WEAKLY PRIME LEFT IDEALS IN NEAR-SUBTRACTION SEMIGROUPS

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Abstract. In this paper we introduce the notion of weakly prime left ideals in near-subtraction semigroups. Equivalent conditions for a left ideal to be weakly prime are obtained. We have also shown that if \((M, L)\) is a weak \(m^*\)-system and if \(P\) is a left ideal which is maximal with respect to containing \(L\) and not meeting \(M\), then \(P\) is weakly prime.

1. Introduction

Schein [7] considered systems of the form \((\phi; \circ, \backslash)\) where \(\phi\) is a set of functions closed under the composition \(\circ\) of functions (and hence \((\phi; \circ)\) is a function semigroup) and the set theoretic subtraction \(\backslash\) (and hence \((\phi; \backslash)\) is a subtraction algebra in the sense of [1]). Zelinka [8] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. Eun Hwan Roh, Kyung Ho Kim, and Jong Geol Lee [6] obtained significant results in subtraction semigroups. The notion of near-subtraction semigroup was studied in [2]. We introduce the notion of weakly prime left ideals in near-subtraction semigroup which is a generalization of prime left ideals and give some characterizations of weakly prime left ideals. In this process the concept of weak \(m^*\)-system has been introduced which plays the same part as the \(m\)-system plays for prime ideals in ring and near-ring theory.

2. Preliminaries

Definition 2.1. A nonempty set \(X\) together with a binary operation \(\leftarrow\) is said to be a subtraction algebra if it satisfies the following identities:

\begin{enumerate}
  \item \(x - (y - x) = x\).
  \item \(x - (x - y) = y - (y - x)\).
  \item \((x - y) - z = (x - z) - y\) for every \(x, y, z \in X\).
\end{enumerate}

The subtraction determines an order relation on \(X\) : \(a \leq b \iff a - b = 0\), where \(a - a = 0\) is an element that does not depend on the choice of \(a \in X\).
Example 2.2. Let $A$ be any nonempty set. Then $(P(A), \setminus)$ is a subtraction algebra, where \(P(A)\) denotes the power set of $A$ and \(\setminus\) denotes the set theoretic subtraction.

In a subtraction algebra the following holds:

1. $x - 0 = x$ and $0 - x = 0$.
2. $(x - y) - x = 0$.
3. $(x - y) - y = x - y$.
4. $(x - y) - (y - x) = x - y$.

Following [4], we have the following definition of subtraction semigroup.

Definition 2.3. A nonempty set $X$ together with two binary operations \(-\) and \(\cdot\) is said to be a subtraction semigroup if it satisfies the following:

1. $(X; -)$ is a subtraction algebra.
2. $(X; \cdot)$ is a semigroup.
3. $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$ for every $x, y, z \in X$.

Note that it is clear that $0x = 0$ and $x0 = 0$ for every $x \in X$.

Example 2.4. Let $\Gamma$ be a subtraction algebra. Then the set $M_\Gamma(\Gamma)$ of all homomorphisms of $\Gamma$ into $\Gamma$ is a subtraction semigroup under point wise subtraction and composition of mappings.

Definition 2.5. Let $(X, - , \cdot)$ be a subtraction semigroup. A nonempty subset $I$ of $X$ is called a left (right) ideal if $x - y \in I$, for every $x \in I$, $y \in X$ and $XI \subseteq I$ ($IX \subseteq I$). If $I$ is both a left and right ideal then $I$ is an ideal, denoted by $I \leq X$, where $AB = \{ab | a \in A, b \in B\}$ for any nonempty subsets $A, B$ of $X$.

Remark 2.6. Let $X$ be a subtraction algebra and $I \subseteq X$. Then the following are equivalent:

(i) $(\forall x \in I, y \in X) \: x - y \in I$.

(ii) $x \leq y$ and $y \in I \Rightarrow x \in I$.

Proof. (i)⇒(ii) Let $y \in I$ and $x \in X$ such that $x \leq y$. Hence $x - y = 0$. Then $x = x - (x - y) = y - (y - x) \in I$ by (i).

(ii)⇒(i) Let $x \in I$ and $y \in X$. Since $(x - y) - x = 0$, $(x - y) \leq x$. Hence by (ii) $x - y \in I$.

Definition 2.7 ([2]). A nonempty set $X$ together with two binary operations \(-\) and \(\cdot\) is said to be a near-subtraction semigroup if

1. $(X; -)$ is a subtraction algebra,
2. $(X; \cdot)$ is a semigroup and
3. $(x - y)z = xz - yz$ for every $x, y, z \in X$.

