GEOMETRY OF CONTACT STRONGLY PSEUDO-CONVEX CR-MANIFOLDS

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Abstract. As a natural generalization of a Sasakian space form, we define a contact strongly pseudo-convex CR-space form (of constant pseudo-holomorphic sectional curvature) by using the Tanaka-Webster connection, which is a canonical affine connection on a contact strongly pseudo-convex CR-manifold. In particular, we classify a contact strongly pseudo-convex CR-space form \((M, \eta, \varphi)\) with the pseudo-parallel structure operator \(h = 1/2L_\xi \varphi\), and then we obtain the nice form of their curvature tensors in proving Schur-type theorem, where \(L_\xi\) denote the Lie derivative in the characteristic direction \(\xi\).

1. Introduction

A contact manifold \((M, \eta)\) is a smooth manifold \(M^{2n+1}\) together with a global one-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere on \(M\). It means that \(d\eta\) has a maximal rank 2\(n\) on the contact distribution (or subbundle) \(D(= \text{kernel of } \eta)\). This fact arises naturally the characteristic vector field \(\xi\) on \(M\), and then leads to the decomposition \(TM = D \oplus \{\xi\}\).

Given a contact structure \(\eta\), we have two associated structures. One is a Riemannian structure (or metric) \(g\), and then we call \((M, \eta, g)\) a contact Riemannian manifold. The other is an almost CR-structure \((\eta, L)\), where \(L\) is the Levi form associated with an endomorphism \(J\) on \(D\) such that \(J^2 = -I\). In particular, if \(J\) is integrable, then we call it the (integrable) CR-structure. The associated almost CR-structure is said to be pseudohermitian, strongly pseudo-convex if the Levi form is hermitian and positive definite. We call such a manifold a contact strongly pseudo-convex CR-space form.

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pseudo-convex almost CR-manifold. There is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta \otimes \eta,$$

where we denote by the same letter $L$ the natural extension of the Levi form to a $(0,2)$-tensor field on $M$, that is, $i_\xi L = 0$, where $i_\xi$ denotes the interior product by $\xi$. We also denote by $\varphi$ the natural extension of $J$, which means that $\varphi|_D = J$ and $\varphi \xi = 0$. Then the above correspondence may be rephrased by the relation between $(\eta, g)$ and $(\eta, \varphi)$. From this point of view, we have two geometries for a given contact manifold, that is, one is formed by the Levi-Civita connection $\nabla$, the other is derived by the Tanaka-Webster connection $\nabla$, which is a canonical affine connection on a strongly pseudo-convex CR-manifold.

The normality of a contact Riemannian structure is defined in [13] (see, section 2). A normal contact Riemannian manifold is called a Sasakian manifold. A Sasakian structure has another picture, namely, a contact strongly pseudo-convex CR-structure whose characteristic vector field is a Killing vector field with respect to its associated Riemannian structure. In this context, we have two sides for a Sasakian space form: one is defined by a Sasakian manifold with constant $\varphi$-holomorphic sectional curvatures with respect to $\nabla$ and the other is of constant pseudo-holomorphic sectional curvature with respect to $\nabla$. Indeed, in [8] we defined a contact Riemannian space form which extends a Sasakian space form in the Riemannian view point. Corresponding to that, in this paper we introduce a notion, say, a contact strongly pseudo-convex CR-space form, which is a contact strongly pseudo-convex CR-manifold $M$ of constant pseudo-holomorphic sectional curvature $c$ (with respect to $\nabla$), that is, $M$ satisfies for any unit vector field $X$ orthogonal to $\xi$

$$L(\hat{\nabla}(X, \varphi X)\varphi X, X) = c \ (\text{constant}).$$

The main purpose of this paper is to find a proper class of contact strongly pseudo-convex CR-space forms (containing Sasakian space forms) and to study their geometric properties.

In particular, a contact strongly pseudo-convex CR-manifold satisfies CR-integrability or the condition of $\eta$-parallel $\varphi$ (that is, $g((\nabla_X \varphi)Y, Z) = 0$ for all vector fields $X, Y, Z$ orthogonal to $\xi$). We note that it is also equivalent to the condition of pseudo-parallel $\varphi$ which is defined by

$$L((\hat{\nabla}_X \varphi)Y, Z) = 0$$
for all vector fields $X, Y, Z$ orthogonal to $\xi$. Here, it is remarkable that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure. But the converse does not always hold. In fact, there are some examples of contact Riemannian manifolds which have integrable CR-structures but are not Sasakian. Other than all 3-dimensional contact Riemannian manifolds ([17]), we see that their associated CR-structures are integrable for (non-Sasakian) contact $(k, \mu)$-spaces (cf. [3], [8]). This class was introduced in [3] and their spaces are studied intensively in [4], [5] and [9]. In particular, their local classification is given in [5].

We restrict our attention to a more suitable class of contact strongly pseudo-convex CR-manifolds endowed with an additional property, namely, it is imposed by the condition of pseudo-parallel $h$:

$$L((\hat{\nabla}Xh)Y, Z) = 0$$

for all vector fields $X, Y, Z$ orthogonal to $\xi$, where $h$ denotes, up to a scaling factor, the Lie derivative of $\varphi$ in the direction of $\xi$. As concerns this condition, we note that it is also equivalent to $\eta$-parallel $h$ (with respect to $\nabla$), i.e., $g((\nabla_Xh)Y, Z) = 0$ for all vector fields $X, Y, Z$ orthogonal to $\xi$. Recently, E. Boeckx and the present author [6] proved that a contact Riemannian space with $\eta$-parallel $h$ is either a K-contact space (in which case, $h$ vanishes identically) or a $(k, \mu)$-space. A contact strongly pseudo-convex CR-manifold with pseudo-parallel $h$ is called a pseudo-parallel contact strongly pseudo-convex CR-manifold, or shortly, a pseudo-parallel contact CR-space.

In Section 2, we collect preliminary notions and results which are needed in later sections. In Section 3, we study the Tanaka-Webster curvature tensor $\hat{R}$ of a contact strongly pseudo-convex CR-manifold. In Section 4, we classify a pseudo-parallel contact strongly pseudo-convex CR-space form. In more detail, a pseudo-parallel contact strongly pseudo-convex CR-space of constant pseudo-holomorphic sectional curvature $c$ is pseudo-homothetic to one of the following: (1) the (normalized) model spaces of Sasakian space forms, (2) the unit tangent sphere bundle of a space of constant curvature $-1$, or (3) a non-Sasakian Lie group with a special left-invariant contact metric, $SU(2), SL(2, R)$, the group $E(2)$ of rigid motions of Euclidean 2-space, the group $E(1, 1)$ of rigid motions of the Minkowski 2-space (Corollary 4.3). It is remarkable that the case (2) above is neither Sasakian nor a space of constant $\varphi$-holomorphic sectional curvature.
In Section 5, we obtain the curvature form of a contact strongly pseudo-convex CR-manifold with constant pseudo-holomorphic sectional curvature. Finally, in Section 6, for the class of pseudo-parallel contact strongly pseudo-convex CR-manifolds, we prove a Schur-type theorem. Then we have the nice form of the curvature tensor of a pseudo-parallel contact strongly pseudo-convex CR-space form.

2. Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and contact strongly pseudo-convex CR-manifold. We refer to [2] for further details. All manifolds in the present paper are assumed to be connected and of class $C^\infty$.

