).
\]

Thus it follows that

\[
M(z, Qz, t) + a \geq p + qM(z, Tz, t)
\]

and

\[
M(z, Qz, t) \geq \frac{p - a}{1 - q} = 1
\]

for all $t > 0$ so $Qz = z$. Hence, we have $Qz = Tz = z$.

Therefore, $z$ is a common fixed of $P, Q, S$ and $T$.

Uniqueness, let $v$ be second common fixed point of $P, Q, S$ and $T$. Then using inequality (ii), we have

\[
M^2(Pz, Qv, kt) \ast [M(Sz, Pz, kt)M(Tv, Qv, kt)]
\ast M^2(Tv, Qv, kt) + aM(Tv, Qv, kt)M(Sz, Qv, 2kt)
\geq \left[pM(Sz, Pz, t) + qM(Sz, Tv, t)\right]M(Sz, Qv, 2kt)
\]

so

\[
M^2(z, v, kt) + aM(z, v, 2kt) \geq \left[p + qM(z, v, t)\right]M(z, v, 2kt)
\]

and

\[
M(z, v, t)M(z, v, 2kt) + aM(z, v, 2kt) \geq \left[p + qM(z, v, t)\right]M(z, v, 2kt).
\]

Thus, it follows that

\[
M(z, v, t) \geq \frac{p - a}{1 - q} = 1
\]

for all $t > 0$ so $z = v$. Hence $P, Q, S$ and $T$ have unique common fixed point. \square
A class of implicit relation

Let \( \{S_\alpha\}_{\alpha \in A} \) and \( \{T_\beta\}_{\beta \in B} \) be the set of all self-mappings of a complete fuzzy metric space \((X, M, *)\).

**Theorem 3.3.** Let \( T, S \) and \( \{P_\alpha\}_{\alpha \in A}, \{Q_\beta\}_{\beta \in B} \) be self-mappings of a complete fuzzy metric space \((X, M, *)\) satisfying:

(i) there exist \( \alpha_0 \in A \) and \( \beta_0 \in B \) such that \( P_{\alpha_0}(X) \subseteq T(X) \), \( Q_{\beta_0}(X) \subseteq S(X) \),

(ii) there exists a constant \( k \in (0, 1) \) such that
\[
M^2(P_\alpha x, Q_\beta y, kt) \ast [M(Sx, P_\alpha x, kt)M(Ty, Q_\beta y, kt)]
\]
\[
\geq [pM(Sx, P_\alpha x, t) + qM(Sx, Ty, t)]M(Sx, Q_\beta y, 2kt)
\]
for every \( x, y \in X \) and every \( \alpha \in A, \beta \in B \) and \( t > 0 \), where \( 0 < p, q < 1, 0 \leq a < 1 \) such that \( p + q - a = 1 \).

(iii) the pairs \( (P_{\alpha_0}, S) \) and \( (Q_{\beta_0}, T) \) are weak compatible of type \( (\gamma) \).

Then \( P_\alpha, S, Q_\beta \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** By Theorem 3.2 \( S, T, P_{\alpha_0} \) and \( Q_{\beta_0} \) for some \( \alpha_0 \in A, \beta_0 \in B \) have a unique common fixed point in \( X \). That is there exist a unique \( a \in X \) such that \( T(a) = S(a) = P_{\alpha_0}(a) = Q_{\beta_0}(a) = a \). Let there exists \( \lambda \in B \) such that \( \lambda \neq \beta_0 \) then we have
\[
M^2(P_{\alpha_0} a, Q_\lambda a, kt) \ast [M(Sa, P_{\alpha_0} a, kt)M(Ta, Q_\lambda a, kt)]
\]
\[
\geq [pM(Sa, P_{\alpha_0} a, t) + qM(Sa, Ta, t)]M(Sa, Q_\lambda a, 2kt)
\]
then
\[
M^2(a, Q_\lambda a, kt) \ast [M(a, a, kt)M(a, Q_\lambda a, kt)]
\]
\[
\geq [pM(a, a, t) + qM(a, a, t)]M(a, Q_\lambda a, 2kt)
\]
so
\[
M^2(a, Q_\lambda a, kt) \ast M(a, Q_\lambda a, kt)
\]
\[
\geq [p + q]M(a, Q_\lambda a, 2kt).
\]
Since \( M(x, y, \cdot) \) is non-decreasing for all \( x, y \in X \), we have
\[
M(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)
\]
\[
+ aM(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)
\]
\[
\geq [p + q]M(a, Q_\lambda a, 2kt).
\]
That is
\[ M(a, Q_\lambda a, kt) \geq \frac{p + q}{1 + a} = 1. \]

Hence for every \( \lambda \in B \) we have \( Q_\lambda(a) = a = T(a) = S(a) \). Similarly for every \( \gamma \in A \) we get \( P_\gamma(a) = a \). Therefore for every \( \gamma \in A, \lambda \in B \) we have
\[ P_\gamma(a) = Q_\lambda(a) = T(a) = S(a) = a. \]

\[ \square \]

References


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