DIAGONAL LIFTS OF TENSOR FIELDS OF TYPE (1,1) ON CROSS-SECTIONS IN TENSOR BUNDLES AND ITS APPLICATIONS

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Abstract. The main purpose of this paper is to investigate diagonal lift of tensor fields of type (1,1) from manifold to its tensor bundle of type (p,q) and to prove that when a manifold $M$ admits a Kählerian structure $(\phi, g)$, its tensor bundle of type $(p, q)$ admits an complex structure.

1. Introduction

Let $M$ be $n$-dimensional differentiable manifold of class $C^\infty$, $T^p_q(M)$ its tensor bundle of type $(p, q)$, and $\pi$ the natural projection $T^p_q(M) \to M$. Let $x^i$, $i = 1, \ldots, n$ be local coordinates in neighborhood $U$ of a point $x$ of $M$. Then a tensor $t$ of type $(p, q)$ at $x \in M$ which is an element of $T^p_q(M)$ is expressible in the form

$$ (x^i, t_{j_1 \cdots j_q}^{i_1 \cdots i_p}) = (x^i, x_j^\bar{j}), \quad \bar{j} = n + 1, \ldots, n + n^{p+q}, $$

whose $t_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ are components of $t$ with respect to the natural frame $\partial_j$. We may consider $(x^i, x_j^\bar{j})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T^p_q(M)$. To a transformation of local coordinates of $M$:

$$ x^j' = x^j(x^i), \quad x_j^\bar{j}' = A^{i_1}_{j_1} \cdots A^{i_p}_{j_p} A^{j_1}_{i_1} \cdots A^{j_q}_{i_q} = A^{(i)}_{(j)} x^3, $$

where

$$ A^{(i)}_{(j)} = A^{i_1}_{j_1} \cdots A^{i_p}_{j_p} A^{j_1}_{i_1} \cdots A^{j_q}_{i_q}, \quad A^{i_1}_{j_1} = \frac{\partial x^{i_1}}{\partial x^{j_1}}. $$

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The Jacobian of (1.1) is given by the matrix

\[
(1.2) \begin{pmatrix}
\frac{\partial x'}{\partial x^j} & \frac{\partial x'}{\partial x^j} & \frac{\partial x''}{\partial x^j} \\
\frac{\partial x'}{\partial x^j} & \frac{\partial x'}{\partial x^j} & \frac{\partial x''}{\partial x^j} \\
\frac{\partial x''}{\partial x^j} & \frac{\partial x''}{\partial x^j} & \frac{\partial x''}{\partial x^j}
\end{pmatrix} = \begin{pmatrix}
A_{j}^{j'} & A_{j}^{k'} & 0 \\
 t_{i}^{(i)} \partial_{j}(A_{i}^{j'}) & A_{i}^{k'} & A_{i}^{l'} A_{(j')}
\end{pmatrix},
\]

where

\[J = (j, j), \ J = 1, \ldots, n + n^{0+q}, \ t_{i}^{(i)} = t_{i_{1} \cdots k_{q}}.\]

We denote by \(\mathcal{D}(M_{n})\) the module over \(F(M_{n})\) of \(C^\infty\) tensor fields of type \((p, q)\) \((F(M_{n})\) is ring of real-valued \(C^\infty\) functions on \(M_{n}\)). If \(\alpha \in \mathcal{D}(M_{n})\), it is regarded, in a natural way, by contraction, as a function in \(T_{q}^{p}(M_{n})\), which we denote by \(\alpha\). If \(\alpha\) has the local expression

\[\alpha = \alpha_{i_{1} \cdots i_{p}} \partial_{j_{1}} \otimes \cdots \otimes \partial_{j_{q}} \otimes dx^{i_{1}} \otimes \cdots \otimes dx^{i_{q}}\]

in a coordinate neighborhood \(U(x') \subset M_{n}\), then \(\alpha \circ \pi = \pi A \circ \alpha\).

