INTEGRATION FORMULAS INVOLVING FOURIER-FEYNMAN TRANSFORMS VIA A FUBINI THEOREM

TIMOTHY HUFFMAN, DAVID SKOG, AND DAVID STORVICK

Abstract. In this paper we use a general Fubini theorem established in [13] to obtain several Feynman integration formulas involving analytic Fourier-Feynman transforms. Included in these formulas is a general Parseval’s relation.

1. Introduction and preliminaries

Let $C_{0}[0,T]$ denote one-parameter Wiener space, that is the space of $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_{0}[0,T]$ and let $m$ denote Wiener measure. $(C_{0}[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional $F$ by

$$\int_{C_{0}[0,T]} F(x)m(dx).$$

A subset $E$ of $C_{0}[0,T]$ is said to be scale-invariant measurable (s.i.m.) [8,15] provided $\rho E \in \mathcal{M}$ for all $\rho > 0$, and a s.i.m. set $N$ is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals $F$ and $G$ are equal s-a.e., we write $F \approx G$. For a rather detailed discussion of s.i.m. and its relation with other topics see [15]. It was also pointed out in [15, p. 170] that the concept of s.i.m., rather than Borel measurability
or Wiener measurability, is precisely correct for the analytic Fourier-Feynman transform theory and the analytic Feynman integration theory. Segal [19] gives an interesting discussion of the relationship between scale change in \( C_0[0,T] \) and certain questions in quantum field theory.

Throughout this paper we will assume that each functional \( F \) (or \( G \)) we consider satisfies the conditions:

\[
(1.1) \quad F : C_0[0,T] \to \mathbb{C} \text{ is defined } s-a.e. \text{ and is } s.i.m..
\]

\[
(1.2) \quad \int_{C_0[0,T]} |F(\rho x)| m(dx) < \infty \text{ for each } \rho > 0.
\]

Let \( C_+ \) and \( C^-_+ \) denote the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part respectively. Let \( F \) satisfy conditions (1.1) and (1.2) above, and for \( \lambda > 0 \), let

\[
J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2} x) m(dx).
\]

If there exists a function \( J^*(\lambda) \) analytic in \( C_+ \) such that \( J^*(\lambda) = J(\lambda) \) for all \( \lambda > 0 \), then \( J^*(\lambda) \) is defined to be the analytic Wiener integral of \( F \) over \( C_0[0,T] \) with parameter \( \lambda \), and for \( \lambda \) in \( C_+ \) we write

\[
(1.3) \quad \int_{C_0[0,T]} F(x)m(dx) = J^*(\lambda).
\]

Let \( q \neq 0 \) be a real parameter and let \( F \) be a functional whose analytic Wiener integral exists for all \( \lambda \in C_+ \). If the following limit exists, we call it the analytic Feynman integral of \( F \) with parameter \( q \) and we write

\[
(1.4) \quad \int_{C_0[0,T]} F(x)m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]} F(x)m(dx)
\]

where \( \lambda \to -iq \) through values in \( C_+ \). Finally, for notational purposes, we let

\[
(1.5) \quad \int_{C_0[0,T]} F(x)m(dx) \equiv \begin{cases} \int_{C_0[0,T]} F(x)m(dx), & \lambda \in C_+ \\ \int_{C_0[0,T]} F(x)m(dx), & \lambda = -iq \in C^-_+ - C_+ \end{cases}
\]

The following Fubini theorem established in [13], plays a major role in this paper.
Theorem 1. Assume that $F$ satisfies conditions (1.1) and (1.2) above and is such that its analytic Feynman integral \( \int_{C_0[0,T]}^{an} F(x)m(dx) \) exists for all $q \in \mathbb{R} - \{0\}$. Then for all $a, b \in \mathbb{R}$ and all $(\lambda, \beta) \in \mathbb{C}^+ \times \mathbb{C}^+$ with $\lambda + \beta \neq 0$,

$$
\int_{C_0[0,T]}^{an_{\lambda,\beta}} \left( \int_{C_0[0,T]}^{an_{\lambda}} F(ay + bz)m(dy) \right) m(dz) = \int_{C_0[0,T]}^{an_{\lambda,\beta}} F(x)dx
$$

$$
= \int_{C_0[0,T]}^{an_{\lambda}} \left( \int_{C_0[0,T]}^{an_{\beta}} F(ay + bz)m(dz) \right) m(dy).
$$

In section 2 below we use Theorem 1 to help us establish several Feynman integration formulas involving Fourier-Feynman transforms. Finally, in section 3 we establish additional integration formulas including a general Parseval's relation.

