THE VALUES OF AN EULER SUM AT THE NEGATIVE INTEGERS AND A RELATION TO A CERTAIN CONVOLUTION OF BERNOULLI NUMBERS

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THE VALUES OF AN EULER SUM AT THE NEGATIVE INTEGERS AND A RELATION TO A CERTAIN CONVOLUTION OF BERNOULLI NUMBERS

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Abstract. The paper deals with the values at the negative integers of a certain Dirichlet series related to the Riemann zeta function and with the expression of these values in terms of Bernoulli numbers.

1. Introduction

We consider the function

\[ h(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s}, \]

where

\[ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \]

are the Harmonic numbers and \( \text{Re } s > 1 \). The function \( h(s) \) was studied by many authors, starting with Euler, who evaluated this series in a closed form when \( s = k \) is a positive integer. An elementary derivation of Euler’s formula can be found, for instance, in [3]. For general \( s \) this function was investigated by Apostol-Vu [1] and Matsuoka [5], who provided an analytic extension to all complex numbers and discussed its values and poles at the negative integers.

In this note we shall find a relation between the values \( h(1 - n) \) and the numbers \( A_n, n = 1, 2, \ldots \), defined as the convolution

\[ A_n = \sum_{k+j=n} \frac{B(k) B(j)}{k! j!}, \quad k = 1, 2, \ldots; \quad j = 0, 1, \ldots, \]

where \( B(n) = B_n \) are the Bernoulli numbers for \( n \neq 1 \) and \( B(1) = -B_1 = \frac{1}{2} \). Thus

\[ \frac{z e^z}{e^z - 1} = \frac{-z}{e^{-z} - 1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!} z^n. \]
The notation \( B(n) \) is used here in order to avoid the negative sign in \( B_1 \) and thus make possible the representation (4). We have also the expansion

\[
\log \left( \frac{e^z - 1}{z} \right) = \sum_{n=1}^{\infty} \frac{B(n)}{n!} \frac{z^n}{n}. \tag{5}
\]

The product of the functions in (4) and (5) is the generating function of the numbers \( A_n \),

\[
\sum_{n=1}^{\infty} A_n z^n = \frac{ze^z \log \left( \frac{e^z - 1}{z} \right)}{e^z - 1}. \tag{6}
\]

The relation between \( A_n \) and the values of \( h(s) \) is based on the evaluation of the following integral

\[
F(s) = \frac{\Gamma(1 - s)}{2\pi i} \int_L \frac{z^{s-1}e^z}{e^z - 1} \log \left( \frac{e^z - 1}{z} \right) \, dz, \tag{7}
\]

where \( L \) is the Hankel contour consisting of three parts: \( L = L_- \cup L_+ \cup L_\varepsilon \), with \( L_- \) the “lower side” (i.e., \( \arg(z) = -\pi \)) of the ray \((-\infty, -\varepsilon), \varepsilon > 0\), traced left to right, and \( L_+ \) the “upper side” (\( \arg(z) = \pi \)) of this ray traced right to left. Finally, \( L_\varepsilon = \{z = \varepsilon e^{i\theta} : -\pi \leq \theta \leq \pi\} \) is a small circle traced counterclockwise and connecting the two sides of the ray. This contour was used, for instance, in [2].

We note that convolutions like (3) appear in the Matiyasevich version of Miki’s identity - see [6]; see also Yu. Matiyasevich, Identities with Bernoulli numbers, http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernoulli.htm

General reference for the Bernoulli numbers, the Riemann-zeta function, and the Gamma and digamma functions is [7].

2. Main results

The main results of this article are given in the following theorem and the three corollaries.

**Theorem 1.** For \( \Re s > 1 \),

\[
F(s) = h(s) - \zeta(s + 1) + \psi(s)\zeta(s) + \zeta'(s),
\]

where \( \zeta(s) \) is the Riemann-zeta function and \( \psi(s) = \Gamma'(s)/\Gamma(s) \) is the digamma function.

As \( F(s)/\Gamma(1 - s) \) is an entire function (from (7)), this provides an extension of the right hand side in (8) to all complex \( s \).

The proof of the theorem is given in Section 3.

It is easy to see that when \( s \) is a negative integer or zero, the integration in (7) can be reduced to \( L_\varepsilon \) only, as the integrals on \( L_+ \) and \( L_- \) cancel each other. This way for the coefficients \( A_n \) of the Taylor series (6) we have

\[
(n - 1)! A_n = F(1 - n) \tag{9}
\]
for \( n = 1, 2, \ldots \). We shall evaluate the right hand side of (8) when \( s = 1 - n \) by considering the three cases: \( n > 1 \) odd, \( n = 1 \), and \( n \) even. The results are organized in three corollaries. Before listing these corollaries, we recall two properties of the Riemann zeta-function. For \( m = 1, 2, \ldots \), \( \zeta(-2m) = 0 \) and \( \zeta(1 - 2m) = -\frac{B_{2m}}{2m} \).

