ON GENERALIZED JORDAN LEFT DERIVATIONS IN RINGS

MOHAMMAD ASHRAF AND SHAKIR ALI

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Abstract. In this paper, we introduce the notion of generalized left derivation on a ring $R$ and prove that every generalized Jordan left derivation on a 2-torsion free prime ring is a generalized left derivation on $R$. Some related results are also obtained.

1. Introduction

Throughout the present paper $R$ will denote an associative ring with centre $Z(R)$. Recall that $R$ is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is said to be a left derivation (resp. Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (resp. $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. Clearly, every left derivation on a ring $R$ is a Jordan left derivation but the converse need not be true in general; (see for example [18, Example 1.1]). First author together with Rehman [4] proved that a Jordan left derivation on a 2-torsion free prime ring is a left derivation. Further in [5], authors together with Rehman proved that if $R$ is a 2-torsion free prime ring and $\delta : R \to R$ is an additive mapping such that $\delta(u^2) = 2u\delta(u)$ for all $u$ in a square closed Lie ideal $U$ of $R$, then either $U \subseteq Z(R)$ or $\delta(U) = \{0\}$. During the last two decades, there has been ongoing interest concerning the relationship between the left derivation and Jordan left derivation on a prime ring (cf. [1, 4, 5, 7, 9, 14, 17, 18] and reference therein).

Following [12], an additive mapping $F : R \to R$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in R$. Clearly, every generalized derivation on a ring is a...
generalized Jordan derivation. But the converse statement does not hold in
general (see e.g., [6]). It is shown in [3] that if $R$ is a ring with a commutator
which is not a divisor of zero, then every generalized Jordan derivation on
$R$ is a generalized derivation. It should be mentioned that the result in [3]
concerning generalized Jordan derivation has been improved in [2] and [6] by
authors together with Rehman. More related results have also been obtained
in [8], [13], and [15], where further references can be found.

Inspired by the definition of generalized derivation, we introduce the notion
of generalized left derivation as follows: an additive mapping $G : R \rightarrow R$
is called a generalized left derivation (resp. generalized Jordan left derivation)
if there exists a Jordan left derivation $\delta : R \rightarrow R$ such that
$G(xy) = xG(y) + y\delta(x)$ (resp. $G(x^2) = xG(x) + x\delta(x)$) holds for all $x, y \in R$. It is obvious to
see that every generalized left derivation on a ring $R$ is a generalized Jordan
left derivation. But the converse need not be true in general. The following
example justifies this fact:

**Example 1.1.** Let $S$ be a ring such that the square of each element in $S$ is
zero, but the product of some nonzero elements in $S$ is nonzero. Next, let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in S \right\}.$$ 

Define a map $G : R \rightarrow R$ such that

$$G \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then, we can find an associated Jordan left derivation $\delta : R \rightarrow R$ such that

$$\delta \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

It is straightforward to check that $G$ is a generalized Jordan left derivation but
not a generalized left derivation.

In the present paper, our aim is to establish set of conditions under which
every generalized Jordan left derivation on a ring is a generalized left derivation.
This lead to the discovery of some new results which can be regarded as a
contribution to the theory of Jordan derivations in rings.

**2. Preliminary results**

To facilitate our discussion, we define a mapping $H : R^2 \rightarrow R$ such that

$$H(x, y) = G(xy) - xG(y) - y\delta(x).$$ 

Since $G$ and $\delta$ both are additive, we have for any $x, y, z \in R$;

$$H(x, y + z) = H(x, y) + H(x, z) \text{ and } H(x + y, z) = H(x, z) + H(y, z).$$
Moreover, if $H$ is zero then $G$ is a generalized left derivation on $R$. We shall make use of commutator identities; $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = (x, z)y + x[y, z]$.

We begin with the following lemmas which are essential for developing the proof of our results.

**Lemma 2.1.** ([14, Proposition 2.2]). Let $R$ be a ring and $X$ be a 2-torsion free left $R$-module. If $\delta : R \to X$ is an additive mapping satisfying $\delta(x^2) = 2x\delta(x)$ for all $x \in R$, then

(i) $\delta(x^2y) = x^2\delta(y) + (xy + yx)\delta(x) + x\delta(xy - yx)$ for all $x, y \in R$,

(ii) $\delta(xy^2) = x^2\delta(y) + (3yx - xy)\delta(x) - x\delta(xy - yx)$ for all $x, y \in R$,

(iii) $[x, y]\delta([x, y]) = 0$ for all $x, y \in R$,

(iv) $(x^2y - 2xyz + yx^2)\delta(y) = 0$ for all $x, y \in R$.