2. Fourier-Feynman transforms

The concept of an $L_1$ analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [5], Cameron and Storvick introduced an $L_2$ analytic FFT. In [14], Johnson and Skoug developed an $L_p$ analytic FFT for $1 \leq p \leq 2$ which extended the results in [1,5] and gave various relationships between the $L_1$ and the $L_2$ theories. In [10], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and in [11,12] obtained various results involving the FFT and the convolution product. Also see [7,16 and 18] for further work on these topics.

In this paper, for simplicity, we restrict our discussion to the case $p = 1$; however most of our results hold for all $p \in [1,2]$. Also, throughout this section, we will assume that the functionals $F : C_0[0,T] \rightarrow C$ satisfy the hypotheses of our Fubini theorem, namely Theorem 1 above.

For $\lambda \in \mathbb{C}^+$ and $y \in C_0[0,T]$, let

$$
(T_\lambda(F))(y) = \int_{C_0[0,T]}^{anw_{\lambda}} F(y + x)m(dx).
$$

Then for $q \in \mathbb{R} - \{0\}$ (see [10,p.663]), the $L_1$ analytic FFT, $T_q^{(1)}(F)$ of
$F$, is defined by the formula ($\lambda \in C_+$)

\[(T_q^{(1)}(F))(y) = \lim_{\lambda \to -iq} (T_\lambda(F))(y)\]

for s-a.e. $y \in C_0[0,T]$ whenever this limit exists. That is to say,

\[(T_q^{(1)}(F))(y) = \int_{C_0[0,T]}^{anf_q} F(y + x)m(dx)\]

for s-a.e. $y \in C_0[0,T]$. We note that if $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(F) \approx T_q^{(1)}(G)$.

In equations (2.4), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13) below, we establish various analytic Feynman integration formulas involving Fourier-Feynman transforms.

**Theorem 2.** Let $F$ be as in Theorem 1 above and let $r > 0$ be given. Then for all $q_1$ and $q_2$ in $\mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$,

\[\int_{C_0[0,T]}^{anf_{rq_2}} (T_q^{(1)}(F))(\sqrt{rz})m(dy) = \int_{C_0[0,T]}^{anf_{rq_1}} F(x)m(dx) = \int_{C_0[0,T]}^{anf_{rq_1}} (T_{q_2}^{(1)}(F))(\sqrt{ry})m(dy).\]

**Proof.** Using equation (2.3) and the first equality in equation (1.6) with $a = 1$, $b = \sqrt{r}$, $\lambda = -iq_1$ and $\beta = -irq_2$, we obtain that

\[\int_{C_0[0,T]}^{anf_{rq_2}} (T_q^{(1)}(F))(\sqrt{rz})m(dy) = \int_{C_0[0,T]}^{anf_{rq_2}} \left( \int_{C_0[0,T]}^{anf_{rq_1}} F(\sqrt{rz} + y)m(dy) \right)m(dy) = \int_{C_0[0,T]}^{anf_{rq_1}} F(x)m(dx).

\[= \int_{C_0[0,T]}^{anf_{rq_2}/(r_1 + r_2)} F(x)m(dx).

\[= \int_{C_0[0,T]}^{anf_{rq_1}/(r_1 + r_2)} F(x)m(dx).

\[= \int_{C_0[0,T]}^{anf_{rq_2}} F(x)m(dx).

