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A CONSTRUCTION OF ONE-FACTORIZATION

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Abstract. In this paper, we construct one-factorizations of given complete graphs of even order. These constructions partition the edges of the complete graph into one-factors and triples. Our new constructions of one-factors and triples can be applied to a recursive construction of Steiner triple systems for all possible orders \( \geq 15 \).

1. Introduction

A graph \( G = (V, E) \) consists of a finite set \( V \) of objects called vertices together with a set \( E \) of unordered pairs of vertices called edges. A subgraph of a graph \( G \) that contains every vertex of \( G \) is called a factor (or a spanning subgraph in [10]). A factorization of \( G \) is a set of factors of \( G \) which are pairwise edge-disjoint (that is, no two have a common edge) and whose union is all of \( G \). Since \( G \) is a factor of itself, \( \{ G \} \) is a factorization of \( G \) so that every graph has a factorization. However, it is more interesting to consider factorizations in which the factors satisfy certain conditions. For a given graph \( G \), a one-factor is a factor which is a regular graph of degree one. In other words, a one-factor is a set of pairwise disjoint edges of \( G \) which between them contain every vertex. A one-factorization of \( G \) is a partition of all the edges into one-factors each of which is, in its turn, a partition of the set of vertices.

One-factors and one-factorizations of the complete graph of even order \( 2n \), written as \( K_{2n} \), have been studied by several authors in [4, 7, 8]. A cyclic graph of order \( 2n \) (see [10]) is a graph whose vertices are the integers modulo \( 2n \) with the property that if \( \{ x, y \} \) is an edge then so is \( \{ x + i, y + i \} \) for \( 1 \leq i \leq 2n - 1 \) when \( |x - y| \neq n \); for \( 1 \leq i \leq n - 1 \) when \( |x - y| = n \). As general backgrounds on one-factorizations of the complete graphs, we refer to [1, 5, 9, 10]. The existence of a one-factor is known for all \( K_{2n} \) and it has been known that a one-factorization exists for \( K_{2n} \) as well (see [6]). As an application, it is

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well-known that a round robin tournament can be expressed as a proper one-factorization of a complete graph of even order (see [8, 10]). In the theory of designs, one-factorization of $K_{2n}$ has been used for constructing a Steiner triple system of order $v$ (written as $STS(v)$), which is a 2-design with $v$ points and blocks of size 3, called triples. We sometimes denote a Steiner triple system by $STS(v)=(V,T)$ for the set of vertices $V$ and the set of triples. In a given Steiner triple system $S$, a Steiner triple system $T$ is called a subsystem of $S$ if every block of $T$ is a block of $S$. J. Doyen and R. Wilson show in [3] that $STS(u)$ is a subsystem of $STS(v)$ for every $u$ and $v$ such that $u, v \equiv 1, 3 \pmod{6}$ and $v > 2u$. Similar recursive construction method of $STS(2v+s)$ for $s=1, 7$ based on a given $STS(v)$ can be found in [2, 8, 10], which shows the existence of Steiner triple system for all feasible orders.

In this paper, we give new constructions of one-factorizations of a cyclic complete graph $K_{2n}$ depending on whether $2n/\gcd(\alpha, 2n)$ is even or odd, where $\alpha$ is a positive integer in the set of all edge differences in $K_{2n}$. For the odd case of $\alpha$, we next give a factorization by modifying our one-factorization in order to obtain an infinite family of $STS(v)$ by applying Doyen-Wilson theorem [3] on a recursive construction scheme. Then, we obtain a family of $STS(v)$s for all possible orders $v \equiv 1, 3 \pmod{6}$ derived from our modified one-factorizations for odd $\alpha$'s and the one-factorizations given in [10] for even $\alpha$'s. These Steiner triple systems are different from the ones given by Bose, seen in [2].

2. Constructions of one-factorization

Let $K_{2n}$ be the cyclic complete graph whose vertex-set is the additive group of residue classes of integers modulo $2n$, that is,

$$Z_{2n} = \{0, 1, \ldots, 2n-1\}.$$

For $\alpha = 1, 2, \ldots, n$, let

$$E_{\alpha} = \{\{i, j\} \mid i - j \equiv \pm\alpha \pmod{2n}\}.$$

Then $\{E_{\alpha} \mid \alpha = 1, 2, \ldots, n\}$ is a partition of the edge-set of $K_{2n}$ and $E_{\alpha}$ is always a one-factor of $K_{2n}$. In this case, $\alpha$ is called the difference of an edge $\{i, j\}$ and $E_{\alpha}$ is said to have $\alpha$-difference.