**Lemma 2.2.** Let $R$ be a 2-torsion free ring and $G : R \to R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta : R \to R$. Then

(i) $G(xy + yx) = xG(y) + yG(x) + x\delta(y) + y\delta(x)$ for all $x, y \in R$,

(ii) $G(xyx) = xyG(x) + 2xy\delta(x) + x^2\delta(y) - yx\delta(x)$ for all $x, y \in R$,

(iii) $G(xyz + zyx) = xyzG(z) + zyG(x) + 2xyz\delta(z) + 2zy\delta(x) + xz\delta(y) + zx\delta(y) - yz\delta(z) - yz\delta(x)$ for all $x, y, z \in R$.

**Proof.** (i) We are given that $G$ is a generalized Jordan left derivation of $R$ such that

$$G(x^2) = xG(x) + x\delta(x) \quad \text{for all } x \in R.$$  

Linearizing (2.1), we get

$$G((x + y)^2) = (x + y)G(x + y) + (x + y)\delta(x + y) = xG(x) + xG(y) + yG(x) + y\delta(x) + x\delta(y) + y\delta(x) + y\delta(y) \quad \text{for all } x, y \in R.$$  

On the other hand, we have

$$G((x + y)^2) = G(x^2 + xy + yx + y^2) = xG(x) + x\delta(x) + G(xy + yx) + yG(y) + y\delta(y) \quad \text{for all } x, y \in R.$$  

Combining (2.2) and (2.3), we get the required result.

(ii) Replacing $y$ by $xy + yx$ in (i), we get

$$G(x(xy + yx) + (xy + yx)x) = xG(xy + yx) + (xy + yx)G(x) + x\delta(xy + yx) + (xy + yx)\delta(x) \quad \text{for all } x, y \in R.$$  

Since, $\delta : R \to R$ is a Jordan left derivation, linearizing $\delta(x^2) = 2x\delta(x)$, we find that

$$\delta(xy + yx) = 2x\delta(y) + 2y\delta(x) \quad \text{for all } x, y \in R.$$
and hence
\[
G(x(xy+yx) + (xy+yx)x) = x^2G(y) + 2xyG(x) + 4xy\delta(y) + 3x^2\delta(y) + yx\delta(x) + yxG(x) \text{ for all } x, y \in R.
\]

Also,
\[
G(x(xy+yx) + (xy+yx)x) = G(x^2y) + 2G(xy^2) = x^2G(y) + yxG(x) + yx\delta(x) + x^2\delta(y) + 2yx\delta(x) + 2G(xy^2) \text{ for all } x, y \in R.
\]

Comparing (2.4), (2.5) and using the fact that char\(R \neq 2\), we obtain
\[
G(xyx) = xyG(x) + 2xy\delta(x) + x^2\delta(y) - yx\delta(x) \text{ for all } x, y \in R.
\]

(iii) Replace \(x\) by \(x+z\) in (2.6), to get
\[
G((x+z)y(x+z)) = xyG(x) + xyG(z) + zyG(x) + zyG(z) + 2xy\delta(x)
\]
\[
+ 2zy\delta(z) + 2zy\delta(x) + x^2\delta(y) + xz\delta(y) + zx\delta(y) + z^2\delta(y) - yx\delta(x) - yz\delta(x) - yz\delta(z)
\]
for all \(x, y, z \in R\).

On the other hand, we have
\[
G((x+z)y(x+z)) = G(xyx) + G(zyz) + G(xyz + zyx)
\]
\[
= xyG(x) + 2xy\delta(x) + x^2\delta(y) - yx\delta(x) + G(xyz + zyx)
+ zyG(x) + 2zy\delta(z) + z^2\delta(y) - yz\delta(x) - yz\delta(z) \text{ for all } x, y, z \in R.
\]

Comparing (2.7) and (2.8), we get (iii). \(\square\)

The following lemma play the key role in the proof of main theorem.

**Lemma 2.3.** Let \(R\) be a 2-torsion free ring and \(G : R \rightarrow R\) be a generalized Jordan left derivation with associated Jordan left derivation \(\delta : R \rightarrow R\). Then
\[
[x, y]H(x, y) = 0 \text{ for all } x, y \in R.
\]

**Proof.** Replace \(z\) by \(xy - yx\) in Lemma 2.2(iii), to get
\[
G(xy(xy - yx) + (xy - yx)yx) = xyG(xy) - xyG(yx) + [x, y]yG(x) + [x, y]\delta([x, y]) + xy\delta([x, y])
+ 2[x, y]\delta(x) + x[x, y]\delta(y) + [x, y]x\delta(y) - y[x, y]\delta(x)
\]
for all \(x, y, z \in R\).
Now, application of Lemma 2.1(iii) yields that
\[
G(xy(xy - yx)) + (xy - yx)yx
\]
\[
= xyG(xy) - xyG(yx) + [x, y]yG(x)
\]
\[
+ 2[x, y]y\delta(x) + xy\delta([x, y]) + x[x, y]y\delta(y)
\]
\[
+ [x, y]x\delta(y) - y[x, y]\delta(x)
\]
for all \(x, y \in R\).

