ALGORITHM FOR THE DETERMINATION OF A LINEAR CRACK IN AN ELASTIC BODY FROM BOUNDARY MEASUREMENTS

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ABSTRACT

In this paper we consider the inverse problem of identifying a linear inclusion inside an elastic body from exterior boundary measurements. Based on the asymptotic formula by Beretta et al. [7], we design an effective reconstruction algorithm to find two end-points and thickness of a linear inclusion. Numerical experiments shows that the algorithm is effective and stable.

INTRODUCTION

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary and \( \sigma = \sigma[P,Q] \subset \Omega \) be a line segment of endpoints, \( P \) and \( Q \). We define \( D_\epsilon \) to be a thin region surrounding \( \sigma \) with thickness \( \epsilon \),
\[
D_\epsilon = \{ x \in \Omega : d(x,\sigma) < \epsilon \}
\]
which represents a linear inclusion of small thickness made of different elastic material.

Suppose that the elasticity tensor for \( \Omega \setminus \overline{D_\epsilon} \) with the Lamé coefficients \( (\lambda_0, \mu_0) \) is given by
\[
(C_0)_{ijkl} = \lambda_0 \delta_{ij} \delta_{k\ell} + \mu_0 (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}), \quad i,j,k,\ell = 1,2,
\]
and let \( (\lambda_1, \mu_1) \) be the Lamé coefficients for the linear inclusion \( D_\epsilon \); then the elasticity tensor for the whole domain can be written as follows,
\[
C_\epsilon = C_0 \chi_{\Omega \setminus D_\epsilon} + C_1 \chi_{D_\epsilon}.
\]

Given a traction field \( g \) on \( \partial \Omega \), the displacement field \( u_\epsilon \) solves the following system

\[
\begin{cases}
\text{div} \left( C_\epsilon \nabla u_\epsilon \right) = 0 & \text{in } \Omega, \\
(C_\epsilon \nabla u_\epsilon) \nu = g & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_\epsilon = 0, \int_\Omega \nabla u_\epsilon - (\nabla u_\epsilon)^T = 0
\end{cases}
\]
where \( \hat{\nabla} u = \frac{1}{2} (\nabla u + (\nabla u)^T) \) is the symmetric deformation tensor and \( \nu \) denotes the outward unit normal to \( \partial \Omega \). We denote by \( u_0 \) the background displacement field, namely the solution to

\[
\begin{align*}
\text{div} \left( C_0 \hat{\nabla} u_0 \right) &= 0 \quad \text{in } \Omega, \\
\left( C_0 \hat{\nabla} u_0 \right) \nu &= g, \quad \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_0 &= 0, \quad \int_{\Omega} \nabla u_0 - (\nabla u_0)^T = 0.
\end{align*}
\]

(5)

We now move our attention to the inverse elastic problem of finding a thin linear elastic inclusion \( D_\epsilon \) in \( \Omega \subset \mathbb{R}^2 \). Beretta and Francini suggested in [7] an asymptotic expansion formula for the boundary measurement \( (u_\epsilon - u_0)|_{\partial \Omega} \) as follows,

\[
(u_\epsilon - u_0)|_{\partial \Omega} = \epsilon w_\sigma(x) + o(\epsilon) \quad \text{as } \epsilon \to 0,
\]

where the first order asymptotic expansion term \( w_\sigma(x) \) for some traction field \( g \) can be explicitly written as a combination of the Neumann function \( N_\epsilon(x) \) and the background solution \( u_0 \).

In section 3 we present two methods to detect the geometry of an inclusion. Both of the methods utilize the fact that \( \epsilon w_\sigma(x) \), which can be easily computed by the boundary measurement \( (u_\epsilon - u_0)|_{\partial \Omega} \), contains all necessary information to reconstruct the linear inclusion parameters such as the end points \( P, Q \) and the thickness \( \epsilon \). A Newton-type iterative method is a direct application of the asymptotic formula, however, it fails in many cases. To overcome this difficulty, we propose a MUSIC-type Elastic Linear Inclusion Reconstruction (MELIR) algorithm which is based on a thorough study of more explicit relationship between the term \( \epsilon w_\sigma(x) \) and the inclusion parameters \( P, Q \) and \( \epsilon \).

Numerical examples for both the MELIR algorithm have been presented in Section 4. Detailed characterization of the MELIR algorithm and its generalization have been also discussed.

REFERENCES