Let $D$ be the set of all differences of $K_{2n}$. Then $D = D_e \cup D_o$, where

$$D_e = \{\alpha \mid 2n/\gcd(\alpha, 2n) \text{ is even, } 1 \leq \alpha \leq n\}$$

and

$$D_o = \{\alpha \mid 2n/\gcd(\alpha, 2n) \text{ is odd, } 1 \leq \alpha \leq n\}.$$

Note that if $\alpha$ is odd, then $\alpha$ must be in $D_e$.

We now give a construction of one-factors of $K_{2n}$ as follows.
Construction 1. (1) Let \( \alpha \in D_e, \alpha \neq n \) and let \( g_\alpha = \gcd(\alpha, 2n) \). Then, as a graph, \( E_\alpha \) consists of \( g_\alpha \) component cycles of length \( \frac{2n}{g_\alpha} \) (see [10]) which means

\[
E_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{ j+i \alpha, j+\alpha +i \alpha \} \mid i = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1 \right\}.
\]

By taking alternate members of the cycles, we have two one-factors \( F_\alpha \) and \( F_\alpha + \alpha \) of the complete graph \( K_{2n} \) as follows:

\[
F_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{ j+i \alpha, j+2\alpha +i \alpha \} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\},
\]

\[
F_\alpha + \alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{ j+2\alpha +i \alpha, j+2\alpha +2i \alpha \} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\}.
\]

That is,

\[
F_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{ j+i \alpha, j+2\alpha +i \alpha \} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\},
\]

\[
F_\alpha + \alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{ j+2\alpha +i \alpha, j+2\alpha +2i \alpha \} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\}.
\]

(2) Let \( \alpha \in D_o \) and \( g_\alpha = \gcd(\alpha, 2n) \). For each \( j = \frac{2n}{\alpha}, l = 0, 1, \ldots, \frac{2n-1}{g_\alpha} \) and each \( k = 1, 2, \ldots, \frac{2n}{g_\alpha} - 1 \), define

\[
A_\alpha(j, k) = \bigcup_{i=0}^{\frac{2n-1}{g_\alpha}} \left\{ \left\{ \frac{g_\alpha k}{2} + i + j, 2n - \frac{g_\alpha k}{2} + i + j \right\}, \{i + j, n + i + j\} \right\}.
\]

Then for each \( j = \frac{2n}{\alpha}, l = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1 \),

\[
F_\alpha + j = \bigcup_{k=1}^{\frac{2n-1}{g_\alpha}} A_\alpha(j, k)
\]

is a one-factor of \( K_{2n} \).

The one-factors shown in (2) of Construction 1 satisfies the following proposition.

Proposition 2. Let \( K_v \) be the cyclic complete graph. If \( v = 2^r \) for some odd number \( p > 1 \) and some \( r \in \mathbb{N} \), then

\[
\bigcup \{ F_2^r + j \mid j = 2^{r-1}l, 0 \leq l \leq p - 1 \} = \bigcup \{ E_\alpha \mid \alpha \in D_o \cup \{ \frac{2n}{2^r} \} \}.
\]

Proof. Let \( v = 2^r p \) for some odd number \( p > 1 \) and some \( r \in \mathbb{N} \). Then \( D_o = \{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \frac{p-1}{2} \times 2^r \} \). For a difference \( \alpha = 2^r \), let \( g_\alpha = \gcd(2^r, 2^r p) = 2^r \). For each \( j = 2^{r-1}l, l = 0, 1, \ldots, p - 1 \) we define \( F_2^r + j \) as (2) in Construction 1. Then for each \( j = 2^{r-1}l \) and \( 0 \leq l \leq p - 1 \), \( F_2^r + j \)
is one-factor and the difference of each edge of $F_{2^r} + j$ is one of $2^r$, $2 \times 2^r$, $3 \times 2^r, \ldots, \left(\frac{p-1}{2}\right) \times 2^r$, and $2^{r-1}p$. Hence there are $p$ one-factors which are pairwise edge-disjoint and the set of differences of all edges of $F_{2^r} + j$ is equal to $D_o \cup \left\{ \frac{v}{2} \right\}$, so that

$$\bigcup \{F_{2^r} + j | j = 2^{r-1}l, \ 0 \leq l \leq p - 1\} \subset \bigcup \{E_\alpha | \alpha \in D_o \cup \left\{ \frac{v}{2} \right\}\}.$$  

Note that the cardinal number of the right side is $(p - 1)v/2 + v/2$ from the definition of $E_\alpha$. Since the cardinal number of left side is $2^{r-1} (p - 1)$, we have the equality of the statement. This completes the proof. □

Note that if $v = 2n$ is a power of 2, then $D_o = \emptyset$ so that $D = D_c$. The following theorem, which is a construction of one-factorization for $v = 2^r$, is known in [4, 7, 8, 10].