Let \(A \in \mathcal{D}^{p}_{q}(M_{n})\). Then there is a unique vector field \(V A \in \mathcal{D}^{p}_{0}(T_{q}^{p}(M_{n}))\) such that for \(\alpha \in \mathcal{D}^{p}_{p}(M_{n})\)

\[V A(\alpha) = \alpha(A) \circ \pi = V(\alpha(A)),\]

where \(V(\alpha(A))\) is the vertical lift of the function \(\alpha(A) \in F(M_{n})\). We call \(V A\) the vertical lift of \(A \in \mathcal{D}^{p}_{q}(M_{n})\) to \(T_{q}^{p}(M_{n})\) (see [2]). The vertical lift \(V A\) has components of the form

\[V A = \begin{pmatrix}
V A_{j}^{j'} \\
V A_{j}^{k'}
\end{pmatrix} = \begin{pmatrix}
0 \\
A_{j_{1} \cdots j_{q}}^{i_{1} \cdots i_{p}}
\end{pmatrix}
\]

with respect to the coordinates \((x', x)\) in \(T_{q}^{p}(M_{n})\).

Let \(\nabla\) be a symmetric affine connection on \(M_{n}\). We define the horizontal lift \(H \nabla = \nabla_{V} \in \mathcal{D}^{p}_{0}(T_{q}^{p}(M_{n}))\) of \(V \in \mathcal{D}^{p}_{0}(M_{n})\) to \(T_{q}^{p}(M_{n})\) [2] by

\[H \nabla(\alpha) = \tau(\nabla \alpha), \ \alpha \in \mathcal{D}^{p}_{p}(M_{n}).\]

The horizontal lift \(H \nabla\) of \(V \in \mathcal{D}^{p}_{0}(M_{n})\) to \(T_{q}^{p}(M_{n})\) has components

\[(1.3) \quad H \nabla = V^{j}_{s} \left(\sum_{\mu=1}^{q} \Gamma_{\mu \nu}^{m} t_{j_{1} \cdots j_{k} \cdots j_{q}}^{i_{1} \cdots i_{p}} - \sum_{\lambda=1}^{p} \Gamma_{m \nu}^{i_{1} \cdots i_{k} \cdots i_{q}} t_{j_{1} \cdots j_{k} \cdots j_{q}}^{i_{1} \cdots i_{p}}\right)
\]

with respect to the coordinates \((x', x)\) in \(T_{q}^{p}(M_{n})\) [1], where \(\Gamma_{ij}^{k}\) are local components of \(\nabla\) in \(M_{n}\).

Suppose that there is given a tensor field \(\xi \in \mathcal{D}^{q}_{q}(M_{n})\). Then the correspondence \(x \rightarrow \xi_{x}, \ \xi_{x}\) being the value of \(\xi\) at \(x \in M_{n}\), determines a mapping \(\sigma_{\xi} : M_{n} \rightarrow T_{q}^{p}(M_{n})\), such that \(\pi \circ \sigma_{\xi} = id_{M_{n}}\), and the \(n\) dimensional submanifold \(\sigma_{\xi}(M_{n})\) of \(T_{q}^{p}(M_{n})\) is called the cross-section determined by \(\xi\). If the
tensor field $\xi$ has the local components $\xi^{l_1 \cdots l_p}_{k_1 \cdots k_q}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\begin{cases}
x^k = x^k \\
x^k = \xi^{l_1 \cdots l_p}_{k_1 \cdots k_q}(x^k)
\end{cases}$$

with respect to the coordinates $(x^k, x^k)$ in $T^p_q(M_n)$. Differentiating (1.4) by $x^j$, we see that $n$ tangent vector fields $B_j$ to $\sigma_\xi(M_n)$ have components

$$(B^K_j) = \left( \frac{\partial x^K}{\partial x^j} \right) = \left( \delta^k_{ij} \right)$$

with respect to the natural frame $\{\partial_k, \partial_k\}$ in $T^p_q(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases}
x^k = \text{const.} \\
t^{l_1 \cdots l_p}_{k_1 \cdots k_q} = t^{l_1 \cdots l_p}_{k_1 \cdots k_q}
\end{cases}$$

