WEAK AND STRONG CONVERGENCE OF THE ISHIKAWA ITERATION PROCESS WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we prove the weak and strong convergence of the Ishikawa iterative scheme with errors to a common fixed point for two asymptotically nonexpansive mappings in a uniformly convex Banach space under a condition weaker than compactness. Our theorems improve and generalize recent known results in literature.

1. Introduction

Let $K$ be a nonempty subset of a real normed linear space $E$. Let $T$ be a self-mappings of $K$. $T$ is said to be asymptotically nonexpansive with constant $\mu_n$ if there exists $\mu_n \in [0, +\infty)$, $\lim_{n \to \infty} \mu_n = 0$, such that

$$\|T^n(x) - T^n(y)\| \leq (1 + \mu_n)\|x - y\|, \quad \forall x, y \in K.$$

$T$ is called nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in K$.

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, but the converse does not hold.

It was proved in [1] that if $E$ is uniformly convex and if $K$ is bounded, closed, and convex, then every asymptotically nonexpansive mapping has a fixed point.

Takahashi and Tamuro([6]) introduced the following iterative schemes known as Ishikawa iterative schemes for a pair of nonexpansive mappings;

$$\begin{cases}
  x_1 = x \in K, \\
  x_{n+1} = a_nSy_n + (1 - a_n)x_n, \\
  y_n = b_nTx_n + (1 - b_n)x_n, \quad n \geq 1,
\end{cases}$$

(1.1)

where $a_n, b_n \in [0, 1]$.

Khan and Hafiz([3]) generalized the scheme (1.1) to the one with errors for a pair of nonexpansive mappings as follows;

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\end{itemize}
\begin{equation}
\begin{cases}
  x_1 = x \in K, \\
  x_{n+1} = a_nSx_n + b_nx_n + c_nu_n, \\
  y_n = a'_nTx_n + b'_n + c'_nv_n, \quad n \geq 1,
\end{cases}
\end{equation}

where \(\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}\) are sequences in \([0, 1]\) with \(0 < \delta \leq a_n, a_n \leq 1 - \delta < 1, a_n + b_n + c_n = 1 = a_n + b'_n + c_n\) and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(K\).

We further generalize this scheme (1.2) for a pair of asymptotically nonexpansive mappings as follows:

\begin{equation}
\begin{cases}
  x_1 = x \in K, \\
  x_{n+1} = a_nS\bar{x}_n + b_nx_n + c_nu_n, \\
  y_n = a'_nT\bar{x}_n + b'_n + c'_nv_n, \quad n \geq 1,
\end{cases}
\end{equation}

where \(\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}\) are sequences in \([0, 1]\) with \(0 < \delta \leq a_n, a_n \leq 1 - \delta < 1, a_n + b_n + c_n = 1 = a_n + b'_n + c_n\) and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(K\).

In this paper, we study the Ishikawa iterative scheme with error numbers (1.3) for the weak and strong convergence for a pair of asymptotically nonexpansive mappings in a uniformly convex Banach space. Our theorems improve and generalize some previous results.

2. Preliminaries

Let \(E\) be a Banach space and let \(K\) be a nonempty subset of \(E\). Let \(T\) be a mapping of \(K\) into itself. For every \(\varepsilon\) with \(0 \leq \varepsilon \leq 2\), we define the modulus \(\delta(\varepsilon)\) of convexity of \(E\) by

\[\delta(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\} \]

A Banach space \(E\) is said to be uniformly convex if \(\delta(\varepsilon) > 0\). A uniformly convex Banach space is reflexive and strictly convex.

A Banach space \(E\) is said to satisfy Opial’s condition([4]) if \(x_n \rightharpoonup x\) and \(x \neq y\) imply

\[\lim \inf_{n \to \infty} \|x_n - x\| < \lim \inf_{n \to \infty} \|x_n - y\|.
\]

We first prove the following lemma.

**Lemma 2.1.** Let \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\), and \(\{\mu_n\}\) be four nonnegative sequences satisfying

\[a_{n+1} \leq (1 + \gamma_n)(1 + \mu_n)a_n + \beta_n\]

for all \(n \geq 1\). If \(\sum_{n=1}^{\infty} \mu_n < \infty\), \(\sum_{n=1}^{\infty} \gamma_n < \infty\), and \(\sum_{n=1}^{\infty} \beta_n < \infty\), then \(\lim_{n \to \infty} a_n\) exists.