\]
Also using equation (2.3) and the second equality in equation (1.6) with \( a = \sqrt{r}, \ b = 1, \ \beta = -iq_2 \) and \( \lambda = -irq_1 \), we obtain that

\[
\int_{C_0[0,T]}^{\text{an}f_{r_1}} (T_{q_2}^{(1)}(F))(\sqrt{r}y)m(dy) = \int_{C_0[0,T]}^{\text{an}f_{r_1}} \int_{C_0[0,T]}^{\text{an}f_{r_2}} F(\sqrt{r}y + z)m(dz)m(dy)
\]

\[
= \int_{C_0[0,T]}^{\text{an}f_{(r_1, q_2)/(r_2 + r_1)}} F(x)m(dx)
\]

\[
= \int_{C_0[0,T]}^{\text{an}f_{(q_1, q_2)/(q_1 + q_2)}} F(x)m(dx).
\]

Now equation (2.4) follows from equations (2.5) and (2.6). □

Our first corollary below says that the Feynman integral with parameter \( q_2 \) of the FFT with parameter \( q_1 \) equals the Feynman integral with parameter \( q_1 \) of the FFT with parameter \( q_2 \) provided \( q_1 + q_2 \neq 0 \).

**Corollary 1 to Theorem 2.** Let \( F \) be as in Theorem 2. Then for all \( q_1 \) and \( q_2 \) in \( \mathbb{R} = \{0\} \) with \( q_1 + q_2 \neq 0 \),

\[
\int_{C_0[0,T]}^{\text{an}f_{q_2}} (T_{q_1}^{(1)}(F))(z)m(dz) = \int_{C_0[0,T]}^{\text{an}f_{q_1}} (T_{q_2}^{(1)}(F))(y)m(dy).
\]

**Corollary 2 to Theorem 2.** Let \( F \) be as in Theorem 2. Then for all \( q \) in \( \mathbb{R} = \{0\} \),

\[
\int_{C_0[0,T]}^{\text{an}f_{q_1}} (T_{q_2}^{(1)}(F))(y)m(dy) = \int_{C_0[0,T]}^{\text{an}f_{q_1/2}} F(x)m(dx)
\]

\[
= \int_{C_0[0,T]}^{\text{an}f_{q}} F(\sqrt{2}x)m(dx).
\]

**Proof.** The first equality in equation (2.8) follows by letting \( r = 1 \) and \( q_1 = q_2 = q \) in equation (2.4). The second equality follows from the formula

\[
\int_{C_0[0,T]}^{\text{an}f_{q}} F(x)m(dx) = \int_{C_0[0,T]}^{\text{an}f_{q}} F(x/\sqrt{k})m(dx)
\]
established in [13] for \( k > 0 \).

**Corollary 3 to Theorem 2.** Let \( F \) be as in Theorem 2. Then for all \( q \in \mathbb{R} - \{0\} \),

\[
\int_{C_0[0,T]}^{\alpha_n f_{q}} (T^{(1)}_{q/2} F)(z)m(dz) = \int_{C_0[0,T]}^{\alpha_n f_{q}} F(x)m(dx)
\]

\[
= \int_{C_0[0,T]}^{\alpha_n f_{q/2}} (T^{(1)}_{-q} F)(y)m(dy).
\]

**Proof.** Simply choose \( q_1 = q/2, q_2 = -q \) and \( r = 1 \) in equation (2.4).

**Theorem 3.** Let \( F \) be as in Theorem 2 and let \( q_1, q_2, \ldots, q_n \) be elements of \( \mathbb{R} - \{0\} \) with

\[
\sum_{j=1}^{k} \frac{q_1 q_2 \cdots q_k}{q_j} \neq 0 \quad \text{for} \quad k = 2, \ldots, n.
\]

Then, for s.a.e. \( z \in C_0[0,T] \),

\[
(T^{(1)}_{q_n} (T^{(1)}_{q_{n-1}} (\cdots (T^{(1)}_{q_2} (T^{(1)}_{q_1} F)) \cdots )))(z)
\]

\[
= \int_{C_0[0,T]}^{\alpha_n f_{q_n}} F(z + x)m(dx)
\]

\[
= (T^{(1)}_{\alpha_n} F)(z)
\]

where \( \alpha_n = \sum_{j=1}^{n} \frac{q_1 q_2 \cdots q_n}{q_j q_1} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} \).

