Approximation by Interpolating Polynomials in Smirnov-Orlicz Class

Ramazan Akgün and Daniyal M. Israfilov

Abstract. Let Γ be a bounded rotation (BR) curve without cusps in the complex plane $\mathbb{C}$ and let $G := \text{int} \Gamma$. We prove that the rate of convergence of the interpolating polynomials based on the zeros of the Faber polynomials $F_n$ for $\overline{G}$ to the function of the reflexive Smirnov-Orlicz class $E_M(G)$ is equivalent to the best approximating polynomial rate in $E_M(G)$.

1. Introduction and main results

Let Γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. The curve Γ separates the plane into two domains $G := \text{int} \Gamma$ and $G^- := \text{ext} \Gamma$. We denote $D := \{z \in \mathbb{C} : |z| < 1\}$, $T := \partial D$ and $D^- := \text{ext} T$.

Let $w = \phi(z)$ be the conformal map of $G^-$ onto $D^-$ normalized by the conditions

$$\phi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\phi(z)}{z} > 0,$$

and let $\psi := \phi^{-1}$ be its inverse mapping.

When $|z|$ is sufficiently large, $\phi$ has the Laurent expansion

$$\phi(z) = dz + d_0 + \frac{d_1}{z} + \cdots$$

and hence we have

$$[\phi(z)]^n = d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k + \sum_{k<0} d_{n,k} z^k.$$

Received February 18, 2005.

2000 Mathematics Subject Classification: Primary 41A10, 41A50; Secondary 41A05, 41A25, 30C10, 30C15.

Key words and phrases: curves of bounded rotation, Faber polynomials, interpolating polynomials, Smirnov-Orlicz class, Orlicz space, Cauchy singular operator.
The polynomial
\[ F_n(z) := d_n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k \]
is called \(n^{th}\) Faber polynomial with respect to \(G\).

Note that for every natural number \(n\), \(F_n\) is a polynomial of degree \(n\). For further information about the Faber polynomials, it can be seen to monographs [5, Ch. I, Section 6], [14, Ch. II], [15].

By \(L^p(\Gamma), 1 \leq p < \infty\), we denote the set of all measurable complex valued functions \(f\) on \(\Gamma\) such that \(|f|^p\) is Lebesgue integrable with respect to arclength.

Let \(z = \phi_0(w)\) be the conformal map of \(D\) onto \(G\) normalized by the conditions
\[
\phi_0(0) = 0, \quad \phi_0'(0) > 0,
\]
and let \(\gamma_r\) be the image of the circle \(|w| = r, 0 < r < 1\), under the mapping \(\phi_0\).

We say that a function \(f\) analytic in \(G\), belongs to the Smirnov class \(E^p(G), 0 < p < \infty\), if for any \(r \in (0, 1)\) the inequality
\[
\int_{\gamma_r} |f(z)|^p |dz| \leq c < \infty
\]
holds.

Every function in \(E^p(G), 1 < p < \infty\), has nontangential boundary values almost everywhere (a. e.) on \(\Gamma\) and the boundary function belongs to \(L^p(\Gamma)\).

For \(p > 1\), \(E^p(G)\) is a Banach space with respect to the norm
\[
\|f\|_{E^p(G)} := \|f\|_{L^p(\Gamma)} := \left( \int_{\Gamma} |f(z)|^p |dz| \right)^{\frac{1}{p}}.
\]

A continuous and convex function \(M : [0, \infty) \to [0, \infty)\) which satisfies the conditions
\[
M(0) = 0, \quad M(x) > 0 \quad \text{for} \quad x > 0,
\]
\[
\lim_{x \to 0} \frac{M(x)}{x} = 0, \quad \lim_{x \to \infty} \frac{M(x)}{x} = \infty,
\]
is called an \(N\)-function.

The complementary \(N\)-function to \(M\) is defined by
\[
N(y) := \max_{x \geq 0} (xy - M(x)), \quad y \geq 0.
\]
We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition
\[ \int_{\Gamma} M[\alpha |f(z)||dz| < \infty \]
for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the norm
\[ \|f\|_{L_M(\Gamma)} := \sup_{g \in L_N(\Gamma)} \left\{ \int_{\Gamma} |f(z)g(z)||dz| : \rho(g;N) \leq 1 \right\}, \]
where $N$ is the complementary $N$-function to $M$ and
\[ \rho(g;N) := \int_{\Gamma} N[|g(z)||dz|. \]

This norm is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space.