We first consider the case when \( n \) is odd.

**Corollary 1.** Let \( n = 2m + 1, \ m > 0 \). Then

\[
(2m)! A_{2m+1} = h(-2m) - \zeta(1 - 2m) = \frac{1}{2}(1 + \frac{1}{2m}) B_{2m}.
\]

**Proof.** From (9) we have \( (2m)! A_{2m+1} = F(-2m) \). In order to evaluate \( F(-2m) \) we use the well-known property of the digamma function

\[
\psi(s) = \psi(1 - s) - \pi \cot \pi s
\]

to write

\[
\psi(s)\zeta(s) = \psi(1 - s)\zeta(s) - \zeta(s)\pi \cot \pi s.
\]

Now, for \( s = -2m \) we have \( \psi(1 + 2m)\zeta(-2m) = 0 \) and

\[
\zeta(s)\pi \cot \pi s \bigg|_{s=-2m} = \zeta'(-2m).
\]

This follows from the Taylor expansion around \( s = -2m \),

\[
\zeta(s)\pi \cot \pi s = \zeta'(-2m) + \frac{1}{2}\zeta''(-2m)(s + 2m) + O((s + 2m)^2).
\]

Thus from (8) we find

\[
F(-2m) = h(-2m) - \zeta(1 - 2m).
\]

The values \( h(-2m) \) were computed by Matsuoka [5] as

\[
h(-2m) = -\frac{B_{2m}}{4m} + \frac{B_{2m}}{2}.
\]

(Note that Matsuoka worked with the function \( f(s) = h(s) - \zeta(s + 1) \)). Therefore, equation (10) follows from (15) and (16).

\( h(-2m) \) was also evaluated in [1], but incompletely (missing the second term on the right hand side in (16)). \( \square \)

Now let us consider the case \( s = 0 \) in (8), that is, \( n = 1 \) in (9).

**Corollary 2.** In a neighborhood of zero,

\[
h(s) = \frac{1}{2s} + \frac{1}{2}(1 + \gamma) + O(s),
\]

where \( \gamma = -\psi(1) \) is the Euler constant.
Proof. As found in [1] and [5], the function $h(s)$ has a simple pole at $s = 0$ with residue $\frac{1}{2}$. In order to establish (17) we need to evaluate $h(s) - \frac{1}{2}$ at zero. The functions $\zeta(s+1)$ and $\psi(s)\zeta(s)$ have residues 1 and $\frac{1}{2}$ respectively, at zero, and so the function

$$\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s}$$

does not have a pole at $s = 0$. Moreover, one easily finds that around $s = 0$

$$\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s} = \frac{\gamma}{2} + \zeta'(0) + O(s).$$

Next we rewrite (8) in the form

$$h(s) - \frac{1}{2s} = F(s) + (\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s}) - \zeta'(s)$$

and also compute the coefficient $A_1 = \frac{1}{2} = F(0)$ from (3). From (19) and (20) we find

$$\left( h(s) - \frac{1}{2s} \right) |_{s=0} = \frac{1}{2}(1 + \gamma),$$

which proves (17). \qed

Finally, we compute $F(1-n)$ for $n = 2m$.

**Corollary 3.** For $m = 2, 3, \ldots$, in a neighborhood of $s = 1 - 2m$ the function $h(s)$ is represented as

$$h(s) = \frac{\zeta(1-2m)}{s+2m-1} + (2m-1)! A_{2m} - \psi(2m)\zeta(1-2m) + O(s+2m-1),$$

and in a neighborhood of $s = -1$,

$$h(s) = \frac{-1}{12(s+1)} - \frac{1}{8} + \frac{\gamma}{12} + O(s+1).$$

Proof. Apostol–Vu [1] and Matsuoka [5] showed that the function $h(s)$ has simple poles at the negative odd integers $s = 1 - 2m$ with residues $\zeta(1 - 2m)$. The same is true for the function $\zeta(s)\pi \cot \pi s$, as follows from the Taylor expansion at $s = 1 - 2m$,

$$\zeta(s)\pi \cot \pi s = \zeta(1 - 2m) \frac{1}{s+2m-1} + \zeta'(1 - 2m) + O(s+2m-1).$$