**Proof.** Using equation (2.3), and then equation (1.6) repeatedly, we obtain that

\[
(T^{(1)}_{q_n} (T^{(1)}_{q_{n-1}} (\cdots (T^{(1)}_{q_2} (T^{(1)}_{q_1} F)) \cdots )))(z)
\]

\[
= \int_{C_0[0,T]}^{\alpha_n f_{q_n}} \left( \int_{C_0[0,T]}^{\alpha_n f_{q_{n-1}}} \left( \cdots \left( \int_{C_0[0,T]}^{\alpha_n f_{q_2}} \left( \int_{C_0[0,T]}^{\alpha_n f_{q_1}} F(z + y_1 + y_2 + \cdots + y_n)m(dy_1) \right) \right) \right) \right) m(dy_2) \cdots m(dy_{n-1}) m(dy_n)
\]

\[
= \int_{C_0[0,T]}^{\alpha_n f_{q_n}} F(z + x)m(dx)
\]

\[
= (T^{(1)}_{\alpha_n} F)(z)
\]
for s-a.e. $z \in C_0[0,T]$. □

Choosing $q_j = q$ for $j = 1, 2, \ldots, n$, we obtain the following corollary to Theorem 3.

**Corollary 1 to Theorem 3.** Let $F$ be as in Theorem 3 and let $q$ be an element of $\mathbb{R} - \{0\}$. Then for s-a.e. $z \in C_0[0,T]$,

\[(2.11) \quad (T_q^{(1)}(T_q^{(1)}(F)))(z) = (T_{q/2}^{(1)}(F))(z) = \int_{C_0[0,T]} F(z + \sqrt{2x})m(dx),\]

\[(2.12) \quad (T_q^{(1)}(T_q^{(1)}(T_q^{(1)}(F))))(z) = (T_{q/3}^{(1)}(F))(z) = \int_{C_0[0,T]} F(z + \sqrt{3x})m(dx),\]

and in general,

\[(2.13) \quad (T_q^{(1)}(T_q^{(1)}(T_q^{(1)}(T_q^{(1)}(F)))))(z) = (T_{q/4}^{(1)}(F))(z) = \int_{C_0[0,T]} F(z + \sqrt{4x})m(dx).\]

**Corollary 2 to Theorem 3.** Let $F$ be as in Theorem 3 and let $q_1$ and $q_2$ be elements of $\mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$. Then for s-a.e. $z \in C_0[0,T]$,

\[(2.14) \quad (T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))(z) = (T_{q_1q_2/(q_1+q_2)}^{(1)}(F))(z) = (T_{q_1}^{(1)}(T_{q_2}^{(1)}(F)))(z).\]

**Remark 1.** We note that the hypotheses (and hence the conclusions) of Theorems 1-3 and their corollaries above are indeed satisfied by many large classes of functionals. These classes of functionals include:

(a) The Banach algebra $S$ defined by Cameron and Storvick in [6]; also see [9,12,18].

(b) Various spaces of functionals of the form

\[F(x) = \exp\{ \int_0^T f(t, x(t))dt \} \]
for appropriate \( f : [0, T] \times \mathbb{R} \to \mathbb{C} \); see for example [5,11 and 14].

(c) Various spaces of functionals of the form

\[
F(x) = f\left(\int_0^T \alpha_1(t)dx(t), \ldots, \int_0^T \alpha_n(t)dx(t)\right)
\]

for appropriate \( f \) as discussed in [10,16].

(d) Various spaces of functionals of the form

\[
F(x) = \exp\left\{\int_0^T \int_0^T f(s,t,x(s),x(t))dsdt\right\}
\]

for appropriate \( f : [0, T]^2 \times \mathbb{R}^2 \to \mathbb{C} \); see for example [12].

