THE QUASI-HADAMARD PRODUCTS OF CERTAIN
p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

MOHAMED KAMAL AOUF

Reprinted from the
Bulletin of the Korean Mathematical Society
Vol. 44, No. 4, November 2007

©2007 The Korean Mathematical Society
THE QUASI-HADAMARD PRODUCTS OF CERTAIN
p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

MOHAMED KAMAL AOUFF

Abstract. The object of the present paper is to show quasi-Hadamard
products of certain $p$-valent functions with negative coefficients in the
open unit disc. Our results are the generalizations of the corresponding
results due to Yaguchi et al. [10], Aouf and Darwish [3], Lee et al. [5]
and Sekine and Owa [9].

1. Introduction

Let $A_p(n)$ be the class of functions of the form :

\[ f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in \mathbb{N} = \{1, 2, \ldots\}) \]

which are analytic and $p$-valent in the unit disc $U = \{z : |z| < 1\}$. A function
$f(z) \in A_p(n)$ is said to be a member of the class $P_{p}^*(n, \alpha, \beta)$ if it satisfies

\[ \left| \frac{f(z)}{z^p} - p \right| < \beta \quad (z \in U) \]

for some $\alpha(0 \leq \alpha < p)$ and $\beta(0 < \beta \leq 1)$. The class $P_{p}^*(n, \alpha, \beta)$ was studied by
Aouf [1, 2].

We note that :

(i) For $\beta = 1$, the class $P_{p}^*(n, \alpha, 1) = P_{p}^*(n, \alpha) = \{f(z) \in A_p(n) : \Re \left\{ \frac{f(z)}{z^p} \right\} > \alpha (z \in U), 0 \leq \alpha < p \}$ was studied by Yaguchi et al. [10] and Owa and Aouf [7];

(ii) For $\beta = n = 1$, the class $P_{p}^*(1, \alpha, 1) = P_{p}^*(\alpha) = \{f(z) \in A_p : \Re \left\{ \frac{f(z)}{z^p} \right\} > \alpha (z \in U), 0 \leq \alpha < p \}$ was studied by Lee et al. [5],

Aouf and Darwish [4] and Aouf [3].
For $\beta = p = 1$, the class $P_p^*(n, \alpha, 1) = \tilde{C}(\alpha, n) = \{ f(z) \in A_1(n) : \text{Re}\{f'(z)\} > \alpha (z \in U), 0 \leq \alpha < 1 \}$ was studied by Sekine and Owa [9];

(iv) For $\beta = p = n = 1$, the class $P_p^*(1, \alpha, 1) = \tilde{C}(\alpha, 1)$ was studied by Sarangi and Urelagaddi [8] and Owa [6].

For functions $f_j(z) \in A_p(n)$ defined by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; j \in N),$$

we denote by $(f_1 \ast f_2)(z)$ the quasi-Hadamard product of functions $f_1(z)$ and $f_2(z)$, that is,

$$ (f_1 \ast f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k .$$

For $\beta = 1$ Yaguchi et al. [10] proved the following results:

**Theorem A.** If $f_j(z) \in P_p^*(n, \alpha_j, 1) = P_p^*(n, \alpha_j)(j=1, 2)$, then $(f_1 \ast f_2)(z) \in P_p^*(n, \gamma)$, where

$$ \gamma = p - \frac{\prod_{j=1}^{2} (p - \alpha_j)}{p + n} .$$

The result is sharp.

**Theorem B.** If $f_j(z) \in P_p^*(n, \alpha_j)(j=1, 2)$, then the function

$$h(z) = z^p - \sum_{k=p+n}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k$$

is in the class $P_p^*(n, \gamma)$, where

$$ \gamma = p - \frac{2(p - \alpha_0)^2}{p + n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2\}) .$$

The result is sharp.

For $\beta = n = 1$, Lee et al. [5] have shown that:

**Theorem C.** If $f_j(z) \in P_p^*(1, \alpha, 1) = P_p^*(\alpha)(j=1, 2)$, then $(f_1 \ast f_2)(z) \in P_p^*(\gamma)$, where

$$ \gamma = p - \frac{(p - \alpha)^2}{p + 1} .$$

The result is sharp.
Also for $\beta = n = 1$, Aouf and Darwish [3] have proved the following results:

**Theorem D.** If $f_j(z) \in P^*_p(\alpha_j)(j = 1, 2)$, then $(f_1 \ast f_2)(z) \in P^*_p(\gamma)$, where

\[
\gamma = p - \frac{\prod_{j=1}^{2}(p - \alpha_j)}{p + 1}.
\]

The result is sharp.