Example 2.8. Let $\Gamma$ be a subtraction algebra. Then the set $M(\Gamma)$ of all mappings of $\Gamma$ into $\Gamma$ is a near-subtraction semigroup under point wise subtraction and composition of mappings. $M(\Gamma)$ is not a subtraction semigroup, because
if $f_{\delta} : \Gamma \to \Gamma$ is given by $f_{\delta}(\gamma) = \delta$ for all $\gamma \in \Gamma$ ($f_{\delta}$ is a constant map), then for any $g, h \in M(\Gamma)$, $f_{\delta} = f_{\delta} \circ (g - h) \neq f_{\delta} \circ g - f_{\delta} \circ h = 0$.

**Example 2.9.** Let $\Gamma = \{0, 1\}$ in which “−” is defined by

\[
\begin{array}{c|ccc}
- & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

Then $\Gamma$ is a subtraction algebra. Now $M(\Gamma) = \{0, a, b, 1\}$ where $0, a, b, 1$ are all functions from $\Gamma$ to $\Gamma$. $M(\Gamma)$ is a near-subtraction semigroup under pointwise subtraction and composition and we have

\[
\begin{array}{c|ccc}
- & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & b \\
b & b & 0 & 0 & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\quad \quad
\begin{array}{c|ccc}
0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 \\
a & a & a & a \\
b & a & 0 & 1 & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

**Definition 2.10** ([2]). A near-subtraction semigroup $X$ is said to be zero-symmetric if $x0 = 0$ for every $x \in X$.

**Example 2.11.** Let $\Gamma$ be a subtraction algebra. Then $M_{0}(\Gamma) = \{f : \Gamma \to \Gamma | f(0) = 0\}$ is a zero-symmetric near-subtraction semigroup under pointwise subtraction and composition of mappings.

Now we introduce the notion of an ideal which is different from that of [2].

**Definition 2.12.** Let $(X, - , \cdot)$ be a near-subtraction semigroup. A nonempty subset $I$ of $X$ such that $x - y \in I$ for every $x \in I$, $y \in X$ is called

(1) a left ideal if $xi - x(x' - i) \in I$ for all $x, x' \in X$ and $i \in I$ denoted by $I \triangleleft_{l} X$.

(2) a right ideal if $IX \subseteq I$ denoted by $I \triangleleft_{r} X$.

(3) an ideal if $I$ is both a left and right ideal denoted by $I \triangleleft X$.

**Note:**

(1) Suppose if $X$ is a subtraction semigroup and $I$ is a left ideal of $X$, then for $i \in I$ and $x, x' \in X$, we have $xi - x(x' - i) = xi - (xx' - xi) = xi \in I$ by Property 1 of subtraction algebra. Thus we have $XI \subseteq I$.

(2) If $X$ is a zero-symmetric near-subtraction semigroup and $I$ is a left ideal of $X$, then for $i \in I$ and $x \in X$, we have $xi - x(0 - i) = xi - 0 = xi \in I$.

**Remark 2.13.** Let $X$ be a zero-symmetric near-subtraction semigroup. Let $I$ be a subset of $X$ such that $x - y \in I$ for every $x \in I$, $y \in X$. Then the following are equivalent:

(i) $XI \subseteq I$.

(ii) $xi - x(x' - i) \in I$ for all $x, x' \in X$ and $i \in I$. 

Let $I$ be a left ideal of $X$. For every $x \in I$, $y \in X$, we have $xi - x(x' - i) \in I$ for all $x, x' \in X$ and $i \in I$.

(ii)$\Rightarrow$(i) Since $X$ is zero-symmetric and $I$ is a left ideal of $X$ by Note (2), we have $XI \subseteq I$.

**Definition 2.14.** Let $X$ be a near-subtraction semigroup. For $S \subseteq X$, $< S >$ denotes the ideal of $X$ generated by $S$ which is the intersection of all ideals of $X$ containing $S$ and hence is the smallest ideal of $X$ containing $S$.

$< S >$ denotes the left ideal of $X$ generated by $S$ which is the intersection of all left ideals of $X$ containing $S$ and hence is the smallest left ideal of $X$ containing $S$.

3. Weakly prime left ideals

Unless stated otherwise throughout this paper $X$ stands for a zero-symmetric near-subtraction semigroup.