Remark 2. In a unifying paper [17], Lee obtains some similar results for several different integral transforms including the Fourier-Feynman transform. However, the results in this paper hold for much more general functionals \( F \). For example, in our notation, Lee requires the functional \( F(x + \lambda y) \) to be an entire function of \( \lambda \) over \( \mathbb{C} \) for each \( x \) and \( y \) in \( C_0[0,T] \) whereas we don’t even require \( F \) to be a continuous function. The classes of functionals studied by Yeh in [20] and Yoo in [21] for the Fourier-Wiener transform are similar to those used by Lee in [17].

3. Further applications

First we state the definition of the convolution product of two functionals \( F \) and \( G \) on \( C_0[0,T] \) as given by Huffman, Park and Skoug in [10, p. 663]. This definition is different than the definition given by Yeh in [20] and used by Yoo in [21]. In [20] and [21], Yeh and Yoo study the relationship between their convolution product and Fourier-Wiener transforms. For \( \lambda \in \mathbb{C}^+ \), the convolution product (if it exists) of \( F \) and \( G \) is defined by the formula (see equation (1.5) above)

\[
(F * G)_\lambda(y) = \int_{C_0[0,T]} F(x + \lambda y)(\frac{x+y}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})m(dx)
\]

for s-a.e. \( y \in C_0[0,T] \). When \( \lambda = -iq \), we usually denote \( (F * G)_\lambda \) by \( (F * G)_q \).
In our first theorem of this section, we show that the Fourier-Feynman transform of the convolution product is a product of their transforms.

**Theorem 4.** Let $F : C_0[0,T] \to \mathbb{C}$ be as in Theorem 1 and assume that $G : C_0[0,T] \to \mathbb{C}$ satisfies the same conditions as $F$; i.e., $G$ is s.i.m., $\int_{C_0[0,T]} |G(\rho x)| \, m(dx) < \infty$ for all $\rho > 0$ and the analytic Feynman integral of $G$ exists for all $q \in \mathbb{R} - \{0\}$. Furthermore assume that $T_q^{(1)}((F \ast G)_q)$ exists for all $q \in \mathbb{R} - \{0\}$. Then for all $q \in \mathbb{R} - \{0\}$,

\[ (3.2) \quad T_q^{(1)}((F \ast G)_q)(z) = (T_q^{(1)}(F))(z/\sqrt{2})(T_q^{(1)}(G))(z/\sqrt{2}) \]

for s-a.e. $z$ in $C_0[0,T]$.

**Proof.** Because of the assumptions on $F$ and $G$, all three of the transforms in equation (3.2) exist; thus we only need to establish the equality. For $\lambda > 0$, using (2.1) and (3.1), we see that

\[
(T_\lambda((F \ast G)_\lambda))(z) = \int_{C_0[0,T]} (F \ast G)_\lambda(z + \lambda^{-1/2}y)m(dy)
\]

\[
= \int_{C_0[0,T]} \int_{C_0[0,T]} F\left(\frac{z}{\sqrt{2}} + \lambda^{-1/2}\left(\frac{y+x}{\sqrt{2}}\right)\right)G\left(\frac{z}{\sqrt{2}} + \lambda^{-1/2}\left(\frac{y-x}{\sqrt{2}}\right)\right)m(dx)m(dy)
\]

for s-a.e. $z \in C_0[0,T]$. But $w_1 = (y + x)/\sqrt{2}$ and $w_2 = (y - x)/\sqrt{2}$ are independent standard Wiener processes, and hence

\[
(T_\lambda((F \ast G)_\lambda))(z) = \int_{C_0[0,T]} \left( \int_{C_0[0,T]} F(z/\sqrt{2} + w_1/\sqrt{\lambda})G(z/\sqrt{2} + w_2/\sqrt{\lambda}) \right)m(dw_2)m(dw_1)
\]

\[
= \int_{C_0[0,T]} F(z/\sqrt{2} + w_1/\sqrt{\lambda})m(dw_1)
\]

\[
\int_{C_0[0,T]} G(z/\sqrt{2} + w_2/\sqrt{\lambda})m(dw_2)
\]

\[
= (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2})
\]

for s-a.e. $z \in C_0[0,T]$. Now by analytic extensions through $C_+$ we obtain that

\[ (3.3) \quad (T_\lambda((F \ast G)_\lambda))(z) = (T_\lambda(F))(z/\sqrt{\lambda})(T_\lambda(G))(z/\sqrt{2}) \]
holds throughout $\mathbb{C}_+$. Finally, equation (3.2) follows from equation (3.3) by letting $\lambda \to -iq$, since all three of the transforms in (3.2) exist. □