Using (12) in (8), we obtain the representation

$$h(s) = \zeta(s)\pi \cot \pi s + F(s) + \zeta(s+1) - \psi(1-s)\zeta(s) - \zeta'(s),$$

and substituting (24) in this, we get

$$h(s) - \frac{\zeta(1-2m)}{s+2m-1} = F(s) + \zeta(s+1) - \psi(1-s)\zeta(s) - \zeta'(s) + \zeta'(1-2m) + O(s+2m-1).$$
Now, evaluating both sides of (26) at $s = 1 - 2m$,

$$h(s) - \frac{\zeta(1 - 2m)}{s + 2m - 1}$$

at $s = 1 - 2m$,

$$F(1 - 2m) + \zeta(2 - 2m) - \psi(2m)\zeta(1 - 2m) + O(s + 2m - 1)$$

and as $F(1 - 2m) = (2m - 1)! A_{2m}$ and $\zeta(2 - 2m) = 0$, we obtain (22).

When $m = 1$, we have $\zeta(2 - 2m) = \zeta(0) = -\frac{1}{2}$, $\zeta(-1) = \frac{1}{12}$, $\psi(2) = 1 - \gamma$, and by direct computation from (3), $A_2 = \frac{7}{24}$. Thus (23) follows from (27).

3. Proof of Theorem 1

Here we evaluate the integral in (7)

$$I(s) = \frac{1}{2\pi i} \int_{L} z^{s-1} e^z e^{z-1} \log \left( \frac{e^z - 1}{z} \right) \, dz,$$

where the contour $L$ is as described in Section 1. We choose $\text{Re } s > 1$ and set $\varepsilon \to 0$. The integral over $L_\varepsilon$ becomes zero, as the function

$$\frac{ze^z}{e^z - 1} \log \left( \frac{e^z - 1}{z} \right)$$

is holomorphic in a neighborhood of zero. Noticing that $z = xe^{-\pi i}$ on $L_-$ and $z = xe^{\pi i}$ on $L_+$, we find that

$$-I(s) = \frac{e^{-\pi is}}{2\pi i} \int_{0}^{\infty} x^{s-1} e^{-x} \log \left( \frac{1 - e^{-x}}{x} \right) \, dx$$

$$+ \frac{e^{\pi is}}{2\pi i} \int_{0}^{\infty} x^{s-1} e^{-x} \log \left( \frac{1 - e^{-x}}{x} \right) \, dx$$

$$= \frac{\sin \pi s}{\pi} \int_{0}^{\infty} x^{s-1} e^{-x} \log \left( \frac{1 - e^{-x}}{x} \right) \, dx.$$  

Next,

$$\int_{0}^{\infty} x^{s-1} e^{-x} \log \left( \frac{1 - e^{-x}}{x} \right) \, dx$$

$$= \int_{0}^{\infty} x^{s-1} e^{-x} \log(1 - e^{-x}) \, dx - \int_{0}^{\infty} x^{s-1} e^{-x} \log x \, dx.$$  

We shall evaluate the two integrals on the right hand side in (31) one by one. First we use the expansion

$$\frac{\log(1 - e^{-x})}{1 - e^{-x}} = -\sum_{n=1}^{\infty} H_n e^{-nx}$$
where
\begin{align*}
\int_0^\infty \frac{x^{s-1} e^{-x}}{1 - e^{-x}} \log(1 - e^{-x}) \, dx &= -\sum_{n=1}^\infty H_n \int_0^\infty x^{s-1} e^{-(n+1)x} \, dx = -\Gamma(s) \sum_{n=1}^\infty \frac{H_n}{(n+1)^s} \\
&= -\Gamma(s)(\psi(s)\zeta(s) + \zeta'(s)).
\end{align*}

(33)

Next, differentiating for \( s \) the representation
\begin{equation}
\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx,
\end{equation}

we obtain
\begin{equation}
\int_0^\infty \frac{x^{s-1}}{e^x - 1} \log x \, dx = \Gamma'(s)\zeta(s) + \Gamma(s)\zeta'(s) = \Gamma(s)(\psi(s)\zeta(s) + \zeta'(s)).
\end{equation}

(35)

From (31), (33) and (35)
\begin{equation}
\int_0^\infty \frac{x^{s-1} e^{-x}}{1 - e^{-x}} \log \left( \frac{1 - e^{-x}}{x} \right) \, dx = -\Gamma(s)(\psi(s)\zeta(s) + \zeta'(s)),
\end{equation}

and therefore,
\begin{equation}
I(s) = \frac{1}{\pi} \Gamma(s) \sin(\pi s)(\psi(s)\zeta(s) + \zeta'(s)) + \psi(s)\zeta(s) + \zeta'(s)).
\end{equation}

(37)

Finally, (8) follows from here in view of the identity
\begin{equation}
\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s}.
\end{equation}

(38)

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