We note that (see, for example, [12, p.51])
\[ L_M(\Gamma) \subset L^1(\Gamma). \]

An $N$-function $M$ satisfies the $\Delta_2$-condition if
\[ \limsup_{x \to \infty} \frac{M(2x)}{M(x)} < \infty. \]

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the $N$-function $M$ and its complementary function $N$ both satisfy the $\Delta_2$-condition [12, p.113].

Let $\Gamma_r$ be the image of the circle $\{w \in \mathbb{C} : |w| = r, 0 < r < 1\}$ under some conformal map of $\mathbb{D}$ onto $G$ and let $M$ be an $N$-function.

**Definition 1.** The class of functions which are analytic in $G$ and satisfy the condition
\[ \int_{\Gamma_r} M[|f(z)||dz| < \infty \]
uniformly in $r$ is called the Smirnov-Orlicz class and denoted by $E_M(G)$.

The Smirnov-Orlicz class is a generalization of the familiar Smirnov class $E^p(G)$. In particular, if $M(x) := x^p, 1 < p < \infty$, then the Smirnov-Orlicz class $E_M(G)$ determined by $M$ coincides with the Smirnov class $E^p(G)$. 
Since (see [10]) $E_M(G) \subset E^1(G)$, every function in the class $E_M(G)$ has the nontangential boundary values a.e. on $\Gamma$ and the boundary value function belongs to $L_M(\Gamma)$. Hence the $E_M(G)$ norm can be defined as:

\[(1) \quad \|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}, \quad f \in E_M(G).\]

Let $\gamma$ be an oriented rectifiable curve. For $z \in \gamma$, $\delta > 0$ we denote by $s_+(z, \delta)$ (respectively $s_-(z, \delta)$) the subarc of $\gamma$ in the positive (respectively negative) orientation of $\gamma$ with the $z$ starting point and arc length from $z$ to each point less than $\delta$.

If $\gamma$ is a smooth curve and

\[
\lim_{\delta \to 0} \left\{ \int_{s_-(z, \delta)} |d_\varsigma \arg (\varsigma - z)| + \int_{s_+(z, \delta)} |d_\varsigma \arg (\varsigma - z)| \right\} = 0
\]

holds uniformly for $z \in \gamma$, then it’s said [16] that $\gamma$ is of vanishing rotation ($VR$).

As follows from this definition, the $VR$ condition is stronger than smoothness. In [16] L. Zhong and L. Zhu also proved that there exists a smooth curve which is not of $VR$.

On the other hand, if the angle of inclination $\theta(s)$ of tangent to $\gamma$ as a function of the arclength $s$ along $\gamma$ satisfies the condition

\[
\int_0^\delta \frac{\omega(t)}{t} dt < \infty,
\]

where $\omega(t)$ is the modulus of continuity of $\theta(s)$, then [16] $\gamma$ is $VR$.

Approximation properties of the Faber and generalized Faber polynomials in the different functional spaces are well known (see for example: [1]-[2], [4]-[8] and also [5, Chapter 1, pp. 42–57], [15]). In this work we investigate the convergence property of the interpolating polynomials based on the zeros of the Faber polynomials in the reflexive Smirnov-Orlicz class. This problem isn’t new. It was studied by several authors. In their work [13] under the assumption $\Gamma \in C(2, \alpha), 0 < \alpha < 1$, X. C. Shen and L. Zhong obtain a series of interpolation nodes in $G$ and show that interpolating polynomials and the best approximating polynomial have the same order of convergence in $E^p(G)$, $1 < p < \infty$. In [17] considering $\Gamma \in C(1, \alpha)$ and choosing the interpolation nodes as the zeros of the Faber polynomials L. Y. Zhu obtain similar result.

In the above cited works $\Gamma$ does not admit corners. Many domains in the complex plane may have corners or cusps. When $\Gamma$ is a piecewise $VR$ curve without cusps, L. Zhong and L. Zhu [16] showed that the
interpolating polynomials based on the zeros of the Faber polynomials converge in the Smirnov class $E^p (G)$, $1 < p < \infty$.