$t^{l_1 \cdots l_p}_{k_1 \cdots k_q}$ being considered as parameters. Thus, on differentiating with respect to $x^j = t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$, we see that $n^{p+q}$ tangent vector fields $C_j$ to the fibre have components

$$\begin{cases}
x^k = \text{const.} \\
t^{l_1 \cdots l_p}_{k_1 \cdots k_q} = t^{l_1 \cdots l_p}_{k_1 \cdots k_q}
\end{cases}$$

with respect to the natural frame $\{\partial_k, \partial_k\}$ in $T^p_q(M_n)$, where $\delta$ is the Kronecker symbol.

We consider in $\pi^{-1}(U) \subset T^p_q(M_n)$, $n + n^{p+q}$ local vector fields $B_j$ and $C_j$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_j\}$ along $\sigma_\xi(M_n)$, which is called the adapted $(B, C)$-frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. Taking account of (1.3) and also (1.5) and (1.6), we can easily prove that, the lifts $V^A$ and $HV$ have along $\sigma_\xi(M_n)$ components of the form

$$(V^A) = \left( \begin{array}{c} V^j \\ \bar{V}^j \end{array} \right) = \left( \begin{array}{c} 0 \\ A^{i_1 \cdots i_p}_{j_1 \cdots j_q} \end{array} \right),$$

$$(HV) = \left( \begin{array}{c} H^j \\ \bar{H}^j \end{array} \right) = \left( \begin{array}{c} V^j \\ -\nabla_V \xi^{i_1 \cdots i_p}_{j_1 \cdots j_q} \end{array} \right)$$

with respect to the adapted $(B, C)$-frame, where $(\nabla_V \xi^{i_1 \cdots i_p}_{j_1 \cdots j_q})$ are local components of $\nabla_V \xi$ in $M_n$. 
Let $A, B \in \mathfrak{X}_p(M_n)$, $V, W \in \mathfrak{X}_0(M_n)$ and $\varphi \in \mathfrak{X}_1(M_n)$. Let $R$ denotes the curvature tensor field of the connection $\nabla$. Then (see [1])

\begin{equation}
\begin{aligned}
[\nabla A, V B] &= 0 \\
[H V, V A] &= V \nabla V A \\
[H V, \gamma \varphi - \gamma \varphi] &= \tilde{\gamma}(L_V \varphi + (\nabla \varphi) - \varphi(\nabla V)) - \gamma(L_V \varphi + (\nabla \varphi) - \varphi(\nabla V)) \\
[H V, H W] &= H [V, W] + (\gamma - \gamma)R(V, W),
\end{aligned}
\end{equation}

where $\tilde{\gamma} \varphi - \gamma \varphi$ is a vector field in $T_p^p(M_n)$ defined by

\begin{equation}
\tilde{\gamma} \varphi - \gamma \varphi = \left(\sum_{\nu=1}^{q} i_{1 \nu \cdot \cdot \cdot \nu} \varphi_{m}^{m} - \sum_{\lambda=1}^{p} i_{1 \lambda \cdot \cdot \cdot \lambda} \varphi_{m}^{m}\right).
\end{equation}

2. Diagonal lifts of affinor fields on a cross-section

Let $\varphi \in \mathfrak{X}_1(M_n)$. We define a tensor field $\nabla \varphi \in \mathfrak{X}_1(T_p^p(M_n))$ along the cross-section $\sigma_{\xi}(M_n)$ by

\begin{equation}
\begin{aligned}
D \varphi(\nabla V) &= H (\varphi(V)), \forall V \in \mathfrak{X}_0(M_n) \\
D \varphi(V A) &= -V (\varphi(A)), \forall A \in \mathfrak{X}_p(M_n),
\end{aligned}
\end{equation}

where $\varphi(A) = C(\varphi \otimes A) \in \mathfrak{X}_p(M_n)$ and call $D \varphi$ the diagonal lift of $\varphi \in \mathfrak{X}_1(M_n)$ to $T_p^p(M_n)$ along $\sigma_{\xi}(M_n)$. Then, from (2.1) we have