**Theorem E.** If $f_j(z) \in P^*_p(\alpha)(j = 1, 2, 3)$, then $(f_1 \ast f_2 \ast f_3)(z) \in P^*_p(\gamma)$, where

\[
\gamma = p - \frac{(p - \alpha)^3}{(p + 1)^2}.
\]

The result is sharp.

**Theorem F.** If $f_j(z) \in P^*_p(\alpha)(j = 1, 2)$, then the function

\[
h(z) = z^p - \sum_{k=p+1}^{\infty} (a^2_{k,1} + a^2_{k,2})z^k
\]

is in the class $P^*_p(\gamma)$, where

\[
\gamma = p - \frac{2(p - \alpha)^2}{p + 1}.
\]

The result is sharp.

Further for $\beta = p = 1$, Sekine and Owa [9] proved the following results:

**Theorem G.** If $f_j(z) \in \tilde{C}^*_1(n, \alpha, 1) = \tilde{C}(\alpha, n)(j = 1, 2)$, then $(f_1 \ast f_2)(z) \in \tilde{C}(n, \gamma)$, where

\[
\gamma = 1 - \frac{(1 - \alpha)^2}{n + 1}.
\]

The result is sharp.

**Theorem H.** If $f_j(z) \in \tilde{C}(\alpha, n)(j = 1, 2)$, then the function

\[
g(z) = z - \sum_{k=n+1}^{\infty} (a^2_{k,1} + a^2_{k,2})z^k
\]

is in the class $\tilde{C}(\gamma, n)$, where

\[
\gamma = 1 - \frac{2(1 - \alpha)^2}{n + 1}.
\]

The result is sharp.
In the present paper, we prove some interesting generalizations of the theorems given by Yaguchi et al. [10], Aouf and Darwish [3], Lee et al. [5] and Sekine and Owa [9].

2. Quasi-Hadamard products

To prove our main results of quasi-Hadamard products, we need the following lemma given by Aouf ([1] and [2]).

**Lemma 1.** A function \( f(z) \in A_n(p) \) is in the class \( P^*_p(n, \alpha, \beta) \) if and only if

\[
(2.1) \quad \sum_{k=p+n}^{\infty} (1 + \beta)ka_k \leq 2\beta(p - \alpha).
\]

Applying the above lemma, we derive :

**Theorem 1.** If \( f_j(z) \in P^*_p(n, \alpha_j, \beta) \) (\( j = 1, 2, \ldots, m \)), then \( (f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in P^*_p(n, \gamma, \beta) \), where

\[
(2.2) \quad \gamma = p - \frac{\prod_{j=1}^{m} 2\beta(p - \alpha_j)}{2\beta[(1 + \beta)(p + n)]^{m-1}}.
\]

The result is sharp for the functions

\[
(2.3) \quad f_j(z) = z^p - \frac{2\beta(p - \alpha_j)}{(1 + \beta)(p + n)}z^{p+n} \quad (j = 1, 2, \ldots, m).
\]

**Proof.** For \( m = 1 \), we see that \( \gamma = \alpha_1 \). For \( m = 2 \), Lemma 1 gives

\[
(2.4) \quad \sum_{k=p+n}^{\infty} \frac{k(1 + \beta)}{2\beta(p - \alpha_j)}a_{k,j} \leq 1 \quad (j = 1, 2)
\]

This gives that

\[
(2.5) \quad \sum_{k=p+n}^{\infty} \frac{(1 + \beta)k}{\sqrt{\frac{2}{\prod_{j=1}^{2} 2\beta(p - \alpha_j)}}} \leq 1.
\]