In this work we investigate the convergence property of the interpolating polynomials based on the zeros of the Faber polynomials in the reflexive Smirnov-Orlicz class under the assumption that $\Gamma$ is a $BR$ curve without cusps.

**Definition 2.** [6] Let $\gamma$ be a rectifiable Jordan curve with length $L$ and let $z = z(t)$ be its parametric representation with arclength $t \in [0, L]$. If $\beta(t) := \arg z'(t)$ can be defined on $[0, L]$ to become a function of bounded variation, then $\gamma$ is called of bounded rotation ($\gamma \in BR$) and $\int_{\Gamma} |d\beta(t)|$ is called total rotation of $\gamma$.

If $\gamma \in BR$, then there are two half tangents at each point of $\gamma$. The class of bounded rotation curves is sufficiently wide. For example, a curve which is made up of finitely many convex arcs (corners are permitted), is bounded rotation [5, p.45]. It is easily seen that every $VR$ curve and also a piecewise $VR$ curve considered in [16] is $BR$ curve. Since a $BR$ curve may have cusps or corners, there exists a $BR$ curve which is not a $VR$ curve (for example, a rectangle in the plane).

In the case that all of the zeros of the $n^{th}$ Faber polynomial $F_n (z)$ are in $G$, we denote by $L_n (f, z)$ the $(n - 1)$th interpolating polynomial to $f(z) \in E_M (G)$ based on the zeros of the Faber polynomials $F_n$.

For $f \in E_M (G)$, we denote by

$$E_n^M (f, G) := \inf \left\{ \| f - p_n \|_{E_M (G)} : p_n \text{ is a polynomial of degree } \leq n \right\}$$

the minimal error of approximation of $f$ by polynomials of degree at most $n$.

The main results of this work are the following.

**Theorem.** Let $\Gamma$ be a $BR$ curve without cusps. Then for sufficiently large natural number $n$, the roots of the Faber polynomials are in $G$ and for every $f$ which belongs to reflexive Smirnov-Orlicz class $E_M (G)$,

$$\| f - L_n (f, \cdot) \|_{E_M (G)} \leq c \cdot E_{n-1}^M (f, G)$$

with a positive constant $c$ depending only on $\Gamma$ and $M$.

In particular, when $M(x) := x^p$, $1 < p < \infty$, we have the following result.
Corollary. Let $\Gamma$ be a $BR$ curve without cusps. Then for sufficiently large natural number $n$, the roots of the Faber polynomials are in $G$ and for every $f$ which belongs to Smirnov class $E^p(G)$, $1 < p < \infty$, 
$$
\|f - L_n(f, \cdot)\|_{E^p(G)} \leq c \cdot E_{n-1}(f, G)_p,
$$
where the constant $c > 0$ depend only on $p$ and $G$.

When $\Gamma$ is a piecewise $VR$ curve without cups, this corollary was proved in [16].

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of our interest.

2. Auxiliary results

Let $\Gamma$ be a $BR$ curve without cusps. Then (see, for example, Pommerenke [11])
$$
F_n(z) = \frac{1}{\pi} \int_{\Gamma} [\phi(\zeta)]^n d\zeta \arg(\zeta - z), \quad z \in \Gamma,
$$
where the jump of $\arg(\zeta - z)$ at $\zeta = z$ equals to the exterior angle $\alpha_z \pi$. Hence we have

(3) $F_n(z) - [\phi(z)]^n = \frac{1}{\pi} \int_{\Gamma \setminus \{z\}} [\phi(\zeta)]^n d\zeta \arg(\zeta - z) + (\alpha_z - 1) [\phi(z)]^n,$

and

(4) $0 \leq \max_{z \in \Gamma} |\alpha_z - 1| < 1.$

Lemma 1. [3] Let $\Gamma$ be a $BR$ curve. For any $\epsilon > 0$ and $\theta$, there exists a $\delta > 0$ such that