\begin{equation}
\begin{aligned}
(i) & \quad D \varphi^{K} H \tilde{V} L = H (\varphi(\tilde{V}))^{K}, \\
(ii) & \quad D \varphi^{K} V A L = -V (\varphi(A))^{K},
\end{aligned}
\end{equation}

where $V(\varphi(A)) = \left(\begin{array}{c}
0 \\
V(\varphi(A))^{k}
\end{array}\right) = \left(\begin{array}{c}
0 \\
\varphi^{l}_{m, A_{k_1 \cdot \cdot \cdot k_q}}
\end{array}\right).

First, consider the case where $K = k$. In the case, (i) of (2.2) reduces to

\begin{equation}
D \varphi^{k} H \tilde{V} L + D \varphi^{k} D \tilde{V} L = H (\varphi(\tilde{V}))^{k} = (\varphi(V))^{k} = \varphi^{k} V^{l}.
\end{equation}

Since the right-hand side of (2.3) are functions depending only on the base coordinates $x^l$, the left-hand side of (2.3) are too. Then, since $H \tilde{V} L$ depend on fibre coordinates, from (2.3) we obtain

\begin{equation}
D \varphi^{k} H \tilde{V} L = 0,
\end{equation}

this implies

\begin{equation}
D \varphi^{k} = \varphi^{k}.
\end{equation}

When $K = k$, (ii) of (2.2) can be rewritten, by virtue of (1.7), (2.4) and (2.5), as $0 = 0$.

When $K = k$, (ii) of (2.2) reduces to

\begin{equation}
D \varphi^{k} V A L + D \varphi^{k} V A L = -V (\varphi(\tilde{V}))^{k}
\end{equation}

\end{document}
Thus, the diagonal lift

\[ D_{\tilde{\varphi}_l}^k A_{s_1 \ldots s_p} = -\varphi_{m}^l A_{k_1 \ldots k_q}^{m l_2 \ldots l_p} = -\varphi_{s_1}^l \delta_{s_2}^l \ldots \delta_{s_p}^l \delta_{k_1}^l \ldots \delta_{k_q}^l A_{s_1 \ldots s_p} \]

for all \( A \in \mathfrak{g}_l(M_n) \), which implies

\[ (2.6) \quad D_{\tilde{\varphi}_l}^k = -\varphi_{s_1}^l \delta_{s_2}^l \ldots \delta_{s_p}^l \delta_{k_1}^l \ldots \delta_{k_q}^l \]

where \( x^l = t_{r_1 \ldots r_q}, x^k = t_{k_1 \ldots k_q} \).

Similarly, we have

\[ (2.7) \quad D_{\tilde{\varphi}_l}^k = -\delta_{s_1}^l \ldots \delta_{s_p}^l \varphi_{k_1}^l \delta_{k_2}^l \ldots \delta_{k_q}^l \]

When \( K = \tilde{k}, \) (i) of (2.2) reduces to

\[ (2.8) \quad D_{\tilde{\varphi}_l}^k H \tilde{V}^l + D_{\tilde{\varphi}_l}^k H \tilde{V}^l = H (\varphi(V))^{\tilde{k}}. \]

We will investigate components \( D_{\tilde{\varphi}_l}^k \).

Let \( \xi \in \mathfrak{g}_l(M_n) \). We consider a new \( \Phi \)-operator

\[ (2.9) \quad (\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q} \varphi_\xi^m \nabla_m \xi_{k_1 \ldots k_q} + \{ \varphi_\xi^m \nabla_m \xi_{l_1 \ldots l_p}^{m l_2 \ldots l_p}, p \geq 1 \}
\]

\[ \varphi_\xi^m \nabla_m \xi_{l_1 \ldots l_p}^{m l_2 \ldots l_p}, q \geq 1. \]

