TRANS-SEPARABILITY IN THE STRICT AND COMPACT-OPEN TOPOLOGIES

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Abstract. We give a characterization of trans-separability for the function spaces \((C_b(X, E), \beta), (C(X, E), k)\) and \((C_b(X, E), u)\) in the case of \(E\) any general topological vector space.

1. Introduction

The fundamental result on the characterization of separability of \((C_b(X), u)\) was obtained by M. Krein and S. Krein [12] in 1940. Later, similar results were obtained by Gulick and Schmets [5] and, independently, by Summers [15] for \((C_b(X), k)\) and \((C_b(X), \beta)\). On the other hand, Gulick and Schmets [5] also gave a characterization of seminorm-separability for \((C_b(X), u), (C_b(X), k)\) and \((C_b(X), \beta)\). Characterization of separability for vector-valued function spaces have been considered in [16, 2, 8]. In [9], the author generalised these results by giving a characterization of neighbourhood-separability for the spaces \((C_b(X, E), \beta), (C(X, E), k)\) and \((C_b(X, E), u)\) in the case of \(E\) a ‘semi-convex’ topological vector space (TVS) having non-trivial topological dual \(E'\). The purpose of this note is to extend these results further to the case of \(E\) any general TVS, using the terminology of trans-separability as in [10].

2. Preliminaries

For the convenience of the reader, we recall some terminology so that this note can be read independently of [9, 10]. Let \(X\) denote a completely regular Hausdorff space and \(E\) a non-trivial Hausdorff TVS with \(W\) a base of neighbourhoods of 0. A neighbourhood \(G\) of 0 in \(E\) is called shrinkable [11] if \(rG \subseteq \text{int } G\) for \(0 \leq r < 1\). By ([11], Theorems 4 and 5), every Hausdorff TVS has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional \(\rho_G\) of any such neighbourhood \(G\) is continuous and positively homogeneous.

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Definition 1. The **strict topology** \( \beta \) \([1, 7]\) on \( C_b(X, E) \) is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form
\[
U(\varphi, W) = \{ f \in C_b(X, E) : (\varphi f)(X) \subseteq W \},
\]
where \( \varphi \in B_0(X) \), the set of all bounded scalar-valued functions on \( X \) which vanish at infinity, and \( W \in \mathcal{W} \).

Let \( u \) (resp. \( k \)) denotes the uniform (resp. compact-open topology) on \( C_b(X, E) \) (resp. \( C(X, E) \)). Then \( k \leq \beta \leq u \) on \( C_b(X, E) \).

For any \( \varphi \in B_0(X) \), let \( \| . \|_\varphi \) denote the seminorm on \( C_b(X) \) given by \( \| f \|_\varphi = \sup \{ |\varphi(x)f(x)| : x \in X \} \), \( f \in C_b(X) \). We shall denote by \( C(X) \otimes E \) the vector subspace of \( C(X, E) \) spanned by the set of all functions of the form \( \varphi \otimes a \), where \( \varphi \in C(X), a \in E \), and \( (\varphi \otimes a) = \varphi(x)a, x \in X \).

Recall that a locally convex space \( L \) is called **seminorm-separable** \([5]\) if, for each continuous seminorm \( p \) on \( L \), \((L,p)\) is separable. The following classical result is stated for reference purpose.

**Theorem 1** \([5]\). The following statements are equivalent:
(a) \((C_b(X), \beta)\) is seminorm-separable.
(b) \((C(X), \kappa)\) is seminorm-separable.
(c) Every compact subset of \( X \) is metrizable.

**Definition 2.** A uniform space \( L \) is called **trans-separable** if every uniform cover of \( L \) admits a countable subcover \([6]\). In particular, a TVS \( L \) is trans-separable if, for each neighbourhood \( W \) of 0 in \( L \), the open cover \( \{ a+W : a \in L \} \) of \( L \) admits a countable subcover.

Drewnowski \([3]\) had actually coined the word “trans-separable” and it has been further used by Robertson \([13]\) and Ferrando-Kakol-Pellicer \([4]\). Khan \([9]\) introduced a generalized notion of separability, namely, the neighbourhood-separability in the TVS setting, as follows.

**Definition 3.** Let \( L \) be a TVS, and let \( V \) be a neighbourhood of 0 in \( L \). A subset \( H \) of \( L \) is said to be \( V \)-dense in \( L \) if, for any \( z \in L \) and \( \delta > 0 \), there exists an element \( y \in H \) such that \( y - z \in \delta V \). \( L \) is called **neighbourhood-separable** if, for each neighbourhood \( V \) of 0, there exists a countable \( V \)-dense subset of \( L \).

Another notion of generalized separability may also be considered, as follows.