**Theorem 3.** If $v = 2^r$ for some $r \in \mathbb{N}$, then the cyclic graph $K_v$ has a one-factorization

$${\bar{G}}_{2^r} = \{F_\alpha, F_\alpha + \alpha | \alpha \in D_c, \alpha \neq \frac{v}{2}\} \cup \{E_{\frac{v}{2}}\},$$

where $F_\alpha, F_\alpha + \alpha$ and $E_{\frac{v}{2}}$ are defined in (1) of Construction 1.

We now have a new construction of one-factorizations of the cyclic complete graph $K_{2n}$ from Proposition 2 and Theorem 3.

**Theorem 4.** If $v = 2^r p$ for some $r \in \mathbb{N}$ and odd number $p$, then the cyclic graph $K_v$ has a one-factorization

$${\bar{G}}_{2^r p} = \{F_\alpha, F_\alpha + \alpha | \alpha \in D_c, \alpha \neq \frac{v}{2}\} \cup \{F_{2^r} + 2^{r-1}l | l = 0, 1, \ldots, p - 1\},$$

where $F_\alpha, F_\alpha + \alpha$ and $F_{2^r} + 2^{r-1}l (l = 0, 1, \ldots, p - 1)$ are one-factors defined in Construction 1.

**Proof.** If $p = 1$, it is obvious from Theorem 3. Now we suppose $v = 2^r p$ for $p > 1$ and $r \in \mathbb{N}$ so that $D = \{1, 2, 3, \ldots, 2^{r-1}p\}$. For $\alpha \in D$, let $g_\alpha = \text{gcd}(\alpha, v)$. For $\alpha = 2^r$, by Proposition 2, note that

$$\{F_{2^r} + 2^{r-1}l | l = 0, 1, \ldots, p - 1\}$$

consists of $p$ edge-disjoint one-factors of $K_v$ and

$$\bigcup \{F_{2^r} + 2^{r-1}l | l = 0, 1, \ldots, p - 1\} = \bigcup \{E_\alpha | \alpha \in D_o \cup \left\{ \frac{v}{2} \right\}\}.$$  

For each $\alpha \in D_c - \left\{ \frac{v}{2} \right\}$, the one-factors $F_\alpha$ and $F_\alpha + \alpha$ have $\alpha$-difference by (1) of Construction 1. Since

$$D_o \cup \left\{ \frac{v}{2} \right\} = \left\{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \left(\frac{p-1}{2}\right) \times 2^r, 2^{r-1}p \right\},$$

the cardinal number of $D_c - \left\{ \frac{v}{2} \right\}$ is

$$|D - \left\{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \left(\frac{p-1}{2}\right) \times 2^r, 2^{r-1}p \right\}| = 2^{r-1}p - \frac{p-1}{2} = 1.$$
Thus there are
\[
(2^{r-1}p - \frac{p-1}{2} - 1) \times 2 = 2^r p - p - 1
\]
one-factors of \( K_v \) and the set of all edges of \( \{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \} \)
is equal to the set \( \bigcup \{ E_\alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \} \). Hence the total number of one-factors of \( K_v \) is \( p + 2^r p - p - 1 = 2^r p - 1 = v - 1 \) and the set of all edges of \( \mathcal{F}_{2^r p} \) is equal to the set \( \bigcup \{ E_\alpha \mid \alpha \in D \} \), where
\[
\mathcal{F}_{2^r p} = \{ F_{2^r} + 2^r - l \mid l = 0, 1, \ldots, p - 1 \} \cup \{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \} .
\]
Therefore, the cyclic complete graph \( K_v \) has a one-factorization \( \mathcal{F}_{2^r p} \) which consists of \( v - 1 \) one-factors. \( \square \)

We now apply the previous Theorem 4 to a complete graph \( K_{10} \).

