ON \( j \)-INVARIANTS OF WEIERSTRASS EQUATIONS

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Abstract. A simple proof of the fact that the $j$-invariants for Weierstrass equations are invariant under birational transformations which keep the forms of Weierstrass equations is given by finding a non-trivial explicit birational transformation which sends a normalized Weierstrass equation to the same equation.

1. Introduction

For Legendre forms
\[ y^2 = x(x-1)(x-\lambda) \]
the $j$-invariants ([4], p.55, [2], p.119)
\[ j(\lambda) = 256 \cdot \frac{\lambda^2 - \lambda + 1}{\lambda^2 \cdot (\lambda - 1)^2} \]
are invariant under birational transformations which keep Legendre forms. This is proved in Hartshorne’s book ([3], p.317) using Galois theory and linear series theory. The point is in showing that the group of birational transformations which keep Legendre forms is generated by
\[ \begin{align*}
\text{(i)} & \quad \begin{cases} x' = 1 - x \\ y' = y \end{cases} \\
\text{(ii)} & \quad \begin{cases} x' = \frac{1}{x} \\ y' = \frac{y}{x^2} \end{cases}
\end{align*} \]
which preserve the set \{0, 1, \infty\}, and map $\lambda$ respectively to
\[ \begin{align*}
\text{(i)} & \quad 1 - \lambda, \\
\text{(ii)} & \quad \frac{1}{\lambda}.
\end{align*} \]

The purpose of this paper is to prove the following theorem.

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**Theorem.** For non-singular Weierstrass equations

\[ y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \]

the \( j \)-invariants are defined by

\[ j(a_1, a_2, a_3, a_4, a_6) = \frac{4(b_2^2 - 24b_4)^3}{36b_2b_4b_6 - (32b_4^4 + 108b_6^2) - b_2^2(b_2b_6 - b_4^2)}, \]

where

\[ \begin{align*}
  b_2 &= a_1^2 + 4a_2 \\
  b_4 &= a_1a_3 + 2a_4 \\
  b_6 &= a_3^2 + 4a_6.
\end{align*} \]

Two elliptic curves defined by equations like (5) are isomorphic if and only if they have the same \( j \)-invariant.

In fact, the invariance of the \( j \)-invariants under birational transformations which keep the form of Weierstrass equations is proved by finding an explicit non-trivial birational transformation which sends a normalized Weierstrass equation to the same equation.

We are working over the field of characteristic not equal to two or three.

**Remark.** Using Legendre forms it is proved that the \( j \)-invariant is independent of the choice of the base points of elliptic curves ([4], pp.108–109).

2. Weierstrass equations

Let \( T \) be a Riemann surface of genus one. By the Riemann-Roch theorem for a fixed point \( P_\infty \), there are meromorphic functions, \( x \) of order two with a pole of order two at \( P_\infty \) and \( y \) of order three with a pole of order three at \( P_\infty \). Then there is a relation between \( x \) and \( y \):

\[ f(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0 \]

called the Weierstrass equation ([4], p.46), where the coefficient of \( y^2 \) is normalized as 1 and those of \( x^3 \) is normalized as \(-1\). In general, the Weierstrass equations (8) may be singular, but we assume that the equations are non-singular, i.e., they define elliptic curves, since our surfaces are of genus one. There are many choices of functions \( x, y \) so that the relation is not unique, and there are many birationally equivalent Weierstrass equations.

We choose such a point \( P_0 \neq P_\infty \) on \( T \) as a zero of the discriminant of (8) w.r.t. \( x \). Subtracting the constant values \( x(P_0), y(P_0) \) from the functions \( x, y \) respectively, we may assume that \( a_4 = a_6 = 0 \). The Weierstrass equation then takes the form

\[ y^2 + a_1xy + a_3y - x^3 - a_2x^2 = 0 \]

and the coordinates of the point \( P_0 \) is \((x, y) = (0, 0)\). We call the forms (9) normalized Weierstrass equations.
3. Proof of Theorem

It is known ([4], pp.46–55, 63–65) or checked by direct computation that the isomorphisms

\[(10)\]

\[
x = u^2X + r \\
y = u^3Y + su^2X + t
\]

fixing the point \(P_\infty = (\infty, \infty)\) keep the form of Weierstrass equations and make the \(j\)-invariant (6) invariant.

The Weierstrass equation (5) can be transformed to a normalized Weierstrass equation

\[(11)\]

\[
y^2 + a_1xy + a_3y - x^3 - a_2x^2 = 0 \quad (a_3 \neq 0)
\]

by some isomorphisms (10), and it is transformed to completely the same form

\[(12)\]

\[
Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 = 0
\]

by the birational transformation

\[(13)\]

\[
\begin{align*}
X &= \frac{a_3(x + a_2)}{y} \\
Y &= \frac{a_3x(x + a_2)^2}{y^2} - \frac{a_1a_3(x + a_2)}{y} - a_3
\end{align*}
\]

which transforms the point \(P_\infty\) to the finite point \(P_0 = (0, 0)\). The inverse of (13) is

\[(14)\]

\[
\begin{align*}
x &= \frac{a_3(Y + a_1X + a_3)}{X^2} \\
y &= \frac{a_3(a_3Y + a_1a_3X + a_3^2 + a_2X^2)}{X^3}
\end{align*}
\]

One can easily check these facts by using Maple as follows:

\[
> f := y^2 + a[1]*x*y + a[3]*y - x^3 - a[2]*x^2; \\
> X = a[3]*(x + a[2])/y; \quad Y = a[3]*x*(x + a[2])/y^2 - a[1]*a[3]*
\]

\[
(x + a[2])/y - a[3]; \\
> solve({X = a[3]*(x + a[2])/y, \quad Y = a[3]*x*(x + a[2])/y^2 - a[1]*
\]

\[
a[3]*(x + a[2])/y - a[3]}, \{x, y\}); \\
> factor(subs(%, f));
\]

Of course, the original equation (11) and the transformed equation (12) have the same \(j\)-invariant (6).

The transformations (10) and (13) (cf. [1], p.374) generate all birational transformations which keep the form of Weierstrass equations. Consequently we have the result.
References


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