From (2.9), we have

\[ V'(\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q} = (V \varphi_\xi^m ) \nabla_m \xi_{l_1 \ldots l_p}^{m l_2 \ldots l_p} \varphi_\xi^m \nabla_m \xi_{k_1 \ldots k_q} + \varphi_\xi^m V'(\varphi_\xi)^{m l_2 \ldots l_p}_{k_1 \ldots k_q}, p \geq 1 \]

and

\[ (2.10) \quad V'(\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q} = \varphi_\xi^m \nabla_m \xi_{k_1 \ldots k_q}, q \geq 1. \]

Using (1.8), from (2.10) we have

\[ (\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q} H V^l + \varphi_\xi^l \delta_{s_2}^l \ldots \delta_{s_p}^l \delta_{k_1}^l \ldots \delta_{k_q}^l H V^l = -H (\varphi(V))^{\tilde{k}} \]

and

\[ -(\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q} H V^l = \varphi_\xi^l \delta_{s_2}^l \ldots \delta_{s_p}^l \delta_{k_1}^l \ldots \delta_{k_q}^l H V^l = H (\varphi(V))^{\tilde{k}}. \]

Comparing (2.8) and (2.11) and making use of (2.6), we get

\[ D_{\tilde{\varphi}_l}^k = -(\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q}, p \geq 1. \]

Similarly, we obtain

\[ D_{\tilde{\varphi}_l}^k = -(\Phi \varphi_\xi)^{l_1 \ldots l_p}_{k_1 \ldots k_q}, q \geq 1. \]

Thus, the diagonal lift \( D_{\Phi} \) of \( \varphi \) has along the cross-section \( \sigma_\xi(M_n) \) components

\[ D_{\tilde{\varphi}_l}^k = \varphi_\xi^l \]

\[ D_{\tilde{\varphi}_l}^k = 0, \]

\[ (2.12) \quad D_{\tilde{\varphi}_l}^k = \left\{ \begin{array}{ll}
-\varphi_{s_1}^l \delta_{s_2}^l \ldots \delta_{s_p}^l \delta_{k_1}^l \ldots \delta_{k_q}^l, p \geq 1 \\
-\delta_{s_1}^l \ldots \delta_{s_p}^l \varphi_{k_1}^l \delta_{k_2}^l \ldots \delta_{k_q}^l, q \geq 1,
\end{array} \right. \]
Now, on putting $B^j = C_j$, we write the adapted $(B, C)$-frame of $\sigma_\xi(M_n)$ as $B_j = \{B_j, B_j^\xi\}$. We define a coframe $\tilde{B}^i$ of $\sigma^\xi_\xi(M_n)$ by $\tilde{B}^i(B_j) = \delta_j^i$. From (1.5), (1.6) and $B_j^k \bar{B}_k^j = \delta^i_j$ we see that covector fields $\tilde{B}^i$ have components

$$
\bar{B}^i = (\bar{B}^i_k) = (\delta_k^i, 0)
$$

(2.13) $\bar{B}^i = (\bar{B}^i_K) = (-\partial_k^i \xi_{11}^j \cdots \delta^i_{kj} \delta^j_{kl} \cdots \delta^j_{lp})$

with respect to the natural coframe $(dx^k, dx^p)$. Taking account of

$$
D \varphi^k = D \varphi(dx^K, \partial_k) = \varphi^k_J B_J \odot (dx^K, \partial_k)
$$

$$
= \varphi^k_J dx^K(B_J \bar{B}^i) = \varphi^k_J dx^K(B_J^H \partial_H \bar{B}^i_L)
$$

$$
= D \varphi^k J B^H \delta_H \bar{B}^i_L = D \varphi^k J B^H \bar{B}^i_L
$$

and also (1.5), (1.6), (2.12) and (2.13), we see that $D \varphi$ has along the cross-section $\sigma_\xi(M_n)$ components of the form

$$
D \varphi^k = \varphi^k, \\
D \varphi^k = 0,
$$

$$
D \varphi^k = -(\Phi \xi^1_l \cdots \xi^p_l + \varphi^m \partial_m \xi^1_l \cdots \xi^p_l + \sum_{\mu=1}^q \varphi^1_{\mu} \partial_{\mu}^m \xi^m_{lk} \cdots \xi^m_{lp}, \ p \geq 1
$$