**Definition 3.1.** A left ideal $P$ of $X$ is said to be prime if $L_1L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$ for any left ideals $L_1, L_2$ of $X$.

**Definition 3.2.** A left ideal $P$ of $X$ is said to be weakly prime if $L_1L_2 \subseteq P$ implies $L_1 = P$ or $L_2 = P$ for any left ideals $L_1, L_2$ of $X$ containing $P$.

A prime left ideal is always weakly prime. But the converse need not be true as the following example shows.

**Example 3.3.** Consider the following near-subtraction semigroup.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Here the left ideal $\{0, c\}$ is weakly prime but not prime since $\{0, b\}$ is a left ideal such that $\{0, b\} \subseteq \{0, c\}$.

**Lemma 3.4.** Let $X$ be a near-subtraction semigroup. Then for $A, B \subseteq X, (A : B) = \{x \in X|xB \subseteq A\}$ is an ideal of $X$.

**Proof.** Let $i \in (A : B)$ and $x \in X$. Then for $b \in B$, $(i - x)b = ib - xb \in A$, since $A$ is a left ideal and $ib \in A, \forall b \in B$. Hence $(i - x)B \subseteq A$. Thus $i - x \in (A : B)$ for all $i \in (A : B)$ and $x \in X$. Now let $i \in (A : B)$, $x, x' \in X$, and $b \in B$. Then $iB \subseteq A$ and hence $ib = a$ for some $a \in A$. Now $(xi - x(x' - i))b = xib - x(x'b - ib) = xa - x(x' - a) \in A$ since $A$ is a left ideal. Thus $xi - x(x' - i) \in (A : B)$ for every $i \in (A : B)$, $x, x' \in X$. Now let $i \in (A : B)$, $x \in X$ and $b \in B$. Then $ixb = ib'$ for some $b' \in B$ as $X$ is zero-symmetric and $B$ is a left ideal. Since $iB \subseteq A$, $ixb \in A$. Hence $ix \in (A : B)$ for every $i \in (A : B), x \in X$. Thus $(A : B)X \subseteq (A : B)$. $\qed$
Theorem 3.5. The following are equivalent for a left ideal $P$ of $X$.

(i) $P$ is weakly prime.
(ii) $\forall \ L_1, \ L_2 \subseteq X : (P \cup L_1)(P \cup L_2) \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$.
(iii) $\forall \ L_1, \ L_2 \subseteq X : L_1 \supseteq P$ and $L_1 L_2 \subseteq P$ imply $L_1 = P$ or $L_2 \subseteq P$.
(iv) $\forall \ L_1, \ L_2 \subseteq X : (P \cup L_1)L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$.
(v) $\forall a, \ b \in X : (a \cup P)(b \cup P) \subseteq P$ implies $a \in P$ or $b \in P$, where $< b$ denotes the left ideal of $X$ generated by $b$.

Proof. (i)⇒(ii). Since $(P \cup L_1)$ and $(P \cup L_2)$ are left ideals containing $P$, $L_1 \subseteq P$ or $L_2 \subseteq P$.

(ii)⇒(iii). Let $L_1 \supseteq P$ and $L_1 L_2 \subseteq P$. Clearly $(P \cup L_1)(P \cup L_2) \subseteq P$. Hence $L_1 = P$ or $L_2 \subseteq P$.

(iii)⇒(iv). It is obvious.

(iv)⇒(v). Let $(a \cup P)(b \cup P) \subseteq P$. Then $a(b \cup P) \subseteq P$ and $P(b \cup P) \subseteq P$. Now $a < a \subseteq (P : b \cup P)$ and hence $(a)(b \cup P) \cup P(b \cup P) \subseteq P$. Thus $(a \cup P)(b \cup P) \subseteq P$ and hence by (iv) $a \in P$ or $b \in P$.

(v)⇒(i). Let $L_1, \ L_2$ be left ideals of $X$ containing $P$ such that $L_1 L_2 \subseteq P$. If $L_1 \neq P$, choose $a \in L_1 \setminus P$. Clearly $a \cup P \subseteq L_1$. Now for any $b \in L_2$, $b < b \cup P \subseteq L_2$. Hence $(a \cup P)(b \cup P) \subseteq L_1 L_2 \subseteq P$. Hence by (v) $L_2 \subseteq P$. □

Theorem 3.6. For a left ideal $P$ of $X$ which is not two sided the following are equivalent.

(i) $P$ is weakly prime.
(ii) $PL \subseteq P$ for a left ideal $L$ implies $L \subseteq P$.