**Definition 4.** Let \((L, \tau)\) be a TVS whose topology is generated by a family \( Q(\tau) \) of continuous \( F \)-seminorms \([17]\). Then \((L, \tau)\) is called **\( F \)-seminorm-separable** if \((L, q)\) is separable for each \( q \in Q(\tau) \).

Clearly, separability implies \( F \)-seminorm-separability; the converse holds in metrizable spaces.

The following result establishes the equivalence of all the above notions of generalized separabilities.
Lemma 1 (cf. [10]). Let \((L, \tau)\) be a TVS. The following are equivalent:
(1) \((L, \tau)\) is trans-separable.
(2) \((L, \tau)\) is neighbourhood-separable.
(3) \((L, \tau)\) is \(F\)-seminorm-separable.

Proof. (1) \(\Rightarrow\) (2) Suppose \(L\) is trans-separable, and let \(V\) be a neighbourhood of 0. For each \(n \geq 1\), \(U_n = \{x + n^{-1}V : x \in L\}\) is a uniform cover of \(L\), and so it has a countable subcover \(U_n^* = \{x_k^{(n)} + n^{-1}V : k \in \mathbb{N}\}\). Let \(D = \bigcup_{n=1}^{\infty}\{x_k^{(n)} : k \in \mathbb{N}\}\). To show that \(D\) is \(V\)-dense in \(L\), let \(y \in L\) and \(\delta > 0\). Choose \(N \geq 1\) such that \(N^{-1} \leq \delta\). Since \(U_N\) is a cover of \(L\), \(y \in x_K^{(N)} + N^{-1}V\) for some \(K \in \mathbb{N}\). Then \(y - x_K^{(N)} \in \delta V\). Hence \(L\) is neighbourhood-separable.
(2) \(\Rightarrow\) (3) This is trivial.
(3) \(\Leftrightarrow\) (1) Suppose \((L, q)\) is separable for each \(q \in Q(\tau)\). Let \(\{x + U : x \in L\}\) be any uniform cover of \(L\), where \(U\) is neighbourhood of 0 in \(L\). Choose a balanced neighbourhood \(V\) of 0 in \(L\) with \(V + V \subseteq U\). Choose \(q \in Q(\tau)\) such that \(W = \{x \in L : q(x) < 1\} \subseteq V\). Let \(\{z_n\}\) be a countable dense subsequence in \((L, q)\). Since \(L = \bigcup_{x \in L}(x + W)\), to each \(z_n \in L\), there exists some \(x_n \in L\) such that \(z_n - x_n \in W\). Let \(y \in L\). Choose \(z_k\) such that \(q(y - z_k) < 1\). Then
\[ y - x_k = (y - z_k) + (z_k - x_k) \in W + W \subseteq U, \]
and so \(L = \bigcup_{n \geq 1}(x_n + U)\).

\(\square\)

3. Main results

Theorem 2. Let \(E\) be any non-trivial TVS. Then the following statements are equivalent:
(a) \((C_b(X) \otimes E, \beta)\) is trans-separable.
(b) \((C(X) \otimes E, k)\) is trans-separable.
(c) Every compact subset of \(X\) is metrizable and \(E\) is trans-separable.

Proof. (a) \(\Rightarrow\) (b) This follows from the fact that \(k \leq \beta\) on \(C_b(X) \otimes E\) and that \(C_b(X) \otimes E\) is \(k\)-dense in \(C(X) \otimes E\).
(b) \(\Rightarrow\) (c) This follows from Theorem 1 and the fact that both \((C(X), k)\) and \(E\) are isomorphic to subspaces of \((C(X) \otimes E, k)\) via the maps \(g \rightarrow g \otimes a\) \((0 \neq a \in E\) fixed\) and \(a \rightarrow 1_X \otimes a\), respectively.
(c) \(\Rightarrow\) (a) By Theorem 1, \((C_b(X), \beta)\) is trans-separable. Fix \(\varphi \in B_{\beta}(X)\), \(0 \leq \varphi \leq 1\) and a balanced \(W \in \mathcal{W}\). We need to show that there is a countable set \(H \subseteq C_b(X) \otimes E\) such that \(C_b(X) \otimes E = H + U(\varphi, W)\).

For every pair \(m, n \in \mathbb{N}\) choose a balanced \(U_{m,n} \in \mathcal{W}\) so that, denoting \(V_{m,n} = U_{m,n} + mU_{m,n} + U_{m,n}\), one has
\[ V_{m,n} + \cdots + V_{m,n} \text{ (n-summands)} \subseteq W. \]
Also, choose a countable set \(D_{m,n}\) in \(E\) so that \(E = D_{m,n} + U_{m,n}\). Let \(D\) be the union of all these sets \(D_{m,n}\) \((m, n \in \mathbb{N})\).
Next, for each $k \in \mathbb{N}$ denote $B_k = \{ f \in C_b(X) : \| f \|_\varphi \leq 1/k \}$ and choose a countable set $G_k$ in $C_b(X)$ so that $C_b(X) = G_k + B_k$. Let $G$ be the union of all these sets $G_k$ ($k \in \mathbb{N}$).