A $(2n+1)$-dimensional manifold $M^{2n+1}$ is said to be a contact manifold if it admits a global one-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form $\eta$, there exists a unique vector field $\xi$, called the characteristic vector field, satisfying $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ for any vector field $X$. It is well-known that there also exists a Riemannian metric $g$ and a $(1,1)$-tensor field $\phi$ such that

\begin{align}
\phi X &= g(X, \xi) \\
\phi^2 X &= -X + \eta(X)\xi,
\end{align}

where $X$ and $Y$ are vector fields on $M$. From (2.1), it follows that

\begin{align}
\phi \xi &= 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).
\end{align}

A Riemannian manifold $M$ equipped with structure tensors $(\eta, g)$ satisfying (2.1) is said to be a contact Riemannian manifold or contact metric manifold and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold $M$, we define an operator $h$ by $h = \frac{1}{2}L_\xi \phi$, where $L$ denotes Lie differentiation. Then we may observe that the structural operator $h$ is symmetric and satisfies

\begin{align}
h\xi &= 0, \quad h\phi = -\phi h, \\
\nabla_X \xi &= -\phi X - \phi h X,
\end{align}

where $X$ and $Y$ are vector fields on $M$. From (2.1), it follows that
where $\nabla$ is Levi-Civita connection. We denote by $R$ the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields $X,Y,Z$ on $M$. Along a trajectory of $\xi$, the characteristic Jacobi operator $l = R(\cdot, \xi)\xi$ is also symmetric. Moreover, we have

$$\nabla_\xi h = \varphi - \varphi l - \varphi^2,$$

(2.5)

$$g(R(X,Y)\xi, Z) = g((\nabla_Y \varphi)X - (\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y, Z)$$

(2.6)

for all vector fields $X,Y,Z$ on $M$. A contact Riemannian manifold for which $\xi$ is a Killing vector field, is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold $M$, one may define naturally an almost complex structure $\mathcal{J}$ on $M \times \mathbb{R}$ by

$$\mathcal{J}(X, f \frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where $X$ is a vector field tangent to $M$, $t$ the coordinate of $\mathbb{R}$ and $f$ a function on $M \times \mathbb{R}$. If the almost complex structure $\mathcal{J}$ is integrable, $M$ is said to be normal or Sasakian. It is known that $M$ is normal if and only if $M$ satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. A Sasakian manifold is also characterized by the condition

$$\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$$

(2.7)

for all vector fields $X$ and $Y$ on the manifold and this is equivalent to

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

(2.8)

for all vector fields $X$ and $Y$.

For a contact Riemannian manifold $M = (M; \eta, g)$, the tangent space $T_pM$ of $M$ at each point $p \in M$ is decomposed as $T_pM = D_p \oplus$
\{\xi\}_p (\text{direct sum}), where we denote \(D_\varphi = \{v \in T_pM | \eta(v) = 0\}\). Then \(D : p \to D_p\) defines a distribution orthogonal to \(\xi\). The \(2n\)-dimensional distribution (or subbundle) \(D\) is called the contact distribution (or contact subbundle). Its associated almost CR-structure is given by the holomorphic subbundle

\[\mathcal{H} = \{X - iJX : X \in D\}\]

of the complexification \(TM^\mathbb{C}\) of the tangent bundle \(TM\), where \(J = \varphi|D\), the restriction of \(\varphi\) to \(D\). Then we see that each fiber \(\mathcal{H}_x (x \in M)\) is of complex dimension \(n\) and \(\mathcal{H} \cap \mathcal{H} = \{0\}\). Furthermore, we have \(\mathcal{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}\). We say that the associated CR-structure is integrable if \([\mathcal{H}, \mathcal{H}] \subset \mathcal{H}\). For \(\mathcal{H}\) we define the Levi form by

\[L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY),\]

where \(\mathcal{F}(M)\) denotes the algebra of differential functions on \(M\). Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is strongly pseudo-convex, pseudo-hermitian CR-structure. We call the pair \((\eta, L)\) a strongly pseudo-convex, pseudo-hermitian structure on \(M\). Since \(d\eta(\varphi X, \varphi Y) = d\eta(X, Y)\), we see that \([JX, JY] - [X, Y] \in D\) and \([JX, Y] + [X, JY] \in D\) for \(X, Y \in D\). Furthermore, if \(M\) satisfies the condition

\[[J, J](X, Y) = 0\]

for \(X, Y \in D\), then the pair \((\eta, L)\) is called a strongly pseudo-convex (integrable) CR-structure and \((M; \eta, L)\) is called a strongly pseudo-convex pseudohermitian CR-manifold. It may be easily proved that the almost CR-structure is integrable if and only if \(M\) satisfies the integrability condition \(Q = 0\), where \(Q\) is a \((1,2)\)-tensor field on \(M\) defined by

\[Q(X, Y) = (\nabla_X \varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX)\]

for all vector fields \(X, Y\) on \(M\) (see \[17, Proposition 2.1\]). Taking account of (2.7) we see that for a Sasakian manifold the associated CR-structure is integrable (cf. \[12\]).

Now, we review the generalized Tanaka-Webster connection ([17]) on a contact strongly pseudo-convex almost CR-manifold \(M = (M; \eta, L)\). The generalized Tanaka-Webster connection \(\hat{\nabla}\) is defined by

\[\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi\]
for all vector fields $X, Y$ on $M$. Together with (2.4), \( \hat{\nabla} \) may be rewritten as

\[
(2.10) \quad \hat{\nabla}_X Y = \nabla_X Y + A(X, Y),
\]

where we have put

\[
(2.11) \quad A(X, Y) = \eta(X) \phi Y + \eta(Y) (\phi X + \phi h X) - g(\phi X + \phi h X, Y) \xi.
\]

We see that the generalized Tanaka-Webster connection \( \hat{\nabla} \) has the torsion

\[
\hat{T}(X, Y) = 2g(X, \phi Y) \xi + \eta(Y) \phi h X - \eta(X) \phi h Y.
\]

In particular, for a K-contact manifold (2.11) reduces as follows:

\[
(2.12) \quad A(X, Y) = \eta(X) \phi Y + \eta(Y) \phi X - g(\phi X, Y) \xi.
\]

Furthermore, it was proved that

**Proposition 2.1 ([17]).** The generalized Tanaka-Webster connection \( \hat{\nabla} \) on a contact Riemannian manifold \( M = (M; \eta, g) \) is the unique linear connection satisfying the following conditions:

(i) \( \hat{\nabla} \eta = 0, \hat{\nabla} \xi = 0 \);

(ii) \( \hat{\nabla} g = 0 \);

(iii-1) \( \hat{T}(X, Y) = 2g(X, \phi Y) \xi, X, Y \in D; \)

(iii-2) \( \hat{T}(\xi, \phi Y) = -\phi \hat{T}(\xi, Y), Y \in D; \)

(iv) \( \hat{\nabla}_X (\phi Y) = Q(X, Y), X, Y \in TM. \)

The Tanaka-Webster connection ([14], [20]) on a nondegenerate (integrable) CR-manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and \( Q = 0 \) (CR-integrability). The metric affine connection \( \hat{\nabla} \) is a natural generalization of the Tanaka-Webster connection. In fact, in [1] the authors treat the use of \( \hat{\nabla} \) in the non-integrable case.