Also, we have
\[
G(xy(xy - yx)) + (xy - yx)yx
\]
\[
= G((xy)^2 - xy^2x + yx^2x - (yx)^2)
\]
\[
= G((xy)^2) - G((yx)^2)
\]
\[
= xyG(xy) + xy\delta(xy) - yxG(yx) - yx\delta(yx)
\]
for all \(x, y \in R\).

Combining (2.11) and (2.12), we find that
\[
yxG(yx) - xyG(yx) + [x, y]yG(x) + 2[x, y]y\delta(x)
\]
\[
+ xy\delta([x, y]) + x[x, y]y\delta(y) + x[x, y]x\delta(y) - y[x, y]\delta(x)
\]
\[
+ yx\delta(yx) - xy\delta(xy) = 0
\]
for all \(x, y \in R\).

This implies that
\[
y[x, y]G(yx) + [x, y]yG(x) + [x, y]y\delta(y)
\]
\[
- 2y[x, y]y\delta(x) + x[x, y]y\delta(y) + y[x, y]y\delta(x) + yx\delta(xy)
\]
\[
- xy\delta(yx) = 0
\]
for all \(x, y \in R\).

By Lemma 2.1(iv), we have
\[
x[x, y]y\delta(y) + y[x, y]y\delta(x) + yx\delta(xy) - xy\delta(xy)
\]
\[
= (x^2y - 2xyx + yx^2)y\delta(y) - (y^2x - 2yxy + xy^2)y\delta(x)
\]
\[
= 0
\]
for all \(x, y \in R\)

and
\[
2[x, y]y\delta(x) - 2y[x, y]y\delta(x)
\]
\[
= 2(y^2x - 2yxy + xy^2)y\delta(x) = 0
\]
for all \(x, y \in R\).

Now, in view of (2.15) and (2.16), (2.14) reduces to
\[
y[x, y]G(yx) + [x, y]yG(x) + [x, y]y\delta(y) = 0
\]
for all \(x, y \in R\).

This implies that
\[
[x, y](G(xy) - xG(y) - y\delta(x)) = 0,
\]
\[
\text{i.e., } [x, y]H(x, y) = 0
\]
for all \(x, y \in R\).

\[\square\]
3. Main results

The main results of the present paper states as follows:

**Theorem 3.1.** Let $R$ be a 2-torsion free ring such that $R$ has a commutator which is not a left zero divisor. Let $G : R \rightarrow R$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta : R \rightarrow R$. Then every generalized Jordan left derivation on $R$ is a generalized left derivation on $R$.

**Proof.** By the assumption, for any fixed element $a, b \in R$ such that $[a, b]c = 0$ implies that $c = 0$. By Lemma 2.3, we have

(3.1) \[ H(a, b) = 0. \]

Replacing $x$ by $x + a$ in (2.9) and using (2.9), we obtain

(3.2) \[ [x, y]H(a, y) + [a, y]H(x, y) = 0 \text{ for all } x, y \in R. \]

Linearizing (3.2) on $y$, we find that

(3.3) \[ [x, b]H(a, y) + [a, y]H(x, b) + [a, b]H(x, y) + [a, b]H(x, b) = 0 \text{ for all } x, y \in R. \]

Substituting $a$ for $x$ in (3.3) and using (3.1), we have $2[a, b]H(a, y) = 0$ for all $x, y \in R$. Since $\text{char} R \neq 2$, the last expression yields that $[a, b]H(a, y) = 0$ for all $x, y \in R$ and hence $H(a, y) = 0$ for all $y \in R$. Again, put $b$ for $y$ in (3.2), we find that $H(x, b) = 0$ for all $x \in R$. Therefore, equation (3.3) reduces to $[a, b]H(x, y) = 0$ for all $x, y \in R$ and hence $H(x, y) = 0$ for all $x, y \in R$, i.e., $G(xy) = xG(y) + y\delta(x)$ for all $x, y \in R$. This completes the proof of our theorem. \(\square\)

**Corollary 3.1.** Let $R$ be a 2-torsion free ring such that $R$ has a commutator which is not a left zero divisor. If $\delta : R \rightarrow R$ is a Jordan left derivation, then $\delta$ is a left derivation on $R$.

If the ring $R$ is prime, then we have the following results:

**Proposition 3.1.** Let $R$ be a 2-torsion free prime ring. If $R$ admits a generalized left derivation with associated Jordan left derivation $\delta$, then either $\delta = 0$ or $R$ is commutative.