$$
\varphi^m \partial_m \xi^m_{lk} \cdots \xi^m_{lp}, \ q \geq 1.
$$

Thus, $D \varphi$ has along the cross-section $\sigma_\xi(M_n)$ components of the form

$$
D \varphi^k = \varphi^k, \ D \varphi^k = 0, \ D \varphi^k = \left\{
\begin{array}{ll}
\varphi^m (\sum_{\mu=1}^p \Gamma^p_{\mu} \xi^1_{\mu} \cdots \xi^p_{\mu} + \sum_{\mu=1}^q \Gamma^q_{\mu} \xi^m_{\mu} \xi^m_{\mu} \cdots \xi^m_{\mu}, \ p \geq 1
\\
\varphi^m (\sum_{\mu=1}^p \Gamma^p_{\mu} \xi^1_{\mu} \cdots \xi^p_{\mu} - \sum_{\mu=2}^q \Gamma^q_{\mu} \xi^m_{\mu} \xi^m_{\mu} \cdots \xi^m_{\mu}, \ q \geq 1
\\
\varphi^m (\sum_{\mu=2}^q \Gamma^q_{\mu} \xi^m_{\mu} \xi^m_{\mu} \cdots \xi^m_{\mu} + \sum_{\mu=1}^p \Gamma^p_{\mu} \xi^m_{\mu} \xi^m_{\mu} \cdots \xi^m_{\mu}, \ q \geq 1
\end{array}
\right.
$$

with respect to the natural frame $\{\partial_h, \partial_h\}$ of $\sigma_\xi(M_n)$ in $\pi^{-1}(U) \subset T^p_\xi(M_n)$ [5].

In particular, if we put $p = 0, q = 1$ in (2.14), then $D \varphi^k$ are the components of the diagonal lift of $\varphi$ from manifold to its cotangent bundle with respect to the natural frame $\{\partial_h, \partial_h\}$ of $\sigma_\xi(M_n)$ [6, p.291].
3. Complex structure in $T_q^p(M_n)$

We shall first state the following lemma for later use.

**Lemma ([3]).** Let $\tilde{S}$ and $\tilde{T}$ be tensor fields in $T_q^p(M_n)$ of type $(1,q)$, where $q > 0$, such that

$$\tilde{S}(\tilde{V}_1, \ldots, \tilde{V}_q) = \tilde{T}(\tilde{V}_1, \ldots, \tilde{V}_q)$$

for all vector fields $\tilde{V}_1, \ldots, \tilde{V}_q$ which are of the form $\tilde{V}A$ or $H\tilde{V}$, where $A \in \mathfrak{D}_q^p(M_n)$ and $V \in \mathfrak{D}_q^p(M_n)$. Then $\tilde{S} = \tilde{T}$.

Let $\varphi \in \mathfrak{D}_q^p(M_n)$. We define $H \varphi \in \mathfrak{D}_q^p(T_q^p(M_n))$ along $\sigma_\xi(M_n)$ by

$$H \varphi = \begin{cases} H(\varphi(V)) = \varphi(V), & \forall V \in \mathfrak{D}_q^p(M_n) \\ H(\varphi(A)) = \varphi(A), & \forall A \in \mathfrak{D}_q^p(M_n), \end{cases}$$

where $\varphi(A) = C(\varphi \otimes A) \in \mathfrak{D}_q^p(M_n)$ ([4]).