From Proposition 2.1 we immediately see that the CR-integrability condition \( Q = 0 \) is equivalent to the condition of pseudo-parallel \( \phi \) (with respect to \( \hat{\nabla} \))

\[
L((\hat{\nabla}_X \phi) Y, Z) = 0
\]

for all vector fields $X, Y, Z$ orthogonal to $\xi$. Since we know that $\nabla_\xi \phi = 0$ holds (cf. [2] p. 67) in a contact Riemannian manifold, we see further...
that CR-integrability is also equivalent to the condition of η-parallel ϕ, i.e., \( g((\nabla_X \varphi)Y, Z) = 0 \) for all vector fields \( X, Y, Z \) orthogonal to \( \xi \).

From (2.3), (2.10) and (2.11) we have

\[
(\bar{\nabla}_X h)Y = (\nabla_X h)Y + A(X, hY) - hA(X, Y)
\]

(2.13)

\[
= (\nabla_X h)Y + 2\eta(X)\varphi hY + g((\varphi h + \varphi h^2)X, Y)\xi \\
+ \eta(Y)(\varphi hX + \varphi h^2 X).
\]

In [6] we studied a contact Riemannian manifold which satisfies the condition that \( h \) is η-parallel (with respect to \( \nabla \)), i.e., \( g((\nabla_X h)Y, Z) = 0 \) for any vector fields \( X, Y, Z \) orthogonal to \( \xi \). Also from (2.13) we see that this is equivalent to the condition that

\[ L((\bar{\nabla}_X h)Y, Z) = 0 \]

for any vector fields \( X, Y, Z \) orthogonal to \( \xi \), i.e., \( h \) is pseudo-parallel (with respect to \( \bar{\nabla} \)). We call a contact strongly pseudo-convex CR-manifold with pseudo-parallel \( h \), a pseudo-parallel contact strongly pseudo-convex CR-manifold, or in short, a pseudo-parallel contact CR-space.

Here, we recall the notion of a pseudo-homothetic transformation (or D-homothetic transformation) of a contact metric manifold ([15]). This transformation means a change of structure tensors of the form

\[
\bar{\eta} = a\eta, \quad \bar{\xi} = 1/a\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,
\]

(2.14)

where \( a \) is a positive constant. From (2.14), we have \( \bar{h} = (1/a)h \). By using the well-known formula

\[
2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y) - g(X, [Y, Z]) \\
- g(Y, [X, Z]) + g(Z, [X, Y])
\]

we have

\[
\bar{\nabla}_X Y = \nabla_X Y + E(X, Y),
\]

(2.15)

where \( E \) is the (1,2)-type tensor defined by

\[
E(X, Y) = -(a - 1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a-1}{a}g(\varphi hX, Y)\xi.
\]
Remark 1. (1) From (2.14) and (2.15), we see that the condition of pseudo-parallel $\varphi$ (or $\eta$-parallel $\varphi$) is invariant under a pseudo-homothetic transformation. Indeed by direct computations we have
\[
(\bar{\nabla}_{X}\bar{\varphi})Y = (\nabla_{X}\varphi)Y + (a - 1)\eta(Y)\varphi^{2}X - (a - 1)/ag(X, hY)\xi.
\]

(2) The condition of pseudo-parallel $h$ (or $\eta$-parallel $h$) is also invariant under a pseudo-homothetic transformation. Namely, for a pseudo-homothetic transformation we have
\[
(\bar{\nabla}_{X}\bar{h})Y = 1/a\left((\nabla_{X}h)Y + (a - 1)\eta(Y)h\varphi X + 2(a - 1)\eta(X)h\varphi Y - (a - 1)/ag(\varphi hX, hY)\xi \right).
\]

3. The Tanaka-Webster curvature tensor of a contact CR-manifold

Let $(M; \eta, L)$ be a contact strongly pseudo-convex CR-manifold. In this section, we define the Tanaka-Webster curvature tensor of $\hat{R}$ (with respect to $\hat{\nabla}$) (in the extended meaning) by
\[
(3.1) \quad \hat{R}(X, Y)Z = \hat{\nabla}_{X}(\hat{\nabla}_{Y}Z) - \hat{\nabla}_{Y}(\hat{\nabla}_{X}Z) - \hat{\nabla}_{[X,Y]}Z
\]
for all vector fields $X, Y, Z$ in $M$. Then we have

Proposition 3.1.
\[
\hat{R}(X, Y)Z = -\hat{R}(Y, X)Z,
\]
\[
g(\hat{R}(X, Y)Z, W) = -g(\hat{R}(X, Y)W, Z).
\]

The first identity follows trivially from the definition of $\hat{R}$. Since the connection parallelizes the metric form, (i.e., $\hat{\nabla}g = 0$), we have also the second one by a similar way as the case of Riemannian curvature tensor. We remark that the Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-type identities do not hold, in general. From (3.1), together with $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$, the straightforward computations yield
\[
\hat{R}(X, Y)Z
\]
\[
= R(X, Y)Z + \eta(Z)\left(\varphi P(X,Y) + \varphi(A(hX, hY) - A(Y, hX))
\right. \\
- \varphi h(A(X, Y) - A(Y, X)) \bigg)
\(- g\left(\varphi P(X, Y) + \varphi(A(X, hY) - A(Y, hX))\right) \)
\(- \varphi h(A(X, Y) - A(Y, X)), Z\xi \)
\(- 2g(\varphi X, Y)A(\xi, Z) - \eta(X)A(\varphi hY, Z) + \eta(Y)A(\varphi hX, Z) \)
\(- \eta(X)\varphi A(Y, Z) + \eta(Y)\varphi A(X, Z) + \eta(A(X, Z))(\varphi Y + \varphi hY) \)
\(- \eta(A(Y, Z))(\varphi X + \varphi hX) + g(\varphi X + \varphi hX, A(Y, Z))\xi \)
\(- g(\varphi Y + \varphi hY, A(X, Z))\xi. \)

We put \(P(X, Y) = (\nabla_X h)Y - (\nabla_Y h)X\), then we see that \(P\) is a \((1, 2)\)-type tensor field on \(M\). By using (2.1), (2.2), (2.3) and (2.11) we have

\[
\hat{R}(X, Y)Z = R(X, Y)Z + B(X, Y)Z, \tag{3.2}
\]

where

\[
B(X, Y)Z = \eta(Z)\varphi P(X, Y) - g(\varphi P(X, Y), Z)\xi - \eta(Z)(\eta(Y)(X + hX) \)
\[- \eta(X)(Y + hY)) + \eta(Y)g(X + hX, Z)\xi - \eta(X)g(Y + hY, Z)\xi \)
\[+ g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY) \]
\[- 2g(\varphi X, Y)\varphi Z \]

for all vector fields \(X, Y, Z\) in \(M\). From (3.2), by making use of (2.2) and (2.3), we obtain

\[
\hat{R}(X, Y)\xi = R(X, Y)\xi + \varphi P(X, Y) + \eta(X)(Y + hY) - \eta(Y)(X + hX). \tag{3.3}
\]

Now, we give

**Definition 3.2.** Let \((M; \eta, L)\) be a contact strongly pseudo-convex CR-manifold with the associated Levi form \(L\). Then \(M\) is said to be of constant pseudo-holomorphic sectional curvature \(c\) (with respect to the Tanaka-Webster connection) if \(M\) satisfies

\[
L(\hat{R}(X, \varphi X)\varphi X, X) = c
\]

for any unit vector field \(X \perp \xi\). A complete and simply connected contact strongly pseudo-convex CR-manifold of constant pseudo-holomorphic sectional curvature is called a contact strongly pseudo-convex CR-space form.
Here, we recall

**Definition 3.3 ([16]).** Let \((M; \eta, g)\) be a Sasakian manifold. Then \(M\) is called a space of constant \(\varphi\)-holomorphic sectional curvature \(c_0\) if \(M\) satisfies

\[ g(R(X, \varphi X) \varphi X, X) = c_0 \]

for any unit vector field \(X \perp \xi\). A complete and simply connected Sasakian space of constant \(\varphi\)-holomorphic sectional curvature is called a Sasakian space form.

