TOPOLOGICAL ENTROPY OF A SEQUENCE OF MONOTONE MAPS ON CIRCLES

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Abstract. In this paper, we prove that the topological entropy of a sequence of equi-continuous monotone maps $f_1, \ldots, f\infty = \{f_i\}_{i=1}^{\infty}$ on circles is $h(f_1, \infty) = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|$. As applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a diffeomorphism $f$ on a smooth 2-dimensional closed manifold and its extension on the unit tangent bundle have the same entropy.

1. Introduction

The concept of topological entropy was originally introduced by Adler, Konheim, and Mcandrew [1] as an invariant of topological conjugacy and a numerical measure for the complexity of a dynamical system. Later, Bowen [2] and Dinaburg [3] gave an equivalent definition when the space under consideration is metrizable. We can see [12] for the definition and main properties of it. With the development of the study of nonautonomous dynamical systems, recently, Kolyada and Snoha [7] introduced and studied the notion of topological entropy for a sequence of endomorphisms of a compact topological space. For other recent results about entropy one can see [4], [9], [11], etc.

The systems on circle play an important role in the study of one-dimensional dynamical systems. In [5] and [12] the authors studied the entropies of homeomorphism and monotone continuous map on circle respectively. Our purpose is to study the topological entropy of a sequence...
of monotone maps on circles. In section 2, by estimating the cardinal of the spanning set and the separated set, we prove that the topological entropy of a sequence of equi-continuous monotone maps \( f_{1,\infty} = \{ f_i \}_{i=1}^{\infty} \) is \( h(f_{1,\infty}) = \limsup_{n \to \infty} \frac{1}{n} \log \prod_{i=1}^{n} | \deg f_i |. \) In section 3, as applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a \( C^1 \) diffeomorphism \( f \) on a smooth 2-dimensional closed manifold \( M \) and its extension \( D^2 f \) on the unit tangent bundle \( SM \) have the same entropy, i.e., \( h(f) = h(D^2 f) \).

Let \((X,d)\) be a compact metric space and \( \{ f_i \}_{i=1}^{\infty} \) a sequence of continuous maps on \( X \). The identity map on \( X \) will be denoted by \( Id \). Let \( N \) be the set of all positive integers. For any \( i \in N \), let \( f_i^0 = Id \) and for any \( i, n \in N \), let

\[
f_i^n = f_{i+(n-1)} \circ \cdots \circ f_{i+1} \circ f_i, \quad f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+(n-1)}^{-1}.
\]

(\( f^{-1} \) will be applied to sets, we don’t assume that the maps \( f_i \) are invertible). Denote by \( f_{1,\infty} \) the sequence \( \{ f_i \}_{i=1}^{\infty} \) and the dynamical system \((X, \{ f_i \}_{i=1}^{\infty})\). Finally, denote by \( f_{1,\infty}^{[n]} \) the sequence of maps \( \{ f_i^{[n]} = f_{(i-1)n+1} \}_{i=1}^{\infty} \).

Let \( \{ f_i \}_{i=1}^{\infty} \) be a sequence of continuous maps of compact metric space \((X,d)\). For any \( n \in N \), define a new metric \( d_n \) on \( X \) by

\[
d_n(x, y) := \max_{0 \leq s \leq n-1} d(f_1^s(x), f_1^s(y)).
\]

For any \( \varepsilon > 0 \), a subset \( E \subset X \) is said to be an \((n, f_{1,\infty}, \varepsilon)\) spanning set of \( X \), if for any \( x \in X \), there exists \( y \in E \) such that \( d_n(x, y) \leq \varepsilon \). Let \( r(n, f_{1,\infty}, \varepsilon) \) denote the smallest cardinality of any \((n, f_{1,\infty}, \varepsilon)\)-spanning set of \( X \). A subset \( F \subset X \) is said to be an \((n, f_{1,\infty}, \varepsilon)\)-separated set of \( X \), if \( x, y \in F, x \neq y \), implies \( d_n(x, y) > \varepsilon \). Let \( s(n, f_{1,\infty}, \varepsilon) \) denote the largest cardinality of any \((n, f_{1,\infty}, \varepsilon)\)-separated set of \( X \). It’s easy to prove that (similar to the proof for the autonomous system in [12])

\[
r(n, f_{1,\infty}, \varepsilon) \leq s(n, f_{1,\infty}, \varepsilon) \leq r(n, f_{1,\infty}, \varepsilon/2).
\]

**Definition 1.1.** Let \( f_{1,\infty} = \{ f_i \}_{i=1}^{\infty} \) be a sequence of continuous maps of compact metric space \((X,d)\), then the topological entropy of \( f_{1,\infty} \) is defined by

\[
h(f_{1,\infty}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, f_{1,\infty}, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, f_{1,\infty}, \varepsilon).
\]
Furthermore, we can see the equivalent definition using open covers in [7].