**Example 5.** Consider the cyclic complete graph \( K_{10} \). Then \( D_e = \{1, 3, 5\} \) and \( D_o = \{2, 4\} \).

If \( \alpha = 1 \in D_e \), then \( g_\alpha = \gcd(1, 10) = 1 \) and \( j = 0 \); so, we have
\[
F_1 = \{ \{2i, 2i + 1\} \mid i = 0, 1, 2, 3, 4\}
\]
\[
= \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\},
\]
\[
F_1 + 1 = \{\{2i + 1, 2i + 2\} \mid i = 0, 1, 2, 3, 4\}
\]
\[
= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 0\}\} .
\]

If \( \alpha = 3 \in D_e \), then \( g_\alpha = \gcd(3, 10) = 1 \) and \( j = 0 \); so, we have
\[
F_3 = \{\{6i, 6i + 3\} \mid i = 0, 1, 2, 3, 4\}
\]
\[
= \{\{0, 3\}, \{6, 9\}, \{2, 5\}, \{8, 1\}, \{4, 7\}\},
\]
\[
F_3 + 3 = \{\{6i + 3, 6i + 6\} \mid i = 0, 1, 2, 3, 4\}
\]
\[
= \{\{3, 6\}, \{9, 2\}, \{5, 8\}, \{1, 4\}, \{7, 0\}\} .
\]

Now, since \( 10 = 2^r p = 2^4 \times 5 \) for the difference \( \alpha = 2 \) we have \( g_2 = \gcd(2, 10) = 2 \) and \( j = 0, 1, 2, 3, 4 \). Thus
\[
F_2 + 0 = \{\{k, (10 - k)\} \mid 0, 5\} \mid k = 1, 2, 3, 4\}
\]
\[
= \{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{0, 5\}\},
\]
\[
F_2 + 1 = \{\{k + 1, (10 - k) + 1\} \mid 0, 5, 1, 1\} \mid k = 1, 2, 3, 4\}
\]
\[
= \{\{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{1, 6\}\},
\]
\[
F_2 + 2 = \{\{k + 2, (10 - k) + 2\} \mid 0, 5, 2, 2\} \mid k = 1, 2, 3, 4\}
\]
\[
= \{\{3, 1\}, \{4, 10\}, \{5, 9\}, \{6, 8\}, \{2, 7\}\},
\]
\[
F_2 + 3 = \{\{k + 3, (10 - k) + 3\} \mid 0, 5, 3, 3\} \mid k = 1, 2, 3, 4\}
\]
\[
= \{\{4, 2\}, \{5, 1\}, \{6, 10\}, \{7, 9\}, \{3, 8\}\},
\]
\[
F_2 + 4 = \{\{k + 4, (10 - k) + 4\} \mid 0, 5, 4, 4\} \mid k = 1, 2, 3, 4\}
\]
\[
= \{\{5, 3\}, \{6, 2\}, \{7, 1\}, \{8, 10\}, \{4, 9\}\} .
\]

Hence \( \{ F_\alpha, F_\alpha + \alpha \mid \alpha = 1, 3\} \cup \{ F_2 + j \mid j = 0, 1, 2, 3, 4\} \) forms a one-factorization of \( K_{10} \) which consists of \( 9 \) one-factors of \( K_{10} \).

By modifying one-factors in Construction 1, we obtain a new factorization consisting of one-factors and triples for \( v \equiv 0 \pmod{6} \) and \( v > 3 \).
Construction 6. Take $\mathcal{F}_{2r}$ the one-factorization of $K_v$ stated in Theorem 4. If $v \equiv 0 \pmod{6}$, then for each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p - 1$) and each $k = 1, 2, \ldots, p - 1$, define

\[ A_{2r}^*(j, k) = A_{2r}(j, k) - \bigcup_{i=0}^{\frac{v}{6} - 1} \left\{ \left\{ \frac{v}{6} + j + i, \frac{5v}{6} + j + i \right\}, \left\{ \frac{v}{3} + j + i, \frac{2v}{3} + j + i \right\} \right\} \]

\[ \bigcup_{i=0}^{\frac{v}{6} - 1} \left\{ \left\{ \frac{v}{6} + j + i, \frac{v}{3} + j + i \right\}, \left\{ \frac{5v}{6} + j + i, \frac{2v}{3} + j + i \right\} \right\}. \]

For each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p - 1$), let

\[ F_{2r}^* + j = \bigcup_{k=1}^{p-1} A_{2r}^*(j, k) \]

be a one-factor of the cyclic complete graph $K_v$. Define a $\frac{v}{3}$-set of triples as

\[ T_{\frac{v}{3}} = \left\{ \left\{ i, \frac{v}{3} + i, \frac{2v}{3} + i \right\} \mid i = 0, 1, \ldots, \frac{v}{3} - 1 \right\}. \]

Let

\[ \mathfrak{F}_1 = \{ F_{2r}^* + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \} \]

and

\[ \mathfrak{F}_2 = \{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_c - \{ \frac{v}{6}, \frac{v}{2} \} \}, \]

where $F_\alpha$ and $F_\alpha + \alpha$ are one-factors defined in Construction 1. Finally, we define $\mathfrak{F}_v^*$ to be

\[ \mathfrak{F}_v^* = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup T_{\frac{v}{3}} \]

which consists of one-factors and triples.