We are going to show that the countable set $H = H_{\varphi,W}$ of all functions in $C_b(X) \otimes E$ of the form $h = \sum_{i=1}^r g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$ ($i = 1, \ldots, r$, $r \in \mathbb{N}$), is as required.

Take any $f \in C_b(X) \otimes E$. Then $f = \sum_{i=1}^n f_i \otimes a_i$ for some $f_1, \ldots, f_n \in C_b(X)$ and $a_1, \ldots, a_n \in E$. Let $m \in \mathbb{N}$ be such that $\| f_i \|_\varphi \leq m$ for $i = 1, \ldots, n$, and next choose $k \in \mathbb{N}$ so that $k^{-1}a_i \in U_{m,n}$ for $i = 1, \ldots, n$. By the definitions of $D_{m,n}$ and $G_k$, there are $d_1, \ldots, d_n \in D_{m,n}$ and $g_1, \ldots, g_k \in G_k$ such that

$$a_i - d_i \in U_{m,n} \quad \text{and} \quad \| f_i - g_i \|_\varphi \leq 1/k \quad \text{for} \quad i = 1, \ldots, n.$$

Now, for $i = 1, \ldots, n$ and $x \in A$,

$$\varphi(x)[f_i(x)a_i - g_i(x)d_i] = \varphi(x)[f_i(x) - g_i(x)]a_i + \varphi(x)f_i(x)(a_i - d_i)$$

(*) hence (using the fact that $U_{m,n}$ is balanced)

$$\varphi(x)[f_i(x)a_i - g_i(x)d_i] \in U_{m,n} + mU_{m,n} + U_{m,n} = V_{m,n}.$$

In consequence, setting $h = \sum_{i=1}^n g_i \otimes d_i$ we have $h \in H$ and for every $x \in A$,

$$\varphi(x)[f(x) - h(x)] = \sum_{i=1}^n \varphi(x)[f_i(x)a_i - g_i(x)d_i] \in W$$

so that $f - h \in U(\varphi, W)$.

\[\square\]

Remark 1. A somewhat more transparent variant of the above proof that (c) implies (a) can be based on Lemma 1. We need to show that for any $\varphi \in B_0(X), 0 \leq \varphi \leq 1$, and any continuous $F$-seminorm $q$ on $E$, the space $(C_b(X) \otimes E, p_\varphi)$ is separable, where $p_\varphi(f) = \sup_{x \in X} q(\varphi(f)(x))$. Now, let $G$ be a countable subset dense in $(C_b(X), \| \|_\varphi)$, and $D$ a countable set dense in $(E,q)$. Take any $f = \sum_{i=1}^n f_i \otimes a_i \in C_b(X) \otimes E$, and choose $m \in \mathbb{N}$ so that $\| f_i \|_\varphi \leq m$ for each $i$. Given $\varepsilon > 0$, let $g = \sum_{i=1}^n g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$. Assume that $\| f_i - g_i \|_\varphi \leq \delta$ for all $i$ and some as yet unspecified $0 < \delta < 1$. Then, making use of (*), it is easily seen that

$$p_\varphi(f - g) \leq \sum_{i=1}^n (q(\| f_i - g_i \|_\varphi a_i) + q(\| f_i \|_\varphi (a_i - d_i)) + q(\| f_i - g_i \|_\varphi (a_i - d_i))$$

$$\leq \sum_{i=1}^n (q(\delta a_i) + (m + 1)q((a_i - d_i))$$

and this can be made smaller than $\varepsilon$ by taking $\delta$ sufficiently small and choosing the $d_i$'s in $D$ sufficiently close to the $a_i$'s. It follows that the countable set of all $g$'s of the above form is dense in $(C_b(X) \otimes E, p_\varphi)$. 
Remark 2. If $X$ has finite covering dimension or $E$ is locally convex, or $E$ has the approximation property or $E$ is complete metrizable with a basis, then $C_k(X) \otimes E$ is $\beta$-dense in $C_b(X, E)$ and that $C(X) \otimes E$ is $k$-dense in $C(X, E)$ (see [14, 7]). Hence, under these assumptions, the above theorem holds with $C_b(X) \otimes E$, $C(X) \otimes E$ and $C_o(X) \otimes E$ replaced by $C_b(X, E)$, $C(X, E)$ and $(C_o(X, E))$, respectively. It is not known whether or not these ‘density’ results hold for $E$ a locally bounded space. However, we include the following analogue of ([8]; [9], Theorem 3.4) for the reader’s interest.

Theorem 3. Let $X$ be any Hausdorff space and $E$ any locally bounded space. Then $(C_b(X, E), \beta)$ is trans-separable $\iff (C(X, E), k)$ is so.