(5) $\int_{\theta-\delta}^{\theta+\delta} |d_t \arg \left( \psi(e^{i t}) - \psi(e^{i \theta}) \right)| < \epsilon,$

and for any $\eta \in (\theta - \delta, \theta + \delta)$ different from $\theta$

$$
\int_{\eta-\delta}^{\eta+\delta} |d_t \arg \left( \psi(e^{i t}) - \psi(e^{i \eta}) \right)| < \epsilon,
$$

and for any $\eta \in (\theta - \delta, \theta + \delta)$ different from $\theta$

$$
\int_{\eta-\delta}^{\eta+\delta} |d_t \arg \left( \psi(e^{i t}) - \psi(e^{i \eta}) \right)| < \epsilon.
$$
\[ < \epsilon + |\alpha z - 1| \pi, \]

where \(\alpha \pi\) is the external angle to \(\Gamma\) at \(z = \psi (e^{i\theta}).\)

**Lemma 2.** If \(\Gamma\) is a BR curve, then for every \(\epsilon > 0\), there exists a \(\delta > 0\) such that

\[
\int_{s_-(z, \delta) \setminus \{z\}} |d_\kappa \arg (\zeta - z)| + \int_{s_+(z, \delta) \setminus \{z\}} |d_\kappa \arg (\zeta - z)| < \epsilon, \quad z \in \Gamma.
\]

**Proof.** We take arbitrary \(z \in \Gamma\) and fix it. By the change of variable \(\varsigma = \psi (e^{it})\) we get

\[
\int_{s_-(z, \delta) \setminus \{z\}} |d_\kappa \arg (\zeta - z)| = \int_{\theta - \delta}^{\theta - \delta} \left| d_{\psi(e^{it})} \arg \left( \psi (e^{it}) - \psi (e^{i\theta}) \right) \right|
\]

and similarly

\[
\int_{s_+(z, \delta) \setminus \{z\}} |d_\kappa \arg (\zeta - z)| = \int_{\theta + \delta}^{\theta + \delta} \left| d_{\psi(e^{it})} \arg \left( \psi (e^{it}) - \psi (e^{i\theta}) \right) \right|.
\]

For any \(\epsilon > 0\), using (5) we have (6).

**Lemma 3.** For any \(\epsilon > 0\), \(\delta > 0\), there exists a natural number \(k\) such that for \(\theta \in [0, 2\pi]\) there exists a trigonometric polynomial \(T_\theta (t)\).
of \(t\) with degree at most \(k\) satisfying

\[
\int_{0}^{2\pi} |v(t, \theta; \delta) - T_\theta(t)| \, dt < \epsilon.
\]

**Lemma 4.** Let \(\Gamma\) be a BR curve without cusps. Then for arbitrary \(\epsilon > 0\), there exists a positive integer \(n_0\) such that

\[
|F_n(z) - [\phi(z)]^n| < |\alpha_z - 1| + \epsilon, \quad z \in \Gamma
\]

holds for \(n > n_0\).

**Proof.** For any \(\epsilon > 0\), there exists a \(\delta > 0\) such that (6) holds. Let \(s(z) := \{s_-(z, \delta) \cup s_+(z, \delta)\}, \quad z \in \Gamma\). Hence by Lemma 3, for given \(\epsilon\) and \(\delta\) there is a positive integer \(n_0\) such that (7) is valid. By (3) for \(z = \psi(e^{i\theta})\) we have

\[
F_n(z) - [\phi(z)]^n = \frac{1}{\pi} \left\{ \int_{s_+(z, \delta) \setminus \{z\}} + \int_{s_-(z, \delta) \setminus \{z\}} \right\} [\phi(\varsigma)]^n d_\varsigma \arg(\varsigma - z)
\]

\[
+ \frac{1}{\pi} \int_{\Gamma \setminus s(z)} [\phi(\varsigma)]^n d_\varsigma \arg(\varsigma - z) + (\alpha_z - 1) e^{in\theta}
\]

\[
= \frac{1}{\pi} \left\{ \int_{s_+(z, \delta) \setminus \{z\}} + \int_{s_-(z, \delta) \setminus \{z\}} \right\} [\phi(\varsigma)]^n d_\varsigma \arg(\varsigma - z)
\]

\[
+ \frac{1}{\pi} \int e^{int} dt \arg \left( \psi(e^{it}) - \psi(e^{i\theta}) \right) + (\alpha_z - 1) e^{in\theta}
\]

\[
= \frac{1}{\pi} \left\{ \int_{s_+(z, \delta) \setminus \{z\}} + \int_{s_-(z, \delta) \setminus \{z\}} \right\} [\phi(\varsigma)]^n d_\varsigma \arg(\varsigma - z)
\]

\[
+ \frac{1}{\pi} \int_{0}^{2\pi} e^{int} \text{Im}[v(t, \theta; \delta)] \, dt + (\alpha_z - 1) e^{in\theta}.
\]