Proof. (i)⇒(ii). Let $PL \subseteq P$ for some left ideal $L$ of $X$. Then $P \subseteq (P : L)$. Hence $< P \supseteq (P : L)$ so that $< P \supseteq L \subseteq P$. By Theorem 3.5 (iii), $< P \neq P$ or $L \subseteq P$. Since $P$ is not a two sided ideal, $< P \neq P$. Hence $L \subseteq P$.

(ii)⇒(i). Let $L_1, \ L_2$ be left ideals of $X$ such that $(P \cup L_1)L_2 \subseteq P$. Then $PL_2 \cup L_1 L_2 \subseteq P$, which implies $PL_2 \subseteq P$. Hence $L_2 \subseteq P$. By Theorem 3.5 (iv), $P$ is weakly prime. □

Definition 3.7. A non empty subset $M$ of $X$ is said to be an $m^\star$-system if for any $m_1, \ m_2 \in M$, there exists $m_1 < m_1 >$ and $m_2 \in m_2 \in M$ such that $m_1 m_2 < m_1 M$.

Clearly every $m^\star$-system is an $m$-system.

Definition 3.8. A weak $m^\star$-system in $X$ is a pair $(M, \ L)$ where $L$ is a left ideal in $X$ and $M$ is a non empty subset of $X$ such that $L \cap M = \phi$ and $(m \cup L)((< n) \cup L)) \cap M \neq \phi$ for all $m, n \in M$.

A weak $m^\star$-system need not be an $m^\star$-system. In Example 3.3, let $M = \{a, \ b\}$ and $L = \{0, \ c\}$, then $(M, \ L)$ is a weak $m^\star$-system but not an $m^\star$-system since for $b, \ a \in M$, $< b > < a \cap M = \phi$.

Theorem 3.9. A left ideal $P$ is weakly prime if and only if $(X \setminus P, \ P)$ is a weak $m^\star$-system.
Proof. Assume that \( P \) is a weak prime ideal. Let \( X \setminus P = M \) and \( m, n \in M \). We claim that \((m \cup P)(< n) \cup P \) \( \not
subsetneq P \). Suppose not. Then by Theorem 3.5 (v), we have \( m \in P \) or \( n \in P \), which is a contradiction to \( M \cap P = \phi \). Hence \((m \cup P)(< n) \cup P \) \( \not \cap M \neq \phi \). Thus \((M, P)\) is a weak \( m^*\)-system.

Conversely let \((X \setminus P, P)\) be a weak \( m^*\)-system. Let \( L_1, L_2 \) be left ideals of \( X \) containing \( P \) such that \( L_1L_2 \subseteq P \). Suppose \( L_1 \neq P \). Choose \( a \in L_1 \setminus P \). We claim that \( L_2 = P \). Suppose not. Then choose \( b \in L_2 \setminus P \). Now \( a, b \in X \setminus P \).

Since \( a \in L_1 \) and \( P \subseteq L_1 \), we have \( a \cup P \subseteq L_1 \). Similarly \( \langle b \rangle \cup P \subseteq L_2 \). Hence \( (a \cup P) \langle b \rangle \cup P \subseteq L_1L_2 \subseteq P \). Thus \( (a \cup P) \langle b \rangle \cup P \) \( \cap X \setminus P = \phi \), which is a contradiction to \((X \setminus P, P)\) is a weak \( m^*\)-system. Hence \( L_2 = P \).

Theorem 3.10. Let \((M, L)\) be a weak \( m^*\)-system. If \( P \) is a left ideal which is maximal with respect to containing \( L \) and not meeting \( M \), then \( P \) is weakly prime.

Proof. Suppose there exist left ideals \( L_1, L_2 \) of \( X \) properly containing \( P \) such that \( L_1L_2 \subseteq P \). Since \( L_1, L_2 \) are left ideals properly containing \( P \), by the maximality of \( P \) we have \( L_1 \cap M \neq \phi \) and \( L_2 \cap M \neq \phi \). Let \( m_1 \in L_1 \cap M \) and \( m_2 \in L_2 \cap M \). Since \( P \subseteq L_1 \) and \( L \subseteq P \) we have \( L \subseteq L_1 \). Hence \( m_1 \cup L \subseteq L_1 \).

Again since \( P \subseteq L_2 \) and \( L \subseteq P \) we have \( L \subseteq L_2 \). Hence \( m_2 \cup L \subseteq L_2 \). Now \( (m_1 \cup L)(< m_2 \cup L) \cap M = \phi \) which is a contradiction to the fact that \((M, L)\) is a weak \( m^*\)-system. Hence \( L_1L_2 \not\subseteq P \). Thus \( P \) is weakly prime.