Proof. Suppose $(C(X, E), k)$ is trans-separable. Let $\varphi \in B_o(X)$ with $0 \leq \varphi \leq 1$, and let $W \subseteq V$. Let $V$ be a balanced bounded neighbourhood of 0 in $E$, and let $S$ be a closed shrinkable neighbourhood of 0 with $S \subseteq V$. The Minkowski functional $\rho = \rho_S$ of $S$ is continuous and positive homogeneous and, consequently, for each $r > 0$, the function $h_r : E \to E$ defined by

$$h_r(a) = \begin{cases} \frac{a}{r} & \text{if } a \in rS \\ a & \text{if } a \in E \setminus rS \end{cases}$$

is continuous. Further, $h_r(E) \subseteq rS \subseteq rV$, which shows that, for each $f \in C(X, E)$, the function $h_r \circ f \in C_b(X, E)$. Choose $t \geq 1$ such that $V + V \subseteq tS$ and $V + V \subseteq tW$. For each $m = 1, 2, \ldots$, there exists a compact set $K_m \subseteq X$ such that $\varphi(x) < 1/mt^2$ for $x \in X \setminus K_m$. Corresponding to each $K_m$, choose $\{f_{mn} : n \in \mathbb{N}\}$ as a $N(K_m, V)$-dense of $C(X, E)$, where

$$U(K_m, V) = \{ f \in C_b(X, E) : f(K_m) \subseteq W \}.$$ 

We show that $\{h_m \circ f_{mn} : m, n = 1, 2, \ldots\}$ is $\beta$-dense in $C_b(X, E)$. Let $f \in C_b(X, E)$ and $0 \leq \delta \leq 1$. Choose integers $M \geq 1/\delta$ and $N \geq 1$ such that $f(X) \subseteq (M\delta/t)V$ and $(f_{MN} - f)(K_M) \subseteq (\delta/t)V$. Let $y \in X$. If $y \not\in K_M$, then $f_{MN}(y) \in f(y) + (\delta/t)V \subseteq (M\delta/t)V + (M\delta/t)V \subseteq MS$ and so

$$\varphi(y)[h_M \circ f_{MN}(y) - f(y)] = \varphi(y)[f_{MN}(y) - f(y)] \in \delta W.$$ 

If $y \in X \setminus K_M$, then, since $h_M(f_{MN}(y)) \in h_M(E) \subseteq MS$,

$$\varphi(y)[h_M \circ f_{MN}(y) - f(y)] \in \varphi(y)[MS - \frac{M\delta}{t}V] \subseteq \frac{1}{tM} [V + \frac{\delta}{t}V] \subseteq \frac{\delta}{t} [V + V] \subseteq \delta W.$$ 

Thus $h_M \circ f_{MN} - f \in \delta U(\varphi, W)$. Consequently, $(C_b(X, E), \beta)$ is neighbourhood-separable and hence trans-separable by Lemma 1.

The converse follows from the fact that $C_b(X, E)$ is dense in $(C(X, E), k)$, using again the local boundedness of $E$. Indeed, let $f \in C(X, E)$, $K$ a compact subset of $X$ and $W \subseteq V$. Let $V$ and $S$ be as above with $S \subseteq V$. Choose
$r \geq 1$ with $f(K) \subseteq rS$. Then, as in the above part, we have a function $h_r \circ f \in C_b(X, E)$ such that

$$h_r \circ f(x) - f(x) = f(x) - f(x) = 0 \in W$$

for all $x \in K$. \hfill $\square$

Next, we obtain:

**Theorem 4.** Let $E$ be a non-trivial TVS. Then

(a) $(C_b(X) \otimes E, u)$ is trans-separable $\iff X$ is a compact metric space and $E$ is trans-separable.

(b) Suppose $X$ is locally compact. Then $(C_0(X) \otimes E, u)$ is trans-separable $\iff X$ is a $\sigma$-compact metric space and $E$ is trans-separable.

**Proof.** (a) In this case, $(C_b(X), u)$ is trans-separable $\Rightarrow$ it is separable $\Rightarrow X$ is a compact metric space [12]. The proof now follows just as in Theorem 2.

(b) If $X$ is locally compact, then $(C_0(X), u)$ is trans-separable $\Rightarrow$ it is separable $\Rightarrow X$ is a $\sigma$-compact metric space [5, 15]. Again the proof follows just as in Theorem 2. \hfill $\square$

Again we remark that, if $C_b(X) \otimes E$ (resp. $C_0(X) \otimes E$) is $u$-dense in $C_b(X, E)$ ($C_0(X, E)$), the above theorem remains valid with $C_b(X) \otimes E$ ($C_0(X) \otimes E$) replaced by $C_b(X, E)$ ($C_0(X, E)$).

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