To prove the case when \( m = 2 \), we have to find the largest \( \gamma \) such that

\[
(2.6) \quad \sum_{k=p+n}^{\infty} \frac{(1 + \beta)k}{2\beta(p - \gamma)}a_{k,1}a_{k,2} \leq 1,
\]

or such that

\[
(2.7) \quad \frac{\sqrt{a_{k,1}a_{k,2}}}{2\beta(p - \gamma)} \leq \frac{1}{\sqrt{\prod_{j=1}^{2} 2\beta(p - \alpha_j)}} \quad (k \geq p + n).
\]
Further, by using (2.5), we need to find the largest $\gamma$ such that
\begin{equation}
\frac{1}{2\beta(p - \gamma)} \leq \frac{(1 + \beta)k}{\prod_{j=1}^{2} 2\beta(p - \alpha_j)} \quad (k \geq p + n).
\end{equation}

It follows from (2.8) that
\begin{equation}
\gamma \leq p - \frac{\prod_{j=1}^{2} 2\beta(p - \alpha_j)}{2\beta(1 + \beta)k} \quad (k \geq p + n).
\end{equation}

Defining the function $\varphi(k)$ by
\begin{equation}
\varphi(k) = p - \frac{\prod_{j=1}^{2} 2\beta(p - \alpha_j)}{2\beta(1 + \beta)k},
\end{equation}
we see that $\varphi'(k) \geq 0$ for $k \geq p + n$. This implies that
\begin{equation}
\gamma \leq \varphi(p + n) = p - \frac{\prod_{j=1}^{2} 2\beta(p - \alpha_j)}{2\beta(1 + \beta)(p + n)}.
\end{equation}

Therefore, the result is true for $m = 2$.

Suppose that the result is true for any positive integer $m$. Then we have
\begin{equation}
(f_1 * f_2 * \cdots * f_m * f_{m+1})(z) \in P^*_p(n, \lambda, \beta),
\end{equation}
where
\begin{equation}
\lambda = p - \frac{2\beta(p - \gamma)2\beta(p - \alpha_{m+1})}{2\beta(1 + \beta)(p + n)},
\end{equation}
where $\gamma$ is given by (2.2). After a simple calculation, we have
\begin{equation}
\lambda = p - \frac{\prod_{j=1}^{m+1} 2\beta(p - \alpha_j)}{2\beta(1 + \beta)(p + n)^{m+1}}.
\end{equation}

Thus, the result is true for $m + 1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer $m$.

Finally, taking the functions $f_j(z)$ defined by (2.3), we have
\begin{equation}
(f_1 * f_2 * \cdots * f_m)(z) = z^p - \left\{ \prod_{j=1}^{m} \frac{2\beta(p - \alpha_j)}{(1 + \beta)(p + n)} \right\} z^{p+n} = z^p - A_{p+n} z^{p+n},
\end{equation}
which shows that
\begin{equation}
\sum_{k=p+n}^{\infty} \left[ \frac{(1 + \beta)k}{2\beta(p - \gamma)} \right] A_k
\end{equation}
Putting $\alpha_j = \alpha$ ($j = 1, 2, \ldots, m$) in Theorem 1, we have:

**Corollary 1.** If $f_j(z) \in P_p^*(n, \alpha, \beta)(j = 1, 2, \ldots, m)$, then $(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in P_p^*(n, \gamma, \beta)$, where

\begin{equation}
\gamma = p - \frac{2\beta(p - \alpha)[m]}{[1 + \beta](p + n)[m - 1]}. \tag{2.16}
\end{equation}

The result is sharp for the functions

\begin{equation}
f_j(z) = z^p - \frac{2\beta(p - \alpha)}{[1 + \beta](p + n)}z^{p+n} \quad (j = 1, 2, \ldots, m). \tag{2.17}
\end{equation}

Putting $\beta = 1$ in Theorem 1, we have:

**Corollary 2.** If $f_j(z) \in P_p^*(n, \alpha_j, 1) = P_p^*(n, \alpha_j)(j = 1, 2, \ldots, m)$, then $(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in P_p^*(n, \gamma)$, where

\begin{equation}
\gamma = p - \frac{m}{(p + n)[m - 1]}. \tag{2.18}
\end{equation}

The result is sharp for the functions

\begin{equation}
f_j(z) = z^p - \frac{p - \alpha_j}{p + n}z^{p+n} \quad (j = 1, 2, \ldots, m). \tag{2.19}
\end{equation}