**Theorem 3.1.** If $\varphi, \phi \in \mathfrak{D}_q^p(M_n)$, then with respect to symmetric affine connection $\nabla$ in $M_n$, we have

$$D \varphi D \phi + D \phi D \varphi = H(\varphi + \phi \varphi),$$

$$D \varphi H \phi + D \phi H \varphi = H(\varphi \phi + \phi \varphi).$$

**Proof.** If $V \in \mathfrak{D}_q^p(M_n)$ and $A \in \mathfrak{D}_q^p(M_n)$, then we have by using (2.1) and (3.1)

$$D(\varphi \phi)_H = D(\varphi \phi)_H(X) = D^H(\varphi \phi)_H(X) + D(\varphi \phi)_H(X) = H(\varphi \phi)_H(X) + H(\varphi \phi)_H(X) = H(\varphi \phi)_H(X).$$

In a way similar to that of the proof of (3.3), we can prove by using similar device easily. \qed

Putting $\varphi = \phi$ in (3.2), we obtain

$$D \varphi D \varphi + D \varphi D \varphi = H(\varphi \varphi), \quad (D \varphi)^2 = H(\varphi).$$

Since $H(id_{M_n}) = id_{\mathfrak{D}_q^p(M_n)}$, using (3.4), we have

**Theorem 3.2.** If $\varphi$ is almost complex structure in $M_n$, then the diagonal lift $D \varphi$ of $\varphi$ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ is an almost complex structure in $T_q^p(M_n)$.

**Theorem 3.3.** If $\varphi, \phi \in \mathfrak{D}_q^p(M_n)$, then

$$D \varphi (\tilde{\gamma} - \gamma) \phi = (\gamma - \tilde{\gamma})(\varphi \phi).$$

**Proof.** (3.5) can be prove by using local expressions of $D \varphi$ and $(\gamma - \tilde{\gamma})(\varphi \phi)$. \qed
Let $\varphi \in \mathfrak{X}_1(M_n)$ and $N_\varphi$ be the Nijenhuis tensor of $\varphi$:

$$N_\varphi(V,W) = [\varphi V, W] - \varphi [\varphi V, W] - [V, \varphi W] + \varphi^2 [V, W], \ V, W \in \mathfrak{X}_0(M_n).$$

Let now $\tilde{N}_{D,\varphi}$ be the Nijenhuis tensor of $D\varphi$ in $T^\varphi(M_n)$. Then by (1.9) and (3.5), if $V, W \in \mathfrak{X}_0(M_n)$ and $A, B \in \mathfrak{X}_0(M_n)$, we have

$$\tilde{N}_{D,\varphi}(V A, V B) = [D\varphi V, A, D\varphi V B] - D\varphi [D\varphi V A, V B] - D\varphi [V A, D\varphi V B] + (D\varphi)^2 [V A, V B]$$
$$= [V(\varphi(A)), V(\varphi(B))] + D\varphi [V(\varphi(A)), V B] + D\varphi [V A, V(\varphi(B))] +$$
$$+ (D\varphi)^2 [V A, V B]$$
$$= 0,$$

$$\tilde{N}_{D,\varphi}(H V, V B) = [D\varphi^H V, D\varphi^H V B] - D\varphi [D\varphi^H V, V B] - D\varphi [H V, D\varphi^H V B] + (D\varphi)^2 [H V, V B]$$
$$= - [H(\varphi(V)), V(\varphi(B))] - D\varphi [H(\varphi(V)), V B] + D\varphi [H V, V(\varphi(B))] +$$
$$+ (D\varphi)^2 [H V, V B]$$
$$= - V(\varphi(V))\varphi(B) + D\varphi^V(\varphi(V) B) + D\varphi^V(\varphi(V) B) + (D\varphi)^2 V(\varphi(V) B)$$
$$= V(-\varphi(V))\varphi(B) + \varphi(\varphi(V) B) - \varphi(\varphi(V) B) + \varphi^2(\varphi(V) B)$$
$$= V(-\varphi(V))\varphi(B) - \varphi(\varphi(V) B),$$