Now, we prove that

**Proposition 3.4.** The contact strongly pseudo-convex CR-space form is a pseudo-homothetic-invariant.

**Proof.** From (2.15), by long but tedious computations, we get

\[
\begin{align*}
g(\hat{R}(X, \varphi X) \varphi X, X) &= g(R(X, \varphi X) \varphi X, X) - (a - 1)[3 + g(hX, X)] \\
&\quad - \frac{a - 1}{a} [g(\varphi hX, X)^2 + g(hX, X)(g(hX, X) - 1)] \\
&\quad + \frac{(a - 1)^2}{a} g(hX, X)
\end{align*}
\]

(3.4)

for any unit horizontal vector \(X \in D(p)\) (with respect to \(g\), \(p \in M\)). For any unit horizontal vector \(X\), from (3.2), we get

\[
\begin{align*}
\bar{L}(\hat{R}(X, \varphi X) \varphi X, X) \\
&= 3 + \tilde{g}(\hat{R}(X, \varphi X) \varphi X, X) - \tilde{g}(\varphi \tilde{h}X, X)^2 - \tilde{g}(\tilde{h}X, X)^2.
\end{align*}
\]

Along with (2.14) and (3.4), we have

\[
\bar{L}(\hat{R}(X, \varphi X) \varphi X, X) = aL(\hat{R}(X, \varphi X) \varphi X, X).
\]

If we denote by \(\hat{K}(X, \varphi X)\) the pseudo-holomorphic sectional curvature \(L(\hat{R}(X, \varphi X) \varphi X, X)\) for a unit horizontal vector \(X\), then this is rewritten by

\[
\hat{K}(X, \varphi X) = a\hat{K}(X, \varphi X).
\]

Thus, we have proved. \(\square\)

**Remark 2.** Making use of (2.14) and (3.4) we can see that the contact space of constant \(\varphi\)-holomorphic sectional curvature is not a pseudo-homothetic invariant, in general.
4. Pseudo-parallel contact strongly pseudo-convex CR-space form

We start this section by reviewing in brief a $(k, \mu)$-space. In [3], the $(k, \mu)$-nullity distribution of a contact Riemannian manifold $M$, for the pair $(k, \mu) \in \mathbb{R}^2$, is defined by

\[ N(k, \mu) : p \rightarrow N_p(k, \mu) = \{ z \in T_pM | R(x, y)z = (kI + \mu h)(g(y, z)x - g(x, z)y) \text{ for any } x, y \in T_pM \} \]

A $(k, \mu)$-space is a contact Riemannian manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution, that is,

\[ R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \]

where $I$ denotes the identity transformation. It is proved in [3] that the $(k, \mu)$-spaces are invariant under a pseudo-homothetic transformation in the range of $(k, \mu)$. More precisely, a pseudo-homothetic transformation with constant $a$ change $(k, \mu)$ into $(\bar{k}, \bar{\mu})$, where

\[ \bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}. \]

Also, the associated CR-structures of the $(k, \mu)$-spaces are integrable, that is, they are contact strongly pseudo-convex CR-manifolds. This class contains Sasakian manifolds with $k = 1$ and $h = 0$. The unit tangent sphere bundle is a $(k, \mu)$-space if and only if the base manifold is of constant curvature $c$ with $k = c(2 - c)$ and $\mu = -2c$ ([3]). (By virtue of the result of Y. Tashiro [19], we know that for $c \neq 1$, the unit tangent sphere bundle is non-Sasakian.) In [4], [5] the curvature tensor $R$ of contact $(k, \mu)$-space is determined completely for $k < 1$. Furthermore, E. Boeckx [5] classified non-Sasakian $(k, \mu)$-spaces up to a pseudo-homothetic transformation.

In [3], the authors proved following useful formulas:

\[ \nabla_X h Y = [(1 - k)g(X, \varphi Y) - g(X, \varphi h Y)]\xi - \eta(Y)[(1 - k)\varphi X + \varphi h X] - \mu \eta(X)\varphi h Y \]
and

\[ P(X,Y) = (\nabla_X h)Y - (\nabla_Y h)X \]

\[ = (1-k)[2g(X,\varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] + (1-\mu)\left[\eta(X)\varphi hY - \eta(Y)\varphi hX\right]. \]  

(4.4)

Then from (4.3), we immediately see that a \((k,\mu)\)-space has a pseudo-parallel structure. Moreover, together with the result in [6] we have

**Theorem 4.1.** A pseudo-parallel contact strongly pseudo-convex CR-manifold is Sasakian or a \((k,\mu)\)-space.

Now, from (3.2), we have for unit vector field \(X \perp \xi\)

\[ L(\hat{R}(X,\varphi X)\varphi X, X) \]

\[ = 3 + g(R(X,\varphi X)\varphi X, X) - g(\varphi h X, X)^2 - g(h X, X)^2. \]  

(4.5)

Hence, we see that \(M\) is of constant pseudo-holomorphic sectional curvature \(c\) if and only if

\[ K(X,\varphi X)(= g(R(X,\varphi X)\varphi X, X)) \]

\[ = (c-3) + g(\varphi h X, X)^2 + g(h X, X)^2. \]  

(4.6)

We prove

**Theorem 4.2.** Let \(M\) be a contact \((k,\mu)\)-space. Then \(M\) is of constant pseudo-holomorphic sectional curvature \(c\) if and only if (1) \(M\) is Sasakian space of constant \(\varphi\)-holomorphic sectional curvature \(c_0 = (c-3)\), (2) \(\mu = 2\) and \(c = 0\), or (3) \(\text{dim } M = 3\) and \(\mu = (2-c)\).