The set $\mathfrak{F}_v^*$ defined in Construction 6 satisfies the following theorem.

Theorem 7. If $v \equiv 0 \pmod{6}$, then the edges of the cyclic $K_v$ are partitioned into $\frac{v}{3}$ triples and $v - 3$ one-factors.

Proof. By the definition in Construction 6, the set $\mathfrak{F}_1$ consists of $p$ one-factors in which each edge of the set $\bigcup \{ E_\alpha \mid \alpha \in D_c - \{ \frac{v}{6}, \frac{v}{2} \} \}$ occurs exactly once. Note that for each $\alpha \in D_c - \{ \frac{v}{6}, \frac{v}{2} \}$, there are two edge-disjoint one-factors. Thus $\mathfrak{F}_2$ defined in Construction 6 consists of

\[ 2 \times \left( 2^{r-1}p - \frac{p + 1}{2} - 1 \right) - 1 = 2^r p - p - 3 \]

pairwise edge-disjoint one-factors, and the set of all edges of $\mathfrak{F}_2$ is equal to the set $\bigcup \{ E_\alpha \mid \alpha \in D_c - \{ \frac{v}{6}, \frac{v}{2} \} \}$. Hence the total number of pairwise edge-disjoint one-factors is

\[ p + (2^r p - p - 3) = v - 3. \]
From the definition of $T_3$ in Construction 6, $T_3$ consists of $\frac{n}{3}$ triples in which each edge of $E_3$ occurs exactly once.

From Construction 6, we have the following example for $K_{12}$.

**Example 8.** Consider the cyclic complete graph $K_{12}$. Firstly we have an one-factorization which consists of 11 one-factors from Construction 1. Then $D_e = \{1, 2, 3, 5, 6\}$, and $D_o = \{4\}$.

If $\alpha = 1 \in D_e$, then we have

$$F_1 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}\},$$

$$F_1 + 1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 0\}\}.$$

If $\alpha = 2 \in D_e$, then we have

$$F_2 = \{\{0, 2\}, \{4, 6\}, \{8, 10\}, \{1, 3\}, \{5, 7\}, \{9, 11\}\},$$

$$F_2 + 2 = \{\{2, 4\}, \{6, 8\}, \{10, 0\}, \{3, 5\}, \{7, 9\}, \{11, 1\}\}.$$

If $\alpha = 3 \in D_e$, then we have

$$F_3 = \{\{0, 3\}, \{6, 9\}, \{1, 4\}, \{7, 10\}, \{2, 5\}, \{8, 11\}\},$$

$$F_3 + 3 = \{\{3, 6\}, \{9, 0\}, \{4, 7\}, \{10, 1\}, \{5, 8\}, \{11, 2\}\}.$$

If $\alpha = 5 \in D_e$, then we have

$$F_5 = \{\{0, 5\}, \{10, 3\}, \{8, 1\}, \{6, 11\}, \{4, 9\}, \{2, 7\}\},$$

$$F_5 + 5 = \{\{5, 10\}, \{3, 8\}, \{1, 6\}, \{11, 4\}, \{9, 2\}, \{7, 0\}\}.$$

Now, for the differences $\alpha = 4 \in D_o$ we have 3 one-factors

$$F_4 + 0 = \{\{2, 10\}, \{0, 6\}, \{3, 11\}, \{1, 7\}, \{4, 8\}, \{5, 9\}\},$$

$$F_4 + 2 = \{\{4, 0\}, \{2, 8\}, \{5, 1\}, \{3, 9\}, \{6, 10\}, \{7, 11\}\},$$

$$F_4 + 4 = \{\{6, 2\}, \{4, 10\}, \{7, 3\}, \{5, 11\}, \{8, 0\}, \{9, 1\}\}$$

which consist of edges with differences $\alpha = 4, 6$. Hence

$$\{F_\alpha, F_\alpha + \alpha \mid \alpha = 1, 2, 3, 5\} \cup \{F_4 + j \mid j = 0, 2, 4\}$$

forms a one-factorization of $K_{12}$ which consists of 11 one-factors of $K_{12}$.