Let $S^1$ be a circle with the “geodesic” metric, in which $S^1$ has length 1 and the distance between two points is the length of the shortest path joining them. Let $f : S^1 \to S^1$ be a continuous surjective map and $F : \mathbb{R}^1 \to \mathbb{R}^1$ a lift of $f$, we say $f$ is monotone if $F$ is monotone. Denote by $\deg f$ the degree of $f$ (see [13]).

2. The main result

The main result of this paper is:

**Theorem 2.1.** Let $f_1, \infty = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of $S^1$. Then

$$h(f_1, \infty) = \lim_{n \to \infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|.$$ 

We will prove this theorem using the idea in [6]. Let $f : S^1 \to S^1$ be a continuous monotone map, $|\deg f| = k$. Then for any $x \in S^1$, $f^{-1}(x)$ is a set consist of $k$ points, denote $f^{-1}(x) = \{x_1, x_2, \ldots, x_n\}$. Let $\alpha_{f,1} = (x_1, x_2), \ldots, \alpha_{f,k-1} = (x_{k-1}, x_k), \alpha_{f,k} = (x_k, x_1)$. Then we get a finite partition $\xi_f = \{\alpha_{f,1}, \alpha_{f,2}, \ldots, \alpha_{f,k}\}$ of $S^1$, where $f(\alpha_{f,i}) = S^1$ and $\alpha_{f,i} \cap \alpha_{f,j} = \emptyset$ for $1 \leq i \neq j \leq k$.

**Lemma 2.2.** Let $f_1, \infty = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of $S^1$. Then there exists a constant $a > 0$, such that for every $f_i(i \geq 1)$ and any partition $\xi_f_i = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \ldots, \alpha_{f_i,k_i}\}$ of $S^1$ defined as above, we have

$$\text{diam} \alpha_{f_i,j} \geq a, \ 1 \leq j \leq k_i,$$

where $k_i = |\deg f_i|$. 

**Proof.** Since $\{f_i\}_{i=1}^{\infty}$ is equi-continuous, then for $\varepsilon = \frac{1}{2}$, there exists a constant $a > 0$ such that

$$d(x, y) < a \implies d(f_i(x), f_i(y)) < \varepsilon, \ \forall i \in \mathbb{N}, \ x, y \in S^1.$$ 

Note that for every $f_i(i \geq 1)$ and any partition $\xi_f_i = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \ldots, \alpha_{f_i,k_i}\}$ of $S^1$ defined as above, $f_i(\alpha_{f_i,j}) = S^1(1 \leq j \leq k_i)$ and $\text{diam} S^1 = 1$, we have $\text{diam} \alpha_{f_i,j} \geq a, \ 1 \leq j \leq k_i$.  

\[\Box\]
Lemma 2.3. Let \( f_{1,\infty} = \{f_i\}_{i=1}^{\infty} \) be a sequence of equi-continuous monotone maps of \( S^1 \), \( \{\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \ldots, \alpha_{f_i,k_i}\}\}_{i=1}^{\infty} \) be any sequence of partitions of \( S^1 \) defined as above. Then for the new sequence of partitions of \( S^1 \)
\[
\{\xi^n_{f_1} = \{f_1^{-(n-1)}(\alpha_{f_n,j}) \mid \alpha_{f_n,j} \in \xi_f, \ 1 \leq j \leq k_n\}\}_{n=1}^{\infty},
\]
we have
\[
h(f_{1,\infty}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi^n_{f_1} \leq h(f_{1,\infty}) + \log 2.
\]