**Proof.** We let \(M\) be a non-Sasakian contact \((k,\mu)\)-space \((k \neq 1)\). Then we already know that (cf. [3] or [8])

\[ K(X,\varphi X) = (1-2\mu) + \frac{k+1-\mu}{k-1}[g(\varphi h X, X)^2 + g(h X, X)^2]. \]  

(4.7)

Thus, from (4.6) and (4.7), we can deduce the following three cases: (1) \(k = 1\) \((h = 0)\) and \(M\) is a Sasakian space form, (2) \(k < 1\), \(\mu = 2\) and \(c = 0\), (3) If \(\text{dim } M = 3\), then we see that \(g(\varphi h X, X)^2 + g(h X, X)^2 = 1/2(\text{trace of } h^2)\). But, since \(h^2 = (k-1)\varphi^2\) (cf. [3]), we have \(\mu = (2-c)\). \(\square\)
In the proof of Proposition 3.4, we see that for a Sasakian space (whose structure is invariant under a pseudo-homothetic transformation) the constancy of pseudo-holomorphic sectional curvature is invariant under pseudo-homothetic transformations (indeed, \( \bar{c} = c/a, \ a > 0 \)). From (4.2), we also see that a \((k, 2)\)-space is invariant under a pseudo-homothetic transformation and \( I_M = \frac{1 - \mu/2}{\sqrt{1 - k}} = 0 > -1 \). Thus, due to the classification theorem of a \((k, \mu)\)-space in [5], we see that \((k, 2)\)-space is pseudo-homothetic to \( T_1M(-1) \). For the three-dimensional non-Sasakian \((k, \mu)\)-space, the local classification is given in [3]. Further in [5] E. Boeckx showed up there picture up to a pseudo-homothetic invariant \( I_M \), indeed there are \( SL(2, R) \) with \( I_{SL(2, R)} < -1 \) or \( -1 < I_{SL(2, R)} < 1, E(1, 1) \) with \( I_{E(1, 1)} = -1, E(2) \) with \( I_{E(2)} = 1, SU(2) \) with \( I_{SU(2)} > 1 \).

**Corollary 4.3.** Let \( M \) be a pseudo-parallel contact strongly pseudo-convex CR-space. Then \( M \) is of constant pseudo-holomorphic sectional curvature \( c \) if and only if \( M \) is pseudo-homothetic to one of the following:

1. the unit sphere \( S^{2n+1} \) with the natural Sasakian structure with \( c_0 = 1 \) for \( c > 0 \), \( R^{2n+1} \) with Sasakian \( \varphi \)-holomorphic sectional curvature \( c_0 = -3 \) for \( c = 0 \), or \( B^n \times R \) with Sasakian \( \varphi \)-holomorphic sectional curvature \( c_0 = -7 \) for \( c < 0 \), where \( B^n \) is a simply connected bounded domain in \( C^n \) with constant holomorphic sectional curvature \(-4\),

2. the unit tangent sphere bundle of a space of constant curvature \(-1\), or

3. a non-Sasakian Lie group with a special left-invariant contact metric, \( SU(2) \), \( SL(2, R) \), the group \( E(2) \) of rigid motions of Euclidean 2-space, the group \( E(1, 1) \) of rigid motions of the Minkowski 2-space.

The Sasakian structure \((\eta, g)\) on \( R^{2n+1}(x^i, y^i, z) \) \((i = 1, \ldots, n)\) is given by the canonical contact structure
\[
\eta = \frac{1}{2}(dz - \sum_{i} y^i dx^i)
\]
and the Riemannian metric \( g \) given by the quadratic form
\[
ds^2 = \frac{1}{4}(\eta \otimes \eta + \sum_{i}(dx^i)^2 + (dy^i)^2)).
\]

We know that the standard contact metric structure of the unit tangent sphere bundle \( T_1M(1) \) of a space of constant curvature 1 is
Sasakian. However, we can check that it has neither constant $\varphi$-holomorphic sectional curvature nor constant pseudo-holomorphic sectional curvature. As stated already, unit tangent sphere bundles are $(k, \mu)$-spaces if and only if the base manifold is of constant curvature $b$ with $k = b(2 - b)$ and $\mu = -2b$. Thus, we have

**Corollary 4.4.** The standard contact strongly pseudo-convex CR-structure of a unit tangent sphere bundle $T_1M(b)$ of $(n + 1)$-dimensional space of constant curvature $b$ has constant pseudo-holomorphic sectional curvature $c$ if and only if

1. $b = -1$ and $c = 0$, or
2. $n = 1$ and $b = 1/2(c - 2)$.

For a regular (i.e., the foliation defined by the vector field $\xi$ is regular) Sasakian space form $M^{2n+1}(c_0)$ of constant $\varphi$-holomorphic sectional curvature $c_0$, the quotient $M^{2n+1}(c_0)/\xi$ with the induced metric and the complex structure $J$ given by $J\pi_*X = \pi_*\varphi X$ is a complex space form $\tilde{M}^n((c_0 + 3)/4)$, where $\pi : M \to M/\xi$ is the Riemannian submersion. Closing this section we state

**Remark 4.** (1) Three-dimensional non-Sasakian contact $(k, \mu)$-spaces have constant $\varphi$-holomorphic sectional curvature (cf. [3], [8]) and at the same time constant pseudo-holomorphic sectional curvatures $c = (2 - \mu)$.

(2) A contact $(k, 2)$-space $(k \neq 1)$ is non-Sasakian and of non-constant $\varphi$-holomorphic sectional curvature (see (4.7)), but has constant “pseudo-holomorphic sectional curvature (with respect to the Tanaka-Webster connection)”.

### 5. The curvature tensor of a contact strongly pseudo-convex CR-space form

In this section, we study the curvature of a contact strongly pseudo-convex CR-space form. Let $M$ be a contact strongly pseudo-convex CR-manifold. We put

$$C(X, Y)Z = R(X, Y)Z + g(hY, Z)hX - g(hX, Z)hY$$

for all vector fields $X, Y, Z$ on $M$. Then we see that $C$ is a $(1,3)$-type tensor field on $M$. From this, by using the symmetries of the curvature
tensor $R$ and the symmetry of structure tensor $h$, we easily see that $C$ also satisfies the symmetries, that is,

1. $C(X, Y)Z = -C(Y, X)Z$,
2. $g(C(X, Y)Z, W) = -g(C(X, Y)W, Z)$,
3. $g(C(X, Y)Z, W) = g(C(Z, W)X, Y)$,
4. $C(X, Y)Z + C(Y, Z)X + C(Z, X)Y = 0$.

Further, we see, together with (4.6), that $M$ has pointwise constant pseudo-holomorphic sectional curvature $H(p)$, $p \in M$, if and only if

$$g(C(X, \varphi X)\varphi X, X) = H_1(p)$$

for a unit horizontal vector $X$, where we have put $H_1(p) = H(p) - 3$.

First of all, from (2.9), we see that $M$ satisfies

$$g(C(X, Y)Z, W) = \varphi P(Z, Y)(X + hX) - \varphi P(X, Y)$$

for all vector fields $X$ and $Y$. Comparing with (2.7), it follows that a contact strongly pseudo-convex CR-manifold is normal (or Sasakian) if and only if $h = 0$.

Then we have the following

**PROPOSITION 5.1.** For all vector fields $X, Y, Z$ on $M$,

5.1

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$$

for all vector fields $X$ and $Y$. Comparing with (2.7), it follows that a contact strongly pseudo-convex CR-manifold is normal (or Sasakian) if and only if $h = 0$.