Lemma 3.11. For any left ideal \( L \) of \( X \), \( B(L) = \{ y \in L \mid yX \subseteq L \} \) is the largest two-sided ideal of \( X \) containing \( L \).

Proof. Let \( x \in B(L) \) and \( z \in X \). Then \((x - z)r = xr - zr \in L \) for every \( r \in X \).

Hence \( x - z \in B(L) \). Let \( r, r' \in X \) and \( i \in B(L) \). Then \((ri - r(r' - i)x = rix - r(r'x - ix) \in L \) as \( ix \in L \) and \( L \) is a left ideal. Hence \( B(L) \) is a left ideal. Clearly \( B(L)X \subseteq B(L) \).

Suppose that there is a two-sided ideal \( A \) which is contained is \( L \) such that \( B(L) \subseteq A \). Now for \( y \in A, yX \subseteq A \subseteq L \). Hence \( B(L) = A \).

Theorem 3.12. A left ideal \( P \) is prime if and only if there is an \( m^*\)-system \( M \) such that \( P \) is a maximal left ideal not meeting \( M \) and \( B(P) \) is the maximal two-sided ideal not meeting \( M \).

Proof. Suppose \( P \) is a prime left ideal of \( X \). Let \( M = X \setminus P \) and \( x_1, x_2 \in M \). Since \( P \) is prime we have \( < x_1 \cup < x_2 \) \( \not\subseteq P \). Hence there exist \( x_1 \in < x_1 \cup < x_2 \) such that \( x_1x_2 \in M \). Hence \( M \) is an \( m^*\)-system.

Now we claim that \( P \) is a maximal left ideal not meeting \( M \). If there is a left ideal \( L \) such that \( L \cap M = \phi \), then \( L \subseteq X \setminus M \) so that \( L \not\subseteq P \). Hence \( P \) is the maximal left ideal not meeting \( M \). Suppose \( A \) is any two-sided ideal not meeting \( M \), then \( A \subseteq X \setminus M \). Hence \( A \subseteq P \). But \( B(P) \) is the largest two-sided ideal contained in \( P \). Hence \( A \subseteq B(P) \).

Thus \( B(P) \) is the maximal two-sided ideal not meeting \( M \).
Conversely, let $M$ be any $m^*$-system. Let $P$ be a maximal left ideal not meeting $M$ and let $B(P)$ be a maximal two-sided ideal not meeting $M$. We show that $P$ is prime. Suppose that there exist left ideals $L_1$ and $L_2$ in $X$ such that $L_1 L_2 \subseteq P$ with $L_1 \not\subseteq P$, $L_2 \not\subseteq P$. Now $L_1 L_2 \subseteq P$ implies $L_1 \subseteq (P : L_2)$. Since $(P : L_2)$ is an ideal, $< L_1 > \subseteq (P : L_2)$ and thus $< L_1 > L_2 \subseteq P$. Hence $(< L_1 > \cup B(P))(P \cup L_2) \subseteq P$. Since $L_1 \not\subseteq P$ and $L_1 \subseteq < L_1 >$, we have $< L_1 > \not\subseteq P$. Now since $B(P)$ is the largest two-sided ideal contained in $P$ and since $< L_1 > \not\subseteq P$, $< L_1 > \cup B(P)$ is a two-sided ideal properly containing $B(P)$. Hence by the maximality of $B(P)$, there is an $m_1 \in (P \cup B(P)) \cap M$.

Also since $L_2 \not\subseteq P$, $P \cup L_2$ is a left ideal properly containing $P$. Hence by the maximality of $P$ there is an $m_2 \in (P \cup L_2) \cap M$. Since $m_1$, $m_2 \in M$ and $M$ is an $m^*$-system, there exist $m'_1 \in < m_1 >$ and $m'_2 \in < m_2 >$ such that $m'_1 m'_2 \in M$.

But $m'_1 \not\in < m_1 > \cup B(P)$ and $m'_2 \not\in < m_2 > \cup B(P)$, Hence $m'_1 m'_2 \in (P \cup B(P))(P \cup L_2) \subseteq P$. Thus $m'_1 m'_2 \in P \cap M$, a contradiction. Therefore $L_1 L_2 \not\subseteq P$. Hence $P$ is a prime ideal.

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