Since \(e^{int}\) is orthogonal to \(T_\theta(t)\) as \(n > n_0\), we get

\[
F_n(z) - [\phi(z)]^n = \frac{1}{\pi} \left\{ \int_{s_+(z, \delta) \setminus \{z\}} + \int_{s_-(z, \delta) \setminus \{z\}} \right\} [\phi(\varsigma)]^n d_\varsigma \arg(\varsigma - z)
\]
\[ \int_0^{2\pi} e^{it} \text{Im} \left[ iv(t, \theta; \delta) - iT_\theta(t) \right] dt + (\alpha_z - 1) e^{in\theta}, \]

and hence
\[
|F_n(z) - [\phi(z)]^n| \leq \frac{1}{\pi} \left\{ \int_{s_+(z, \delta) \setminus \{z\}} + \int_{s_-(z, \delta) \setminus \{z\}} \right\} |d_\varsigma \arg (\varsigma - z)| + |\alpha_z - 1| \]
\[ + \frac{1}{\pi} \int_0^{2\pi} |v(t, \theta; \delta) - T_\theta(t)| dt. \]

If \( z \) is not a corner of \( \Gamma \), then \( |\alpha_z - 1| = 0 \) and by (6) and (7) our assumption follows. If \( z \) is a corner of \( \Gamma \), then \( 0 \leq |\alpha_z - 1| < 1 \) and hence by (6) and (7) we have (8) again.

For \( z \in \Gamma \) and \( \epsilon > 0 \) let \( \Gamma(z, \epsilon) \) denote the portion of \( \Gamma \) which is inside the open disk of radius \( \epsilon \) centered at \( z \), i.e., \( \Gamma(z, \epsilon) := \{ t \in \Gamma : |t - z| < \epsilon \} \). Further, let \( |\Gamma(z, \epsilon)| \) denote the length of \( \Gamma(z, \epsilon) \). A rectifiable Jordan curve \( \Gamma \) is called a Carleson curve if
\[
\sup_{\epsilon > 0} \sup_{z \in \Gamma} \frac{1}{\epsilon} |\Gamma(z, \epsilon)| < \infty. \]

We consider the Cauchy-type integral
\[
(\mathcal{H}f)(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G
\]
and Cauchy’s singular integral of \( f \in L^1(\Gamma) \) defined as
\[
S_{\Gamma}f(z) := \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(z, \epsilon)} \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in \Gamma.
\]

The linear operator \( S_{\Gamma} : f \to S_{\Gamma}f \) is called the Cauchy singular operator.

**Lemma 5.** [9] Let \( \Gamma \) be a rectifiable Jordan curve and let \( L_M(\Gamma) \) be a reflexive Orlicz space on \( \Gamma \). Then the singular operator \( S_{\Gamma} \) is bounded on \( L_M(\Gamma) \), i.e.,
\[
\|S_{\Gamma}f\|_{L_M(\Gamma)} \leq c_1 \|f\|_{L_M(\Gamma)}, \quad f \in L_M(\Gamma),
\]
for some constant \( c_1 > 0 \) if and only if \( \Gamma \) is a Carleson curve.
3. Proof of Theorem

We proof firstly that for sufficiently large \( n \), all zeros of the Faber polynomials \( F_n \) are in \( G \). Let

\[
\kappa := \max_{z \in \Gamma} |\alpha_z - 1|, \quad z \in \Gamma.
\]

Then by (4) we have \( 0 \leq \kappa < 1 \). Setting \( \epsilon := \frac{1 - \kappa}{2} \) in Lemma 4, for sufficiently large \( n \) we get

\[
|F_n(z) - [\phi(z)]^n| < \frac{1 + \kappa}{2}, \quad z \in \Gamma.
\]

Since \( F_n(z) - [\phi(z)]^n \) is analytic on \( C\overline{G} := \mathbb{C}\setminus \overline{G} \), by the maximum principle we have

\[
|F_n(z) - [\phi(z)]^n| < \frac{1 + \kappa}{2}, \quad z \in C\overline{G},
\]

and therefore

\[
|F_n(z)| \geq |\phi(z)|^n - \frac{1 + \kappa}{2} \geq \frac{1 - \kappa}{2} > 0, \quad z \in C\overline{G}.
\]

This gives to us the first part of the theorem.