Then we have the following

**PROPOSITION 5.1.** For all vector fields $X, Y, Z$ on $M$,

5.2

$$C(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) - \varphi P(X, Y),$$

5.3

$$g(C(\xi, X)Y, Z) = \eta(Z)g(Y + hY, X) - \eta(Y)g(Z + hZ, X) + g(\varphi P(Z, Y), X),$$

5.4

$$R(X, Y)\varphi Z = \varphi R(X, Y)Z - g(Y + hY, Z)(\varphi X + \varphi hX) + g(X + hX, Z)(\varphi Y + \varphi hY) + g(\varphi X + \varphi hX, Z)(Y + hY) - g(\varphi Y + \varphi hY, Z)(X + hX) + g(P(X, Y), Z)\xi - \eta(Z)P(X, Y),$$
and

\[ C(X, Y)\varphi Z - \varphi C(X, Y)Z = R(X, Y)\varphi Z - \varphi R(X, Y)Z \]
\[ \quad + g(hY, \varphi Z)hX - g(hX, \varphi Z)hY \]
\[ \quad - g(hY, Z)\varphi hX + g(hX, Z)\varphi hY. \]  

\[(5.5)\]

**Proof.** First, together with (2.3) we see that

\[ C(X, Y)\xi = R(X, Y)\xi \]

and

\[ g(C(ξ, X)Y, Z) = g(R(ξ, X)Y, Z). \]

But, from (2.6), (5.1) and the fundamental symmetries of the curvature tensor, we compute

\[ R(X, Y)\xi \]

and

\[ g(R(ξ, X)Y, Z). \]

So, we obtain (5.2) and (5.3). The Ricci identity for \( \varphi \) is given as

\[(5.6)\]

\[ R(X, Y)\varphi Z - \varphi R(X, Y)Z = (\nabla^2_{X,Y} \varphi)Z - (\nabla^2_{Y,X} \varphi)Z, \]

where \( \nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \). From (5.1) we have

\[ (\nabla^2_{X,Y} \varphi)Z = -g(Y + hY, Z)(\varphi X + \varphi hX) + g(\varphi X + \varphi hX, Z)(Y + hY) \]
\[ \quad + g((\nabla_X h)Y, Z)\xi - \eta(Z)(\nabla_X h)Y, \]

and thus (5.4) and (5.5) follow easily from this, (5.6) and the definition of the tensor \( C \).

\( \square \)

Now, we prove

**Proposition 5.2.** Let \( M \) be a contact strongly pseudo-convex CR-manifold. Then the necessary and sufficient condition for \( M \) to have pointwise constant pseudo-holomorphic sectional curvature \( H = H(p) \), \( p \in M \), is

\[(5.7)\]

\[ g(R(X, Y)Z, W) \]
\[ = \frac{1}{4} \left\{ H \left[ (g(Y, Z) - \eta(Y)\eta(Z))(g(X, W) - \eta(X)\eta(W)) \right. \right. \]
\[ \quad - (g(X, Z) - \eta(X)\eta(Z))(g(Y, W) - \eta(Y)\eta(W)) \right\} \]
\[ \quad + (H - 4) \left[ g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W) \right. \]
\[ \quad - 2g(\varphi X, Y)g(\varphi Z, W) \left. \right\} \]
\[ \quad + g(hY, Z)(g(X, W) - \eta(X)\eta(W)) \]
\[ \quad - g(hX, Z)(g(Y, W) - \eta(Y)\eta(W)) \]
Similarly, from (5.4) and (5.5) we get

\[ C \text{ and the curvature tensor} \]

for all vector fields \( X, Y, Z, W \) in \( M \).

**Proof.** For \( X, Y \in D \), using the fundamental properties of the tensor \( C \) and the curvature tensor \( R \), (2.1), (2.2) and (2.3), we obtain from (5.4) and (5.5)

\[ g(C(X, \varphi X)Y, \varphi Y) = g(C(X, \varphi Y)Y, \varphi X) + g(C(X, Y)\varphi X, \varphi Y) \]

and

\[ g(C(X, Y)\varphi X, \varphi Y) = g(C(X, Y)X, Y) \]

\[ - g(X, Y)^2 - 2g(hX, Y)^2 - 2g(X, Y)g(hX, Y) + g(X, X)g(Y, Y) \]

\[ + g(hX, X)g(hY, Y) + g(Y, Y)g(hX, X) \]

\[ + 2g(hX, X)g(hY, Y) - g(\varphi X, Y)^2 \]

\[ + 2g(\varphi hX, Y)^2 - 2g(\varphi hX, X)g(\varphi hY, Y) - g(X, X)g(Y, Y) \]

Similarly, from (5.4) and (5.5) we get

\[ g(C(X, \varphi Y)X, \varphi Y) = g(C(X, \varphi Y)Y, \varphi X) \]

\[ + g(X, Y)^2 - 2g(hX, Y)^2 - 2g(\varphi hX, X)g(\varphi hY, Y) - g(X, X)g(Y, Y) \]
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\[ -g(Y,Y)g(hX, X) + g(X, X)g(hY, Y) + 2g(hX, X)g(hY, Y) + g(\varphi X, Y)^2 + 2g(\varphi hX, Y)^2 + 2g(\varphi X, Y)g(\varphi hX, Y) \]

and

\[ g(C(Y, \varphi X)Y, \varphi X) = g(C(X, \varphi Y)Y, \varphi X) + g(X, Y)^2 - 2g(hX, Y)^2 - 2g(\varphi hX, X)g(\varphi hY, Y) - g(X, X)g(Y, Y) + g(Y, Y)g(hX, X) - g(X, X)g(hY, Y) + 2g(hX, X)g(hY, Y) + g(\varphi X, Y)^2 + 2g(\varphi hX, Y)^2 + 2g(\varphi X, Y)g(\varphi hX, Y). \]

We now suppose that \( M \) has a pointwise constant pseudo-holomorphic sectional curvature \( H(p) \), i.e., for any \( X \in D(p) \),

\[ L(\hat{R}(X, \varphi X)\varphi X, X) = H(p)g(X, X)^2. \]

Then together with (4.5) we immediately get

\[ g(C(X, \varphi Y)Y, \varphi X) = H_1(p)g(X, X)^2, \]

where we have put \( H_1(p) = H(p) - 3 \). Substituting \( X \) by \( X + Y \) and \( X - Y \) for \( X, Y \in D \) in (5.12) respectively, and summing them, we get

\[ 2g(C(X, \varphi X)\varphi Y, Y) + C(R(X, \varphi Y)\varphi Y, X) + 2g(C(X, \varphi Y)\varphi X, Y) + g(C(Y, \varphi X)\varphi X, Y) = 2H_1 \{ 2g(X, Y)^2 + g(X, X)g(Y, Y) \}. \]

From (5.8), (5.9), (5.10), (5.11) and (5.13), we get

\[ 3g(C(X, \varphi Y)Y, \varphi X) + g(C(X, Y)X, Y) + 2g(hX, Y)^2 + 2g(X, Y)g(hX, Y) - g(X, X)g(hY, Y) - g(Y, Y)g(hX, X) - 4g(hX, X)g(hY, Y) - 4g(\varphi hX, Y)^2 + 4g(\varphi hX, X)g(\varphi hY, Y) = H_1 \{ 2g(X, Y)^2 + g(X, X)g(Y, Y) \}. \]
Replacing $Y$ by $\varphi Y$ in (5.14) and using (2.1), (2.2) and (2.3), we have

\begin{equation}
3g(C(X, Y)\varphi Y, \varphi X) - g(C(X, \varphi Y)X, \varphi Y) \\
+ 4g(\varphi hX, Y)^2 - 2g(X, \varphi Y)g(hX, \varphi Y) \\
+ g(X, X)g(hY, Y) - g(Y, Y)g(hX, X) \\
+ 4g(hX, X)g(hX, Y) - 4g(hX, Y)^2 - 4g(\varphi X, X)g(\varphi Y, Y) \\
= H_1\{2g(X, \varphi Y)^2 + g(X, X)g(Y, Y)\}.
\end{equation}

(5.15)