**Proof.** Let $G : R \rightarrow R$ be a generalized left derivation with associated Jordan left derivation $\delta : R \rightarrow R$. Then for any $x, y \in R$, we have

(3.4) \[ G(x^2y) = x^2G(y) + 2yx\delta(x) \text{ for all } x, y \in R. \]

On the other hand, we find that

(3.5) \[ G(x^2y) = G(xy) = x^2G(y) + 2xy\delta(x) \text{ for all } x, y \in R. \]

Comparing (3.4) and (3.5), we obtain

(3.6) \[ 2[x, y]\delta(x) = 0 \text{ for all } x, y \in R. \]
Since $R$ is a 2-torsion free, the equation (3.6) implies that $[x, y]δ(x) = 0$ for all $x, y ∈ R$. Replacing $y$ by $yz$ in the last expression, we find that $[x, y]Rδ(x) = \{0\}$ for all $x, y ∈ R$. Thus for each $x ∈ R$, the primeness of $R$ implies that either $[x, y] = 0$ or $δ(x) = 0$ for all $y ∈ R$. Now, we put $A = \{x ∈ R \mid [x, y] = \{0\}\}$ for all $y ∈ R$. Then, clearly $A$ and $B$ are additive subgroups of $R$ whose union is $R$. But a group can not be written as a set theoretic union of two of its proper subgroups and hence we obtain that either $A = R$ or $B = R$. If $A = R$, then $δ(x) = 0$ for all $x ∈ R$. On the other hand, if $B = R$, then $[x, y] = 0$ for all $x, y ∈ R$ and hence $R$ is commutative. The proof of the proposition is complete.

As a special case of above proposition, we have the following result:

**Corollary 3.2.** Let $R$ be a 2-torsion free prime ring. If $R$ admits a nonzero Jordan left derivation $δ$, then $R$ is commutative.

**Theorem 3.2.** Let $R$ be a 2-torsion free prime ring. Let $G : R → R$ be a generalized Jordan left derivation with associated Jordan left derivation $δ : R → R$. Then every generalized Jordan left derivation is a generalized left derivation on $R$.

**Proof.** If the associated Jordan left derivation $δ = 0$, then $G$ is a Jordan left multiplier on $R$. Therefore in view of Proposition 1.4 [19], $G$ is a left multiplier (right centralizer). Hence for $δ = 0$, it is a generalized left derivation.

On the other hand suppose that the associated Jordan left derivation $δ ≠ 0$. Then, by Corollary 3.2, $R$ is commutative. Notice that in view of main theorem of [4], every Jordan left derivation on a 2-torsion free prime ring is a left derivation. Hence by Lemma 2.2(i) and using the fact that $R$ is 2-torsion free prime ring, we find that

$$G(xyz + zyx) = G((xy)z + z(yx))$$

$$= xyG(z) + zG(yx) + xyδ(z) + zyδ(x) + zxδ(y)$$
for all $x, y, z ∈ R$.

Combining (3.7) with Lemma 2.2(iii), we find that

$$zG(yx) + xyδ(z) + zxδ(y) + zyδ(x)$$

$$= zyG(x) + 2xyδ(z) + 2zγδ(x) + xzδ(y) + zyδ(x) − yzδ(x) − yxδ(z)$$
for all $x, y, z ∈ R$.

Since $R$ is commutative, so equation (3.8) reduces to

$$z(G(yx) − yG(x) − xδ(y)) = 0$$
for all $x, y, z ∈ R$.

This implies that

$$(G(yx) − yG(x) − xδ(y))R(G(yx) − yG(x) − xδ(y)) = \{0\}$$
for all $x, y ∈ R$. 

□
Thus, the primeness of $R$ yields that $G(xy) - yG(x) - xG(y) = 0$ for all $x, y \in R$. That is, $G(xy) = xG(y) + yG(x)$ for all $x, y \in R$. Hence, $G$ is a generalized left derivation on $R$. This completes the proof of our theorem. □

The following example demonstrates that $R$ to be prime is essential in the hypotheses of the above theorem.

Example 3.1. Consider the rings $S$ and $R$, as in Example 1.1, and define maps $G, \delta : R \rightarrow R$ in similar manner. Then, it can be easily seen that $G(r^2) = rG(r) = r\delta(r) = s\delta(r) = 0$ for all $r, s \in R$ but $G(rs) \neq 0$ for some nonzero elements $r, s \in R$.

In the end, it is to remark that the above result may be obtained for semiprime ring, but to our knowledge it has not yet been settled.

References

Mohammad Ashraf  
Department of Mathematics  
Aligarh Muslim University  
Aligarh-202002, India  
E-mail address: mashraf80@hotmail.com

Shakir Ali  
Department of Mathematics  
Aligarh Muslim University  
Aligarh-202002, India  
E-mail address: shakir50@rediffmail.com