$$\tilde{N}_{D,\varphi}(H V, H W) = [D\varphi^H V, D\varphi^H W] - D\varphi [D\varphi^H V, H W] - D\varphi [H V, D\varphi^H W] + (D\varphi)^2 [H V, H W]$$
$$= [H(\varphi(V)), H(\varphi(W))] - D\varphi [H(\varphi(V)), H W] - D\varphi [H V, H(\varphi(W))] +$$
$$+ (D\varphi)^2 [H V, H W]$$
$$= H(\varphi(V), \varphi(W)) + (\tilde{\gamma} - \gamma) R(\varphi(V), \varphi(W)) - D\varphi(H(\varphi(V), \varphi(W)),$$
$$+ (\tilde{\gamma} - \gamma) R(\varphi(V), \varphi(W)) - D\varphi(H(\varphi(V), \varphi(W)),$$
$$+ (D\varphi)^2[H V, W] + (\tilde{\gamma} - \gamma) R(\varphi(V), \varphi(W)),$$
$$= H(\varphi(V), \varphi(W)) + (\tilde{\gamma} - \gamma) R((\varphi(V), \varphi(W)) - H(\varphi((\varphi(V), \varphi(W)))$$
$$+ (\tilde{\gamma} - \gamma) R((\varphi(V), \varphi(W))) - H(\varphi(V, \varphi(W))),$$
$$+ (\tilde{\gamma} - \gamma) R((\varphi(V), \varphi(W))) + (\tilde{\gamma} - \gamma) R(V, \varphi(W)),$$
$$+ (D\varphi)^2 V, W] + (\tilde{\gamma} - \gamma) R(V, \varphi(W))$$
$$= H(N_\varphi) + (\tilde{\gamma} - \gamma)(R(\varphi(V), \varphi(W)) - \varphi R(V, \varphi(W))$$
$$- \varphi R(V, \varphi(W) +$$
$$+ \varphi^2 R(V, W)).$$
Summing up, we have the following formulas:

\[
\begin{aligned}
\tilde{N}_{\varphi}(V^AV^B) &= 0 \\
\tilde{N}_{\varphi}(H^V^W) &= V(-\varphi(V)\varphi)B - \varphi(\varphi(V)B) \\
\tilde{N}_{\varphi}(H^V^H) &= H(N^V^W + (\bar{\varphi} - \varphi)(R(\varphi(V), \varphi(W)) - \varphi(R(\varphi(V), \varphi(W) - \varphi(R(\varphi(V), W) + \varphi^2R(V, W).
\end{aligned}
\]

We now suppose that \((\varphi, g)\) is a Kählerian structure in \(M_n\) and \(\nabla\) the Riemannian connection determined by the metric \(g\). Then we see that

(i) \(\varphi\) is an almost complex structure in \(M_n\), i.e., \(\varphi^2 = -I\);

(ii) \(\nabla\varphi = 0\);

(iii) The curvature tensor \(R\) of \(\nabla\) satisfies \(R(\varphi V, \varphi W) = R(V, W)\) for any \(V, W \in \mathfrak{g}(M_n)\) \[7, p.129\].

Thus, from (iii), we get \(R(\varphi V, W) = -R(V, \varphi W)\), since \(\varphi^2 = -I\). Hence again using \(\varphi^2 = -I\), we find

\[R(\varphi V, \varphi W) - \varphi R(V, \varphi W) - \varphi R(\varphi V, W) + \varphi^2 R(V, W) = 0.\]

Therefore it follows, from (3.6) and (ii), that

\[
\begin{aligned}
\tilde{N}_{\varphi}(V^AV^B) &= 0 \\
\tilde{N}_{\varphi}(H^V^W) &= 0 \\
\tilde{N}_{\varphi}(H^V^H) &= 0
\end{aligned}
\]

for any \(V, W \in \mathfrak{g}(M_n)\) and \(A, B \in \mathfrak{g}(M_n)\). Hence, by Lemma, \(\tilde{N}_{\varphi}\) is zero, since \(N\) is skew-symmetric. Thus, \(D\varphi\) is necessary integrable. Summing up, we have

\textbf{Theorem 3.4.} Let \((\varphi, g)\) be a Kählerian structure in \(M_n\) and \(\nabla\) the Riemannian connection determined by the metric \(g\). Then the diagonal lift \(D\varphi\) of \(\varphi\) to \(T^p_0(M_n)\) along \(\sigma_\xi(M_n)\) is an complex structure in \(T^p_0(M_n)\).

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