From (5.15), together with (5.9) and (5.10), we get

\begin{equation}
3g(C(X, Y)Y, X) + g(C(X, \varphi Y)\varphi X, Y) \\
+ 2g(X, Y)^2 + 4g(hX, Y)^2 + 6g(X, Y)g(hX, Y) - 2g(X, X)g(Y, Y) \\
- 3g(X, X)g(hY, Y) - 3g(X, X)g(hX, X) - 4g(hX, X)g(hY, Y) \\
+ 2g(X, \varphi Y)^2 - 4g(\varphi hX, Y)^2 + 4g(\varphi hX, Y)g(\varphi hY, Y) \\
= H_1\{2g(X, \varphi Y)^2 + g(X, X)g(Y, Y)\}.
\end{equation}

(5.16)

From (5.14) and (5.16), we have

\begin{equation}
4g(C(X, Y)Y, X) \\
= (H_1 + 3)\{g(X, X)g(Y, Y) - g(X, Y)^2\} + 3(H_1 - 1)g(X, \varphi Y)^2 \\
- 2\{2g(hX, Y)^2 + 4g(X, Y)g(hX, Y) - 2g(X, X)g(hY, Y) \\
- 2g(Y, Y)g(hX, X) \\
- 2g(hX, X)g(hY, Y) - 2g(\varphi hX, Y)^2 + 2g(\varphi hX, X)g(\varphi hY, Y)\}
\end{equation}

(5.17)

for all $X, Y \in D$. Substituting $X = X + Z$ in (5.17), we obtain

\begin{equation}
4g(C(X, Y)Y, Z) \\
= (H_1 + 3)\{g(X, Z)g(Y, Y) - g(X, Y)g(Y, Z)\} \\
+ 3(H_1 - 1)g(X, \varphi Y)g(Z, \varphi Y) - 4\{g(hX, Y)g(hY, Z) \\
+ g(X, Y)g(hY, Z) \\
+ g(Y, Z)g(hX, Y) - g(X, Z)g(hY, Y) - g(Y, Y)g(hX, Z) \\
- g(hX, Z)g(hY, Y) \\
- g(\varphi hX, Y)g(\varphi hZ, Y) + g(\varphi hX, Z)g(\varphi hY, Y)\}.
\end{equation}

(5.18)
If we substitute $Y = Y + W$ in (5.18) again and use (2.3), then we obtain

\begin{equation}
4\{g(C(X, Y)W, Z) + g(C(X, W)Y, Z)\}
= (H_1 + 3)\{2g(X, Z)g(Y, W) - g(X, Y)g(W, Z) - g(X, W)g(Y, Z)\}
+ 3(H_1 - 1)\{g(X, \varphi Y)g(Z, \varphi W) + g(X, \varphi W)g(Z, \varphi Y)\}
- 4\{g(hX, Y)g(hZ, W) + g(hX, W)g(hZ, Y) + g(X, Y)g(hZ, W)\}
+ g(X, W)g(hZ, Y) + g(Y, )g(hX, W) + g(Z, W)g(hX, Y)
- 2g(X, Z)g(hY, W) - 2g(Y, W)g(hX, Z) - 2g(hX, Z)g(hY, W)
- g(\varphi hX, Y)g(\varphi hZ, W) - g(\varphi hX, W)g(\varphi hZ, Y)
+ 2g(\varphi hX, Z)g(\varphi hY, W)\}
\end{equation}

and we have

\begin{equation}
4\{g(C(X, Z)W, Y) + g(C(X, W)Z, Y)\}
= (H_1 + 3)\{2g(X, Y)g(Z, W)
- g(X, Z)g(W, Y) - g(X, W)g(Z, Y)\}
+ 3(H_1 - 1)\{g(X, \varphi W)g(Y, \varphi Z)\}
+ g(X, Z)g(hY, W) + g(hX, W)g(hY, Z) + g(Y, Z)g(hX, W)
+ g(Y, W)g(hX, Z) - 2g(X, Y)g(hZ, W) - 2g(Z, W)g(hX, Y)
- 2g(hX, Y)g(hZ, W) - g(\varphi hX, Z)g(\varphi hY, W)
- g(\varphi hX, W)g(\varphi hY, Z) + 2g(\varphi hX, Y)g(\varphi hZ, W)\}.
\end{equation}

We subtract (5.20) from (5.19). Then by using the Bianchi-type identity for the curvature-like tensor field $C$ and (2.3), we get

\begin{equation}
4g(C(X, Y)Z, W)
= (H_1 + 3)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}
+ (H_1 - 1)\{g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W)\}
- 2g(\varphi X, Y)g(\varphi Z, W)\}
+ 4\{g(hY, Z)g(X, W) - g(hX, Z)g(Y, W) + g(Y, Z)g(hX, W)
- g(X, Z)g(hY, W) + g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W)
- g(\varphi hY, Z)g(\varphi hX, W) + g(\varphi hX, Z)g(\varphi hY, W)\},
\end{equation}
where \(X, Y, Z, W \in D(p)\). We now let \(X\) be an arbitrary vector field on \(M\). Then we may write
\[
X = X^T + \eta(X)\xi,
\]
where \(X^T\) denotes the horizontal part of \(X\). Then we have for all vector fields \(X, Y, Z, W\) in \(M\):
\[
(5.22)
g(C(X, Y)Z, W) = g(C(X^T, Y^T)Z^T, W^T) + \eta(X)g(C(\xi, Y^T)Z^T, W^T)
+ \eta(Y)g(C(X^T, \xi)Z^T, W^T) + \eta(Z)g(C(X^T, Y^T)\xi, W^T)
+ \eta(W)g(C(X^T, Y^T)Z^T, \xi) + \eta(X)\eta(Y)g(C(\xi, Y^T)\xi, W^T)
+ \eta(Y)\eta(W)g(C(X^T, \xi)Z^T, \xi).
\]
Furthermore, from (5.22), by using (5.2), (5.3), (5.21) and straightforward calculations, we obtain (5.7). \(\Box\)

From (5.7), by using (2.4) and (2.5), we find for the Ricci tensors:
\[
(5.23)
\rho(X, Y)(= \sum_i g(R(e_i, X)Y, e_i))
= \frac{1}{2} \left( (n + 1)H(p) - 4 \right) \left( g(X, Y) - \eta(X)\eta(Y) \right)
+ (2n - 1)g(hX, Y) + g(hX, hY) - \eta(X)\sum_i g(\varphi P(e_i, Y), e_i)
+ \eta(Y)\sum_i g(\varphi P(X, e_i), e_i) + g(\varphi P(\xi, X), Y)
+ \eta(X)\eta(Y)(2n + \text{tr } h^2)
\]
for all vectors \(X\) and \(Y\) in \(T_p M\), where \(\{e_i\} (i = 1, 2, \ldots, 2n + 1)\) is an arbitrary local orthonormal basis for \(T_p M\). Since the trace of \(h\) vanishes, from (5.23), we have for the scalar curvature:
\[
\tau(= \sum_i \rho(e_i, e_i)) = n \left( (n + 1)H - 4 \right) + 2n - \text{tr } h^2,
\]
where we have used \(\sum_i g(\varphi P(e_i, \xi), e_i) = \text{tr } h^2\).
6. Schur-type theorem for a contact strongly pseudo-convex CR-space form