Proof. For any \( x \in S^1 \), let \( B_d(x, \varepsilon) = \{y \in S^1 \mid d(x, y) < \varepsilon\} \). By the definition of \( \xi^n_{f_1} \), for any given \( n \in \mathbb{N} \), there are \( n-1 \) new partitions of \( S^1 \): \( \xi^n_{f_1} = \{\alpha'_{f_i,1}, \alpha'_{f_i,2}, \ldots, \alpha'_{f_i,k_i}\}, \ 1 \leq i \leq n-1 \), such that
\[
\xi^n_{f_1} = \left\{\bigcap_{i=1}^{n} f_1^{-(i-1)}(\alpha'_{f_i,j}) \mid \alpha'_{f_i,j} \in \xi^n_{f_1}, \ 1 \leq j \leq k_i\right\}.
\]
by Lemma 2.2, diam \( \alpha'_{f_i,j} \geq a \) for any \( 1 \leq i \leq n-1, \ 1 \leq j \leq k_i \).

Let \( 0 < \varepsilon < \frac{a}{2} \) (the meaning of \( a \) is in Lemma 2.2), and \( E \) be an \((n, f_{1,\infty}, \varepsilon)\)-spanning set of minimal cardinality of \( S^1 \). It can be seen that for any \( x \in E \) and \( 0 \leq i \leq n-1 \), the \( \varepsilon \)-neighborhood \( B_d(f^n_{1}(x), \varepsilon) \) of \( f^n_{1}(x) \) intersects at most 2 elements of \( \xi_f^n \). So \( \overline{B_d(x, \varepsilon)} \) intersects at most \( 2^n \) elements of \( \xi^n_{f_1} \). By the definition of spanning set, \( \bigcup_{x \in E} \overline{B_d(x, \varepsilon)} = S^1 \), then \( \text{card } \xi^n_{f_1} \leq 2^n \text{card } E \). Therefore,
\[
(1) \qquad \limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi^n_{f_1} \leq h(f_{1,\infty}) + \log 2.
\]

Now we take an arbitrary \( 0 < \varepsilon < \frac{1}{2} \) and choose an \((n, f_{1,\infty}, \varepsilon)\)-separated set \( F \) of maximal cardinality of \( S^1 \). By the definition of separated set, for any \( \alpha \in \xi^n_{f_1} \) and any two adjacent points \( x, y \in \alpha \cap F \), there exists \( j \) with \( 0 \leq j \leq n-1 \) such that \( d(f^n_1(x), f^n_1(y)) > \varepsilon \). Since \( f^n_1 \) is monotone on \( \alpha \), then \( f^n_1(x) \) and \( f^n_1(y) \) are also two adjacent points. Hence, for each \( 0 \leq j \leq n-1 \), there are at most \( M = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 \) pairs adjacent points which are more than \( \varepsilon \) apart in \( f^n_1(\alpha \cap F) \). We claim that there are at most \( nM + 1 \) points in \( \alpha \cap F \). In fact, if there are \( nM + 2 \) points in \( \alpha \cap F \), then there are at least \( nM + 1 \) pairs adjacent points. As mentioned above, for any two adjacent points \( x, y \in \alpha \cap F \), there exists \( j \) with \( 0 \leq j \leq n-1 \) such that \( d(f^n_1(x), f^n_1(y)) > \varepsilon \). This implies that
there exists at least one \(0 \leq s \leq n - 1\) such that \(d(f_s^*(x), f_s^*(y)) > \varepsilon\) for at least \(M+1\) pairs adjacent points. This contradicts with the definition of \(M\).

In such a way, we have \(\text{card}(\alpha \cap F) \leq nM + 1\). Hence, \(\text{card}F \leq (nM + 1)\) \(\text{card} \xi_{f_1^n}\). Furthermore, we have

\[
\frac{1}{n} \log s(n, f_1^\infty, \varepsilon) \leq \frac{1}{n} \log \text{card} \xi_{f_1^n} + \frac{1}{n} \log(nM + 1).
\]

Letting \(n \to \infty\), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log s(n, f_1^\infty, \varepsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}.
\]

Taking limits as \(\varepsilon\) goes to 0 establish the following inequality:

(2) \(h(f_1^\infty) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}\).