Applying Construction 6 to this one-factorization, we have 9 one-factors and 4 triples as follows.

$$F_1^* + 0 = \{\{2, 4\}, \{10, 8\}, \{3, 5\}, \{11, 9\}, \{0, 6\}, \{1, 7\}\},$$

$$F_1^* + 2 = \{\{4, 6\}, \{0, 10\}, \{5, 7\}, \{1, 11\}, \{2, 8\}, \{3, 9\}\},$$

$$F_1^* + 4 = \{\{6, 8\}, \{2, 0\}, \{7, 9\}, \{3, 1\}, \{4, 10\}, \{5, 11\}\},$$

$$F_1, F_1 + 1, F_3, F_3 + 3, F_5, F_5 + 5 \text{ and}$$

$$T_4 = \{\{0, 4, 8\}, \{1, 5, 9\}, \{2, 6, 9\}, \{3, 7, 11\}\}.$$

We define

$$\mathfrak{F}_1^* = \{F_1^* + 0, F_1^* + 2, F_1^* + 4\},$$

$$\mathfrak{F}_2^* = \{F_1, F_1 + 1, F_3, F_3 + 3, F_5, F_5 + 5\},$$

and

$$\mathfrak{F}_{12}^* = \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup T_4.$$
Then \( \mathfrak{F}_{12} \) is a factorization which consists of 9 one-factors and 4 triples from Construction 6.

Next, by modifying one-factorization in Construction 1, we obtain a new factorization consisting of one-factors and triples for \( v \equiv 0 \pmod{2} \), \( v > 7 \) and \( v \neq 10 \) as follows.

**Construction 9.** Take \( \mathfrak{F}_{2p} \) the one-factorization of \( K_v \) stated in Theorem 4. If \( v \equiv 0 \pmod{2} \), \( v > 7 \) and \( v \neq 10 \), then for each \( j = 2^{r-1}l \ (l = 0, 1, \ldots, p-1) \) we have the following one-factor:

\[
F_j^* + j = (F_2 + j - \{(1 + j, (v - 1) + j), (\frac{v}{2} - 1) + j, (\frac{v}{2} + 1) + j\}) \\
\cup \{(1 + j, (\frac{v}{2} - 1) + j), (v - 1) + j, (\frac{v}{2} + 1) + j\}.
\]

Define a \( v \)-set of triples as

\[
T_{1,2,3} = \{\{(i, i + 1, i + 3) \mid i = 0, 1, 2, \ldots, v - 1\}.
\]

Let

\[
\mathfrak{F}_1 = \{F_{2r} + 2^{r-1}l \mid r > 1, l = 0, 1, \ldots, p - 1\}, \\
\mathfrak{F}_1^* = \{F_2 + j \mid j = 0, 1, \ldots, \frac{v}{2} - 1\}, \\
\mathfrak{F}_2 = \{F_2, F_2 + \alpha \mid \alpha \in D_c - \{1, 3, \frac{v}{2} - 2, \frac{v}{2}\}\}, \text{ and} \\
\mathfrak{F}_3 = \{F_2, F_2 + \alpha \mid \alpha \in D_c - \{1, 2, 3, \frac{v}{2}\}\}.
\]

We define \( \mathfrak{F}_v^* \) to be

\[
\mathfrak{F}_v^* = \left\{ \begin{array}{ll}
\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup T_{1,2,3} & \text{if } v = 2p, v \neq 10, \\
\mathfrak{F}_1 \cup \mathfrak{F}_3 \cup T_{1,2,3} & \text{if } v = 2^r p, r > 1
\end{array} \right.
\]

which consists of one-factors and triples.

The set \( \mathfrak{F}_v^* \) defined in Construction 9 satisfies the following theorem.

**Theorem 10.** If \( v \equiv 0 \pmod{2} \) and \( v > 7, v \neq 10 \), then the edges of the cyclic \( K_v \) are partitioned into \( v \) triples and \( v - 7 \) one-factors.