Let $M$ be a pseudo-parallel contact strongly pseudo-convex CR-manifold. Then, since we already know that the pseudo-parallel $h$ is equivalent to the $\eta$-parallel $h$, it follows that

\begin{align*}
0 &= g((\nabla_X h) Y^T, Z^T) \\
&= g((\nabla_X - \eta(X)) h)(Y - \eta(Y))\xi, Z - \eta(Z)\xi) \\
&= g((\nabla_X h) Y, Z) - \eta(X) g((\nabla_\xi h) Y, Z) - \eta(Y) g((\nabla_X h) \xi, Z) \\
&\quad - \eta(Z) g((\nabla_X h) Y, \xi) + \eta(X) g((\nabla_\xi h) \xi, Z) \\
&\quad + \eta(Y) g((\nabla_X h) \xi, \xi) + \eta(Z) \eta(X) g((\nabla_\xi h) Y, \xi) \\
&\quad - \eta(X) \eta(Y) \eta(Z) g((\nabla_\xi h) \xi, \xi).
\end{align*}

From the above equation, by using (2.3), (2.4) and (2.5), we have

\begin{align}
(\nabla_X h) Y &= g((h - h^2)\varphi X, Y)\xi + \eta(Y)(h - h^2)\varphi X \\
&\quad + \eta(X)(\varphi Y - \varphi l Y - \varphi h^2 Y)
\end{align}

for all vector fields $X$ and $Y$. Before we prove the Schur-type theorem we prepare [6].

**Lemma 6.2.** Let $M$ be a pseudo-parallel contact strongly pseudo-convex CR-manifold. Then the eigenvalues of $h$ are constant.

Moreover, from (6.1), we have

\begin{align}
P(X, Y) &= -g((\varphi h^2 + h^2 \varphi) X, Y)\xi \\
&\quad + \eta(Y)(h \varphi - \varphi + \varphi l) X - \eta(X)(h \varphi - \varphi + \varphi l) Y,
\end{align}

\begin{align}
\varphi P(X, Y) &= \eta(Y)(h - \varphi^2 - l) X - \eta(X)(h - \varphi^2 - l) Y.
\end{align}

We prove a Schur-type theorem for this class. Namely,

**Theorem 6.3.** Let $(M^{2n+1}; \eta, L)$ $(n > 1)$ be a pseudo-parallel contact strongly pseudo-convex CR-manifold. If the pseudo-holomorphic sectional curvature (with respect to the Tanaka-Webster connection) at
any point of $M$ is independent of the choice of pseudo-holomorphic section, then it is constant $c$ on $M$ and the curvature tensor is given by

\[
g(R(X,Y)Z,W)
= \frac{1}{4} \left[ c \left( (g(Y,Z) - \eta(Y)\eta(Z))(g(X,W) - \eta(X)\eta(W)) 
- (g(X,Z) - \eta(X)\eta(Z))(g(Y,W) - \eta(Y)\eta(W)) \right) 
+ (c - 4) \left[ g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W) 
- 2g(\varphi X, Y)g(\varphi Z, W) \right] \right]
+ g(hY, Z)(g(X, W) - \eta(X)\eta(W))
- g(hX, Z)(g(Y, W) - \eta(Y)\eta(W))
+ g(hX, W)(g(Y, Z) - \eta(Y)\eta(Z)) - g(hY, W)(g(X, Z) - \eta(X)\eta(Z))
- g(\varphi h Y, Z)g(\varphi h X, W) + g(\varphi h X, Z)g(\varphi h Y, W)
- \eta(X)\eta(Z)g(lY, W) + \eta(X)\eta(W)g(lY, Z)
+ \eta(Y)\eta(Z)g(lX, W) - \eta(Y)\eta(W)g(lX, Z)
\]

for all vector fields $X, Y, Z, W$ in $M$.

\textbf{Proof.} Suppose that $M$ has pointwise constant pseudo-holomorphic sectional curvature $H$. Then, taking account of (6.1), (6.2) and (6.3), from (5.23) we obtain

\[
\rho(X, Y) = \frac{1}{2} \left( (n + 1)H - 4 \right) \left( g(X, Y) - \eta(X)\eta(Y) \right)
+ 2(n - 1)g(hX, Y) + g(h^2X, Y) + g((\varphi^2 + l)X, Y)
+ \eta(X)\eta(Y)(2n - \text{tr } h^2),
\]

\[
\tau = n \left( (n + 1)H - 4 \right) + 2n - \text{tr } h^2.
\]

From (6.1) and by using (2.4) and Lemma 6.2, we have

\[
(\nabla_X \rho)(Y, Z)
= \frac{1}{2} \left( (n + 1)(XH) \right) \left( g(Y, Z) - \eta(Y)\eta(Z) \right)
- \frac{1}{2} \left( (n + 1)H - 4 \right) \left( (\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) \right)
+ 2(n - 1)(g(\nabla_X h)Y, Z) + g((\nabla_X h^2)Y, Z)
\]
+ (\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) + g(\nabla_X l)Y, Z) \\
+ (2n - \text{tr } h^2)\left( (\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) \right),

which yields

\begin{align}
\sum_i (\nabla_{e_i \rho})(X, e_i) \\
= \frac{1}{2} (n+1)\{(XH) - (\xi H)\eta(X)\} + \sum_i g((\nabla_{e_i \rho})X, e_i) \\
= \frac{1}{2} (n+1)\{(XH) - (\xi H)\eta(X)\} - \sum_i g((\nabla_X R)(\xi, e_i)\xi, e_i) \\
- \sum_i g((\nabla_X R)(e_i, X)\xi, e_i) \\
= \frac{1}{2} (n+1)\{(XH) - (\xi H)\eta(X)\} + (\nabla_X \rho)(\xi, \xi) - (\nabla_{e_i \rho})(X, \xi) \\
= \frac{1}{2} (n+1)\{(XH) - (\xi H)\eta(X)\},
\end{align}

where we have used the 2nd Bianchi identity. By the well-known formula

\begin{align}
\nabla_X \tau = 2 \sum_i (\nabla_{e_i \rho})(X, e_i)
\end{align}

for any local orthonormal frame field \{e_i\} (i = 1, 2, \ldots, 2n + 1) and by using (6.6), (6.7) and Lemma 6.2, we have

\begin{align}
(n+1)\{(XH) - (\xi H)\eta(X)\} = n(n+1)XH.
\end{align}

This says that \(\xi H = 0\) and \((n-1)XH = 0\). Since \(n > 1\), we see that \(H\) is constant, say \(c\). By applying (6.1), (6.2) and (6.3) in Proposition 5.2, we obtain (6.4).

So, from the proofs of Proposition 5.2 and Theorem 6.3, we have

**Theorem 6.4.** Let \(M\) be a complete and simply connected pseudo-parallel contact CR-space. Then \(M\) is a contact strongly pseudo-convex CR-space form if and only if the curvature tensor \(R\) is given by (6.4).

We note that a contact strongly pseudo-convex CR-space form is a proper extension of a Sasakian space form \((h = 0)\). Since we already
know that a pseudo-parallel contact CR-space is a \((k, \mu)\)-space, from the results in [4], we see that a pseudo-parallel contact pseudo-convex CR-space form has a locally homogeneous contact Riemannian structure and is a locally \(\varphi\)-symmetric space in the strong sense. (We refer to [4] or [7] for the definition of a locally \(\varphi\)-symmetric space in the strong sense.)

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