Then (1) and (2) yields

\[
h(f_1^\infty) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n} \leq h(f_1^\infty) + \log 2.
\]

Lemma 2.4. Let \(m\) be any given positive integer. Then for the sequence of maps \(g_1^\infty\) defined as above and the relevant sequence of partition \(\{\xi_{g_1^n}\}_{n=1}^\infty\), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{card} \xi_{g_1^n} = m \limsup_{n \to \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}.
\]

Proof. By Lemma 2.2, for any \(i \in \mathbb{N}\), \(\text{card} \xi_{f_i} \leq N := \lceil \frac{1}{a} \rceil + 1\). Then, for any positive integer \(n = lm + j, 0 \leq j \leq m - 1\), we have

\[
\text{card} \xi_{f_1^{lm}} \leq \text{card} \xi_{f_1^n} \leq N^m \text{card} \xi_{f_1^{lm}}.
\]
So
\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi f^n = \limsup_{l \to \infty} \frac{1}{lm} \log \text{card } \xi f^l
\]
\[
= \frac{1}{m} \limsup_{l \to \infty} \frac{1}{l} \log \text{card } \xi g^l.
\]
Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi g^n = m \limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi f^n.
\]

Lemma 2.5. ([7]). If \( f_1, \infty = \{ f_i \}_{i=1}^{\infty} \) is a sequence of equi-continuous maps on a compact metric space, then for any \( m \in \mathbb{N} \), we have
\[
h(f_1, \infty) = m \cdot h(f_1, \infty).
\]

Proof of Theorem 2.1. For any \( \varepsilon > 0 \), take \( m \in \mathbb{N} \) such that \( \frac{\log 2}{m} < \varepsilon \).

Since \( f_1, \infty \) is a sequence of monotone equi-continuous maps on \( S^1 \), as mentioned above, it is easy to see that \( g_1, \infty = f_1, \infty \) is also a sequence of equi-continuous monotone maps on \( S^1 \). By Lemma 2.3, we get
\[
h(f_1, \infty) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi g^n \leq h(f_1, \infty) + \log 2.
\]
Using Lemmas 2.4 and 2.5, and notice the way \( m \) is taken, we get
\[
h(f_1, \infty) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{card } \xi f^n \leq h(f_1, \infty) + \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, noting that \( \text{card } \xi f^n = \prod_{i=1}^{n} | \deg f_i | \), we get immediately
\[
h(f_1, \infty) = \limsup_{n \to \infty} \frac{1}{n} \log \prod_{i=1}^{n} | \deg f_i |.
\]

Corollary 2.6. If \( f_1, \infty = \{ f_i \}_{i=1}^{\infty} \) is a sequence of equi-continuous monotone maps of \( S^1 \), and the absolute values of the degrees of the mappings are the same, denote it by \( k \), then \( h(f_1, \infty) = \log k \).

In particular, (Theorem in [5]) If \( f : S^1 \to S^1 \) is a continuous monotone map, then \( h(f) = \log | \deg f | \).
Corollary 2.7. If every element of the sequence \( \{f_i\}_{i=1}^\infty \) on \( S^1 \) is chosen from a set consisted of finite continuous monotone maps, then
\[
h(f_{1,\infty}) = \lim_{n \to \infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|.
\]

Proof. It is only to note that the continuous map on compact space is uniformly continuous, and finite uniformly continuous maps are equi-continuous. \( \Box \)

Corollary 2.8. Let \( f \) be an expansive map of \( S^1 \), i.e., \( f \) be of \( C^1 \), and for every lift \( F : R^1 \to R^1 \) of it, \( |F'(x)| > 1, \forall x \in R \). If \( \{f_i\}_{i=1}^\infty \) are generated by sufficiently small \( C^1 \)-perturbation of \( f \), then \( h(f_{1,\infty}) = \log |\deg f| \).

Proof. Note that the expansive map of \( S^1 \) is strictly monotone and structurally stable ([13]). Also note the degree of the mapping is an invariant of topological conjugacy. Therefore, if every element of \( \{f_i\}_{i=1}^\infty \) is chosen from the sufficiently small \( C^1 \)-neighborhood of \( f \), then \( \{f_i\}_{i=1}^\infty \) must be a sequence of equi-continuous monotone mappings, and \( \deg f_i = \deg f, \forall i \in N \). From Lemma 2.6, we have \( h(f_{1,\infty}) = \log |\deg f| \). \( \Box \)

3. Applications

Proposition 3.1. ([2]). Let \( X, Y \) be compact metric spaces, \( F : X \to X, f : Y \to Y \) be continuous maps, \( \pi : X \to Y \) be a surjective continuous map, and satisfy \( \pi \circ F = f \circ \pi \), that is, \( f \) and \( F \) are topological semi-conjugate and \( f \) is the factor of \( F \). Then
\[
h(f) \leq h(F) \leq h(f) + \sup_{y \in Y} h(F, \pi^{-1}(y)).
\]