**Proof.** By hypothesis, we obtain
\[ a_{n+m} \leq (1 + \gamma_{n+m-1})(1 + \mu_{n+m-1})a_{n+m-1} + \beta_{n+m-1} \]

\[ \leq e^{\gamma_{n+m-1} + \mu_{n+m-1}}a_{n+m-1} + \beta_{n+m-1} \]

\[ = e^{\gamma_{n+m-1} + \mu_{n+m-1}}[(1 + \gamma_{n+m-2})(1 + \mu_{n+m-2})a_{n+m-2} + \beta_{n+m-2}] + \beta_{n+m-1} \]

\[ \leq e(\gamma_{n+m-1} + \gamma_{n+m-2}) + (\mu_{n+m-1} + \mu_{n+m-2})a_{n+m-2} \]

\[ + e^{\gamma_{n+m-1} + \mu_{n+m-1}}\beta_{n+m-2} + \beta_{n+m-1} \]

\[ \ldots \]

\[ \leq e^{\sum_{k=n}^{n+m-1}(\gamma_k + \mu_k)}a_n + e^{\sum_{k=n}^{n+m-1}(\gamma_k + \mu_k)}\sum_{k=1}^{n+m-1} \beta_k. \]

Thus, from \( \sum_{k=1}^{n} \gamma_k < +\infty, \sum_{k=1}^{n} \mu_k < +\infty, \sum_{k=1}^{n} \beta_k < +\infty, \) we can obtain \( \limsup_{n \to \infty} a_n \leq \liminf_{n \to \infty} a_n. \) So, \( \lim_{n \to \infty} a_n \) exists.

**Lemma 2.2 (see [2])**. Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition, \( \phi \neq K \subset E \) closed and convex, and \( T : K \to K \) asymptotically nonexpansive. Then \( I - T \) is demiclosed with respect to zero.

We also know the following lemma proved by Schu ([5]).

**Lemma 2.3**. Let \( E \) be a uniformly convex Banach space and \( \{\alpha_n\} \) a sequence in \( [\varepsilon, 1 - \varepsilon] \) for some \( \varepsilon \in (0, 1). \) Suppose \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r, \) and \( \limsup_{n \to \infty} \|\alpha_n x_n + (1 - \alpha_n) y_n\| = r \) holds for some \( r \geq 0. \) Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0. \)

### 3. Main Results

In this section, we prove our main theorems. Let \( K \) be a nonempty bounded closed uniformly convex Banach space \( E. \) Let \( S, T : K \to K \) be asymptotically nonexpansive mappings. Let \( F(S) \) denote the set of all fixed points of \( S. \) The following iteration scheme is studied:

\[
\begin{align*}
    x_1 & = x \in K, \\
    x_{n+1} & = a_n S^n y_n + b_n x_n + c_n u_n, \\
    y_n & = \bar{a}_n T^n x_n + \bar{b}_n x_n + \bar{c}_n v_n, \quad n \geq 1,
\end{align*}
\]

(3.1)

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{\bar{a}_n\}, \{\bar{b}_n\}, \{\bar{c}_n\} \) are sequences in \([0, 1]\) with \( 0 < \delta \leq a_n, \bar{a}_n \leq 1 - \delta < 1, a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n, \) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K. \)

**Theorem 3.1**. Let \( E \) be a uniformly convex Banach space satisfying the Opial’s condition and \( K, S, T \) and \( \{x_n\} \) be as taken in Lemma 3.3. If \( F(S) \cap F(T) \neq \phi, \) then \( \{x_n\} \) converges weakly to a common fixed point of \( S \) and \( T. \)

**Theorem 3.2**. Let \( E \) be a uniformly convex Banach space and \( K, \{x_n\} \) be as taken in Lemma 3.3. Let \( S, T : K \to K \) be two asymptotically nonexpansive
mappings satisfying condition (A). If \( F(S) \cap F(T) \neq \emptyset \), then \( \{x_n\} \) converges strongly to a common fixed point of \( S \) and \( T \).

References


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