**Proof.** For \( v = 2^r p \), we have two cases: \( r = 1 \) and \( r > 1 \). If \( r = 1 \) and \( v \neq 10 \), then \( 2 \in D_c \). From the definition of \( \mathfrak{F}_1^* \) in Construction 9, \( \mathfrak{F}_1^* \) consists of \( p \) pairwise edge-disjoint one-factors and the set of all edges of \( \mathfrak{F}_1^* \) is equal to the set

\[
\bigcup \left\{ E_\alpha \mid \alpha \in D_c \cup \left\{ 2 - \frac{v}{2}, \frac{v}{2} \right\} \right\}.
\]

Note that for each \( \alpha \in D_c - \{1, 3, \frac{v}{2} - 2, \frac{v}{2}\} \), there are edge-disjoint two one-factors \( F_\alpha \), \( F_\alpha + \alpha \). Thus \( \mathfrak{F}_2^* \) defined in Construction 9 contains \( 2 \times (2^{r-1}p - \frac{v}{2} - 1 - 3) \) pairwise edge-disjoint one-factors and the set of all edges of \( \mathfrak{F}_2^* \) is equal to

\[
\bigcup \left\{ E_\alpha \mid \alpha \in D_c - \left\{ 1, 3, \frac{v}{2} - 2, \frac{v}{2} \right\} \right\}.
\]

Hence for the case \( r = 1 \), \( \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \) consists of

\[
p + 2 \times (2^{r-1}p - \frac{v}{2} - 1 - 3) = v - 7
\]
pairwise edge-disjoint one-factors in which each edge of the set
\[ \bigcup \{ E_\alpha \mid \alpha \in D - \{1, 2, 3\} \} \]
appears exactly once.

Now, we suppose \( r > 1 \); then, \( 2 \in D_e \). From the definition of \( \mathcal{F}_3^* \) in Construction 9, there are \( 2 \times (2^{r-1}p - \frac{p-1}{2} - 1 - 3) = 2^r p - p - 7 \) one-factors in which every edges of
\[ \bigcup \{ E_\alpha \mid \alpha \in D_e - \{1, 2, 3, v\} \} \]
appears exactly once. From Proposition 2, \( \mathcal{F}_1 \) defined in Construction 9 consists of \( p \) one-factors in which each edge of \( \bigcup \{ E_\alpha \mid \alpha \in D_o \cup \{v\} \} \) occurs exactly once. Hence, for \( r > 1 \), \( \mathcal{F}_1 \cup \mathcal{F}_3^* \) consists of \( p + (2^r p - p - 7) = v - 7 \) one-factors and \( v \) triples in which each edge of \( K_v \) appears exactly once. Then, from the definition of \( T_{1,2,3} \), we have \( v \) edge-disjoint triples and \( v \) sets of all of them is equal to \( \{ E_\alpha \mid \alpha = 1, 2, 3 \} \). In all, for \( v \equiv 0 \pmod{2} \), \( v > 7 \) and \( v \neq 10 \),
\[ \mathcal{F}_v^* = \left\{ \begin{array}{ll} \mathcal{F}_1 \cup \mathcal{F}_3 \cup T_{1,2,3} & \text{if } v = 2p, v \neq 10, \\
\mathcal{F}_1 \cup \mathcal{F}_3 \cup T_{1,2,3} & \text{if } v = 2^r p 
\end{array} \right. \]
consists of \( v - 7 \) one-factors and \( v \) triples in which each edge of \( K_v \) appears exactly once. \[ \square \]

Note that Construction 6 and 9 directly imply the Theorem 7 and 10, respectively, which can be seen in [2, 7, 9].

Now we remark the excluded case that \( v = 10 \) from Construction 9. In \( K_{10} \), we have three one-factors and ten triples as follows;
\[ \{(0, 5), (1, 7), (2, 6), (3, 9), (4, 8)\}, \]
\[ \{(1, 6), (2, 8), (3, 7), (4, 0), (5, 9)\}, \]
\[ \{(2, 7), (3, 8), (4, 9), (0, 6), (1, 5)\}, \]
and \( \{i, i + 1, i + 3 \mid i = 0, 1, 2, \ldots, 10 - 1\} \).

Including this construction for \( K_{10} \), we finally have the following theorem.

**Theorem 11.** If \( v \equiv 0 \pmod{2} \) and \( v > 7 \), then the edges of the cyclic \( K_v \) are partitioned into \( v \) triples and \( v - 7 \) one-factors.

We remark that the result in Theorem 11 can be also seen in [2, 7, 9] with different approaches.

As an example of Construction 9, we have the following factorization of \( K_{14} \).