Putting $n = 1$ in Corollary 1, we have:

**Corollary 3.** If $f_j(z) \in P_p^*(1, \alpha, \beta) = P_p^*(\alpha, \beta)(j = 1, 2, \ldots, m)$, then $(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in P_p^*(\gamma, \beta)$, where

\begin{equation}
\gamma = p - \frac{2\beta(p - \alpha)[m]}{2\beta([1 + \beta](p + 1))[m - 1]}. \tag{2.20}
\end{equation}

The result is sharp for the functions

\begin{equation}
f_j(z) = z^p - \frac{2\beta(p - \alpha)}{[1 + \beta](p + 1)}z^{p+1} \quad (j = 1, 2, \ldots, m). \tag{2.21}
\end{equation}

Putting $\beta = n = 1$ in Corollary 1, we have:
Corollary 4. If \( f_j(z) \in P_p^*(1, \alpha, 1) = P_p^*(\alpha)(j = 1, 2, \ldots, m) \), then \((f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in P_p^*(\gamma)\), where
\[
(2.22) \quad \gamma = p - \frac{(p - \alpha)^m}{(p+1)^{m-1}}.
\]
The result is sharp for the functions
\[
(2.23) \quad f_j(z) = z^p - \frac{p - \alpha}{p+1} z^{p+1} \quad (j = 1, 2, \ldots, m).
\]

Putting \( \beta = p = 1 \) in Corollary 1, we have:

Corollary 5. If \( f_j(z) \in P_1^*(n, \alpha, 1) = \tilde{C}(n, \alpha)(j = 1, 2, \ldots, m) \), then \((f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in \tilde{C}(n, \gamma)\), where
\[
(2.24) \quad \gamma = 1 - \frac{(1 - \alpha)^m}{(1+n)^{m-1}}.
\]
The result is sharp for the functions
\[
(2.25) \quad f_j(z) = z - \frac{1 - \alpha}{1+n} z^{1+n} \quad (j = 1, 2, \ldots, m).
\]

Remark 1. (i) Corollary 4 (when \( \beta = n = 1 \)) is the generalization of Theorem E given by Aouf and Darwish [3];
(ii) Corollary 2 is the generalization of Theorem A given by Yaguchi et al. [10]. Also Corollary 2 (when \( n = 1 \)) is the generalization of Theorem D given by Aouf and Darwish [3];
(iii) Corollary 4 is the generalization of Theorem C given by Lee et al. [5];
(iv) Corollary 5 is the generalization of Theorem G given by Sekine and Owa [9].

Theorem 2. If \( f_j(z) \in P_p^*(n, \alpha_j, \beta)(j = 1, 2, \ldots, m) \) and
\[
(2.26) \quad h(z) = z^p - \sum_{k=p+n}^{\infty} (\sum_{j=1}^m a_{k,j}^2) z^k,
\]
then \( h(z) \in P_p^*(n, \gamma, \beta) \), where
\[
(2.27) \quad \gamma = p - m \left[ \frac{2\beta(p - \alpha_0)^2}{2\beta(1+\beta)(p+n)} \right] (\alpha_0 = \min\{\alpha_1, \alpha_2, \ldots, \alpha_m\}) .
\]
The result is sharp for the functions \( f_j(z) \) given by (2.3).

Proof. Since Lemma 1 gives
\[
(2.28) \quad \sum_{k=p+n}^{\infty} \left\{ \frac{(1+\beta)k}{2\beta(p-\alpha_j)} \right\}^2 a_{k,j}^2 \leq \left\{ \sum_{k=p+n}^{\infty} \frac{(1+\beta)k}{2\beta(p-\alpha_j)} a_{k,j} \right\}^2 \leq 1
\]
for \( j = 1, 2, \ldots, m \), we have
\[
\sum_{k=p+n}^{\infty} \frac{1}{m} \left\{ \frac{(1 + \beta)k}{2\beta(p - \alpha_j)} \right\}^2 \left( \sum_{j=1}^{m} a_{k,j}^2 \right) \leq 1 .
\]

Note that we have to find the largest \( \gamma \) such that
\[
\sum_{k=p+n}^{\infty} \left\{ \frac{(1 + \beta)k}{2\beta(p - \gamma)} \right\}^2 \left( \sum_{j=1}^{m} a_{k,j}^2 \right) \leq 1 .
\]

This implies that
\[
\gamma \leq p - \frac{m[2\beta(p - \alpha_0)]^2}{2\beta(1 + \beta)k} \quad (k \geq p + n) ,
\]
that is, that
\[
\gamma \leq p - \frac{m[2\beta(p - \alpha_0)]^2}{2\beta(1 + \beta)(p + n)} ,
\]
which completes the proof of Theorem 2.