Let \( P_{n-1}(z) \) be the \((n-1)th\) best approximating polynomial to \( f \) in \( E_M(G) \). Then

\[
\|f - L_n(f, \cdot)\|_{E_M(G)} = \|f - P_{n-1} - L_n(f - P_{n-1}, \cdot)\|_{E_M(G)} \leq (1 + \|L_n\|) \|f - P_{n-1}\|_{E_M(G)}
\]

because \( L_n(f, z) \) is a linear interpolating polynomial operator. Now we only need to show that, for large values of \( n \), \( L_n(f, z) \) is uniformly bounded in the reflexive Smirnov-Orlicz class \( E_M(G) \).

Choosing the interpolation nodes as the zeros of the Faber polynomials we have for \( z' \in G \)

\[
f(z') - L_n(f, z') = \frac{F_n(z')}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{F_n(\varsigma)(\varsigma - z')} d\varsigma
\]

\[
= F_n(z') \left( \mathcal{H} \left[ \frac{f}{F_n} \right] \right)(z').
\]
Taking the limit $z' \to z \in \Gamma$ along all nontangential paths inside of $\Gamma$ we get by (1)

$$
\|f - L_n(f, \cdot)\|_{E M(G)} = \left\| F_n(z) \cdot \left( \mathcal{H} \left[ \frac{f}{F_n} \right] \right)(z) \right\|_{L M(\Gamma)} \\
\leq \left\{ \max_{z \in \Gamma} |F_n(z)| \right\} \cdot \left\| \mathcal{H} \left[ \frac{f}{F_n} \right] \right\|_{L M(\Gamma)},
$$

and later by Lemma 5,

$$
\|f - L_n(f, \cdot)\|_{E M(G)} \leq c_1 \cdot \left\{ \max_{z \in \Gamma} |F_n(z)| \right\} \cdot \left\| \frac{f}{F_n} \right\|_{L M(\Gamma)} \\
\leq c_1 \cdot \left\{ \max_{z, \xi \in \Gamma} \left| \frac{F_n(z)}{F_n(\xi)} \right| \right\} \cdot \|f\|_{L M(\Gamma)}.
$$

From (9),

$$
\frac{1 - \kappa}{2} < |F_n(z)| < \frac{3 + \kappa}{2}, \quad z \in \Gamma
$$

and hence

$$
\|f - L_n(f, \cdot)\|_{E M(G)} \leq c_1 \cdot \frac{3 + \kappa}{1 - \kappa} \cdot \|f\|_{L M(\Gamma)}.
$$

Since

$$
\|L_n(f, \cdot)\|_{E M(G)} \leq \|f\|_{E M(G)} + \|f - L_n(f, \cdot)\|_{E M(G)} \\
\leq \left( 1 + c_1 \cdot \frac{3 + \kappa}{1 - \kappa} \right) \cdot \|f\|_{L M(\Gamma)},
$$

by choosing $c_2 := 1 + c_1 \cdot \frac{3 + \kappa}{1 - \kappa}$ we obtain that $\|L_n\| \leq c_2$ and therefore we conclude by (2)

$$
\|f - L_n(f, \cdot)\|_{E M(G)} \leq (1 + c_2) \|f - P_{n-1}\|_{E M(G)} \\
= c_3 \cdot E^M_{n-1}(f, G),
$$

where $c_3 := 1 + c_2$.

\[\square\]

References

424 Ramazan Akgun and Daniyal M. Israfilov


Department of Mathematics
Faculty of Art-Science
Balikesir University
10100 Balikesir, Turkey
E-mail: mdaniyal@balikesir.edu.tr
rakgun@balikesir.edu.tr