**Corollary 1 to Theorem 4.** Let $F$ be as in Theorem 4. Then for all $q \in \mathbb{R} - \{0\}$,

$$(T_q^{(1)}((F * 1)_q))(z) = (T_q^{(1)}(F))(z/\sqrt{2}).$$

Furthermore, if $T_q^{(1)}((F * F)_q)$ exists, then

$$(T_q^{(1)}((F * F)_q))(z) = [(T_q^{(1)}(F))(z/\sqrt{2})]^2.$$
\[ \int_{C_0[0,T]}^{anf_q} (T_q^{(1)}(F))(z/\sqrt{2})(T_q^{(1)}(G))(z/\sqrt{2})m(dz) \]

\[ = \int_{C_0[0,T]}^{anf_q} (T_q^{(1)}((F * G)_q))(z)m(dz) \]

\[ = \lim_{p \to 0^+} \int_{C_0[0,T]}^{anw_{p+q^2}/2p} (T_{p-q}(F * G)_{p-q}(z)m(dz)) \]

\[ = \lim_{p \to 0^+} \int_{C_0[0,T]}^{anw_{p+q^2}/2p} (F * G)_{p-q}(w)m(dw) \]

\[ = \lim_{p \to 0^+} \int_{C_0[0,T]}^{anf_q} (F * G)_{p-q}(\sqrt{2p/p^2 + q^2})m(dw) \]

Our first corollary gives an alternative form of Parseval’s relation.

**Corollary 1 to Theorem 5.** Let \( F \) and \( G \) be as in Theorem 5. Then for all \( q \) in \( \mathbb{R} - \{0\} \),

\[ \int_{C_0[0,T]}^{anf_q} (T_{q/2}^{(1)}(F))(z)(T_{q/2}^{(1)}(G))(z)m(dz) \]

\[ = \int_{C_0[0,T]}^{anf_q} F(x)G(-x)m(dx). \]
Corollary 2 to Theorem 5. Let $F$ be as in Theorem 5 and assume that $T_q^{(1)}((F \ast F)q)$ exists for all $q \in \mathbb{R} - \{0\}$. Then
\[
\int_{C_0[0,T]} (T_q^{(1)}(F))(z/\sqrt{2})^2 m(dz) = \int_{C_0[0,T]} F(x/\sqrt{2})F(-x/\sqrt{2}) m(dx).
\]

Theorem 6. Let $F$ be as in Theorem 1. Furthermore, assume that $F$ is continuous on $C_0[0,T]$. Then for all $q$ in $\mathbb{R} - \{0\}$,
\[
(T_{-q}^{(1)}(T_q^{(1)}(F)))(y) = F(y)
\]
for s-a.e. $y \in C_0[0,T]$.

Proof. Proceeding as in the proof of Theorem 5 and using equations (2.2), (2.1), (1.6) and the continuity of $F$, we obtain that for s-a.e. $y \in C_0[0,T]$,
\[
(T_{-q}^{(1)}(T_q^{(1)}(F)))(y) = \lim_{p \to 0^+} (T_{p+q}(T_{p-q}(F)))(y)
\]
\[
= \lim_{p \to 0^+} \int_{C_0[0,T]} F(y + z + x)m(dx) m(dz)
\]
\[
= \lim_{p \to 0^+} \int_{C_0[0,T]} F(y + w)m(dw)
\]
\[
= \lim_{p \to 0^+} \int_{C_0[0,T]} F(y + \sqrt{\frac{2p}{p^2 + q^2}}w)m(dw)
\]
\[
= \int_{C_0[0,T]} F(y)m(dw)
\]
\[
= F(y).
\]

Corollary 1 to Theorem 6. Let $E$ be the class of all continuous functionals $F : C_0[0,T] \to \mathbb{C}$ satisfying the hypotheses of Theorem 1. Let $T_0^{(1)}$ denote the identity map; i.e., $T_0^{(1)}(F) \approx F$. Then
\[
\left\{T_q^{(1)}\right\}_{q \in \mathbb{R}}
\]
forms an abelian group acting on $E$ with $(T_q^{(1)})^{-1} = T_{-q}^{(1)}$.