**Example 12.** For the cyclic complete graph \( K_{14} \), we have a factorization which consists of 7 one-factors and 14 triples.
From (2) in Construction 1, we first have 7 one-factors

\[ F_2^* + 0 = \{\{1, 13\}, \{2, 12\}, \{3, 11\}, \{4, 10\}, \{5, 9\}, \{6, 8\}, \{0, 7\}\} \]

\[ F_2^* + 1 = \{\{2, 0\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{7, 9\}, \{1, 8\}\} \]

\[ F_2^* + 2 = \{\{3, 1\}, \{4, 0\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{8, 10\}, \{2, 9\}\} \]

\[ F_2^* + 3 = \{\{4, 2\}, \{5, 1\}, \{6, 0\}, \{7, 13\}, \{8, 12\}, \{9, 11\}, \{3, 10\}\} \]

\[ F_2^* + 4 = \{\{5, 3\}, \{6, 2\}, \{7, 1\}, \{8, 0\}, \{9, 13\}, \{10, 12\}, \{4, 11\}\} \]

\[ F_2^* + 5 = \{\{6, 4\}, \{7, 3\}, \{8, 2\}, \{9, 1\}, \{10, 0\}, \{11, 13\}, \{5, 12\}\} \]

\[ F_2^* + 6 = \{\{7, 5\}, \{8, 4\}, \{9, 3\}, \{10, 2\}, \{11, 1\}, \{12, 0\}, \{6, 13\}\} \]

for the edge difference \( \alpha \in D_0 \cup \{7\} = \{2, 4, 6\} \cup \{7\} \) and 6 one-factors

\[ F_1^*, F_1^* + 1, F_3^*, F_3^* + 3, F_5^*, F_5^* + 5 \]

for the edge difference \( \alpha \in D_c - \{7\} = \{1, 3, 5, 7\} - \{7\} \).

By modifying these 13 one-factors, we construct new factorization consisting of 7 one-factors and 14 triples as follows.

\[ F_2^* + 0 = \{\{1, 6\}, \{2, 12\}, \{3, 11\}, \{4, 10\}, \{5, 9\}, \{13, 8\}, \{0, 7\}\} \]

\[ F_2^* + 1 = \{\{2, 7\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{0, 9\}, \{1, 8\}\} \]

\[ F_2^* + 2 = \{\{3, 8\}, \{4, 0\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{1, 10\}, \{2, 9\}\} \]

\[ F_2^* + 3 = \{\{4, 9\}, \{5, 1\}, \{6, 0\}, \{7, 13\}, \{8, 12\}, \{9, 11\}, \{3, 10\}\} \]

\[ F_2^* + 4 = \{\{5, 10\}, \{6, 2\}, \{7, 1\}, \{8, 0\}, \{9, 13\}, \{3, 12\}, \{4, 11\}\} \]

\[ F_2^* + 5 = \{\{6, 11\}, \{7, 3\}, \{8, 2\}, \{9, 1\}, \{10, 0\}, \{4, 13\}, \{5, 12\}\} \]

\[ F_2^* + 6 = \{\{7, 12\}, \{8, 4\}, \{9, 3\}, \{10, 2\}, \{11, 1\}, \{5, 0\}, \{6, 13\}\} \]

and

\[ T_{1,2,3} = \{\{1, i+1, i+3\} \mid i = 0, 1, \ldots, 13\}\]

We define

\[ \mathfrak{F}_1^* = \{F_2^* + j \mid j = 0, 1, \ldots, 6\} \]

\[ \mathfrak{F}_2^* = \emptyset. \]

Then we have a factorization

\[ \mathfrak{F} = \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup T_{1,2,3} \]

which consists of 7 one-factors and 14 triples.

We now apply our one-factors and triples given from Constructions 1, 6, and 9 to the well-known recursive construction of STS suggested by J. Doyen and R. Wilson [3] described as follows.

**Theorem 13.** If there is a STS\( (v) \), then there is a STS\( (2v+s) \) with the original STS\( (v) \) as a subsystem for \( s = 1, \) or 7. If \( v \equiv 3 \) (mod 6) and \( s = 3 \), then there is a STS\( (2v+s) \) with the original STS\( (v) \) as a subsystem.

Theorem 13 (also shown in [2, 8, 10]) guaranties that STS\( (2v+s) \) is obtained from combining the given subsystem STS\( (v) \) with our one-factorization of \( K_{v+s} \) for each \( s = 1, 7, \) and STS\( (2v+s) \) is also guarantied for the case when \( s = 3 \) and \( v \equiv 3 \) (mod 6).
A CONSTRUCTION OF ONE-FACTORIZATION

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