Putting \( \alpha_j = \alpha \ (j = 1, 2, \ldots, m) \) in Theorem 2, we have:

**Corollary 6.** If \( f_j(z) \in P_p^*(n, \alpha, \beta)(j = 1, 2, \ldots, m) \) and \( h(z) \) is defined by (2.26), then \( h(z) \in P_p^*(n, \gamma, \beta) \), where
\[
\gamma = p - \frac{m[2\beta(p - \alpha_0)]^2}{2\beta(1 + \beta)(p + n)} .
\]
The result is sharp for the functions \( f_j(z) \) defined by (2.17).

Putting \( \beta = 1 \) in Theorem 2, we have:

**Corollary 7.** If \( f_j(z) \in P_p^*(n, \alpha_j)(j = 1, 2, \ldots, m) \) and \( h(z) \) is defined by (2.26), then \( h(z) \in P_p^*(n, \gamma) \), where
\[
\gamma = p - \frac{m(p - \alpha_0)^2}{p + n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2, \ldots, \alpha_m}\} .
\]
The result is sharp for the functions \( f_j(z) \) defined by (2.19).

Putting \( \beta = 1 \) in Corollary 6, we have:

**Corollary 8.** If \( f_j(z) \in P_p^*(n, \alpha)(j = 1, 2, \ldots, m) \) and \( h(z) \) is defined by (2.26), then \( h(z) \in P_p^*(n, \gamma) \), where
\[
\gamma = p - \frac{m(p - \alpha)^2}{p + n} .
\]
The result is sharp for the functions $f_j(z)$ defined by

$$f_j(z) = z^p - \frac{p - \alpha}{p + n} z^{p+n} \ (j = 1, 2, \ldots, m).$$

Putting $n = 1$ in Corollary 6, we have:

**Corollary 9.** If $f_j(z) \in P_p^*(\alpha, \beta)(j = 1, 2, \ldots, m)$ and $h(z)$ is defined by (2.26) with $n = 1$, then $h(z) \in P_p^*(\gamma, \beta)$, where

$$\gamma = p - \frac{m[2\beta(p - \alpha)]^2}{2\beta(1 + \beta)(p + 1)}.$$

The result is sharp for the functions $f_j(z)$ defined by (2.21).

Putting $\beta = n = 1$ in Corollary 6, we have:

**Corollary 10.** If $f_j(z) \in P_p^*(\alpha)(j = 1, 2, \ldots, m)$ and $h(z)$ is defined by (2.26) with $n = 1$, then $h(z) \in P_p^*(\gamma)$, where

$$\gamma = p - \frac{m(p - \alpha)^2}{p + 1}.$$

The result is sharp for the functions $f_j(z)$ defined by (2.23).

Putting $\beta = p = 1$ in Corollary 6, we have:

**Corollary 11.** If $f_j(z) \in \tilde{C}(n, \alpha)(j = 1, 2, \ldots, m)$ and $h(z)$ is defined by (2.26) with $p = 1$, then $h(z) \in \tilde{C}(n, \gamma)$, where

$$\gamma = 1 - \frac{m(1 - \alpha)^2}{1 + n}.$$

The result is sharp for the functions $f_j(z)$ defined by (2.25).

**Remark 2.**

(i) Corollary 7 is the generalization of Theorem B given by Yaguchi et al. [10];

(ii) Corollary 10 is the generalization of Theorem F given by Aouf and Darwish [3];

(iii) Corollary 11 is the generalization of Theorem H given by Sekine and Owa [9].

**Acknowledgement.** The author would like to thank the referee of the paper for his helpful suggestions.

**References**


Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail address: mkaouf127@yahoo.com