**Remark 3.** Looking at equation (3.2) above, together with its proof, it is quite tempting to conjecture that for s-a.e. $z$ in $C_0[0, T]$,

\[
(T_q^{(1)}((F \ast G)_{q_2}))(z) = \frac{1}{(T_{q_1 + q_2}^{(1)}(F))(z/\sqrt{2})T_{q_1 + q_2}^{(1)}(G))(z/\sqrt{2})}
\]

provided that $q_1 + q_2 \neq 0$. But in general, equation (3.8) holds if and only if $q_1 = q_2 = q$, in which case equation (3.8) reduces to equation (3.2). The proof given above to establish equation (3.2) fails to work for equation (3.8) since for $\lambda > 0$ and $\beta > 0$,

\[
\begin{align*}
(T_{\lambda}(F \ast G_{\beta}))(z) &= \int_{C_0[0,T]} \int_{C_0[0,T]} F(\frac{z}{\sqrt{2}} + \frac{w_1}{\sqrt{2\lambda}} + \frac{w_2}{\sqrt{2\beta}}) G(\frac{z}{\sqrt{2}} + \frac{w_1}{\sqrt{2\lambda}} - \frac{w_2}{\sqrt{2\beta}}) \, m(dx)m(dy),
\end{align*}
\]

while $w_1 = \frac{x}{\sqrt{2\lambda}} + \frac{y}{\sqrt{2\beta}}$ and $w_2 = \frac{x}{\sqrt{2\lambda}} - \frac{y}{\sqrt{2\beta}}$ are independent processes if and only if $\lambda = \beta$.

In particular (see section 3 of [12] for the appropriate definitions), for $F$ and $G$ in the Banach algebra $S$ with corresponding finite Borel measures $f$ and $g$ in $M(L_2[0, T])$ and using equations (3.2) and (3.5) of [12], it is easy to see that

\[
\begin{align*}
(T_{z_1 \sqrt{2}q_1}^{(1)}(F))(z/\sqrt{2}) &= \int_{L_2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \int_0^T v(t)dt + \frac{i(q_1 + q_2)}{4q_1 q_2} \int_0^T v^2(t)dt \right\} df(v),
\end{align*}
\]

\[
\begin{align*}
(T_{z_1 \sqrt{2}q_2}^{(1)}(G))(z/\sqrt{2}) &= \int_{L_2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \int_0^T w(t)dt + \frac{i(q_1 + q_2)}{4q_1 q_2} \int_0^T w^2(t)dt \right\} dg(w),
\end{align*}
\]
and that

\[(3.11)\]

\[
(T^{(1)}_{q_1}((F \ast G)_{q_2}))(z) = \int_{L^2_{2}[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \int_0^T [v(t) + w(t)]dz(t) \right\} \\
\quad \cdot \exp\left\{-\frac{i(q_1 + q_2)}{4q_1q_2} \int_0^T [v^2(t) + w^2(t)]dt \right\} \\
\quad \cdot \exp\left\{\frac{i(q_1 - q_2)}{2q_1q_2} \int_0^T v(t)w(t)dt \right\} df(v)dg(w). \]

Now a careful examination of equations (3.9), (3.10) and (3.11) shows that equation (3.8) holds if and only if \(q_1 = q_2\).

References

Integration formulas involving Fourier-Feynman transforms


Timothy Huffman
Department of Mathematics
Northwestern College
Orange City, IA 51041, USA
E-mail: timh@nwciowa.edu

David Skoug
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0323, USA
E-mail: dskoug@math.unl.edu

David Storvick
School of Mathematics
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: storvick@math.umn.edu