Let \( X, Y \) be compact metric spaces. A continuous map \( F : X \times Y \to X \times Y \) is called a skew-product, if there exist a continuous map \( f \) of \( X \) and a set of continuous maps \( \{g_x \mid x \in X\} \) of \( Y \) which depend on \( x \) continuously, such that \( F(x, y) = (f(x), g_x(y)), \forall x \in X, y \in Y \). By Proposition 3.1, we can get that: for the skew-product \( F : X \times Y \to X \times Y \), we have
\[
h(f) \leq h(F) \leq h(f) + \sup_{x \in X} h(F, \pi^{-1}(x)),
\]
where \( \pi : X \times Y \to X, (x, y) \mapsto x \) is the natural projection.
Proposition 3.2. ([10]). If \( f \) is a piecewise monotone continuous self-map of \( I \), then
\[
h(f) = \lim_{n \to \infty} \frac{1}{n} \log C_n,
\]
where \( C_n \) denotes the number of pieces of monotonicity of \( f^n \).

Corollary 3.3. (1) Let \( F(x, y) = (f(x), g_x(y)) \) be a skew product of annular \( I \times S^1 \). If \( f \) is piecewise monotone, \( \{g_x \mid x \in I\} \) is a sequence of equi-continuous monotone maps, then
\[
\lim_{n \to \infty} \frac{1}{n} \log C_n \leq h(F) \leq \lim_{n \to \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|.
\]

(2) Let \( F(x, y) = (f(x), g_x(y)) \) be a skew product of torus \( S^1 \times S^1 \). If \( \{f\} \cup \{g_x \mid x \in S^1\} \) is a sequence of equi-continuous monotone maps, then
\[
\log |\deg f| \leq h(F) \leq \log |\deg f| + \sup_{x \in S^1} \limsup_{n \to \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|.
\]

Proof. Firstly, note that for any skew product \( F : X \times Y \to X \times Y \), and any \( x \in X \), we have
\[
h(F, \pi^{-1}(x)) = h(\{g_{f^i-1(x)}\}_{i=1}^{\infty}).
\]
From Propositions 3.1, 3.2 and Theorem 2.1, we can get (1). From Proposition 3.1, Corollary 2.6 and Theorem 2.1, we can get (2). \( \square \)

Let \((M, \rho)\) be a smooth 2-dimensional closed manifold (i.e., \( M \) is compact and without boundary), \( TM \) be the tangent bundle of \( M \). We denote \(|\cdot|\), \( \|\cdot\| \) and \( d(\cdot, \cdot) \), respectively, the norm on \( TM \), the operator norm and the metric on \( M \) induced by the Riemannian metric. Denote by \( SM = \bigcup_{x \in M} S_xM \) the unit tangent bundle of \( M \), where \( S_xM = \{u \in T_xM \mid |u| = 1\} \). Note that \( SM \) is a compact metric space and its metric \( d \) can be derived from \( \rho \). That is, the restriction of \( d \) on \( S_xM \) is consistent with the restriction of the metric of \( T_xM \), which derived from the inner product \( \rho_x \), on \( S_xM \).

Let \( f : M \to M \) be a \( C^1 \) diffeomorphism, \( Df : TM \to TM \) be the tangent map of \( f \). Let \( D^2f : SM \to SM \), \( u \mapsto \frac{Df(x)_{u}}{|Df(x)_{u}|} \), \( u \in T_xM \). Then \((SM, D^2f)\) is a compact topological system, we also call it the extension...
of $f$ on the unit tangent bundle. One can see [8] for some connections of the dynamics between $f$ and its extension $D^2 f$.

**Proposition 3.4.** Let $f : M \rightarrow M$ be a $C^1$ diffeomorphism on a smooth two-dimensional closed Riemannian manifold $M$, and $D^2 f$ be its extension on the unit tangent bundle $SM$. Then

$$h(f) = h(D^2 f).$$

**Proof.** Let $\pi : SM \rightarrow M$, $u \mapsto x$, $u \in S_x M$ be the natural projection. It is easy to verify that $\pi \circ D^2 f = f \circ \pi$. By Proposition 3.1, we have

$$h(f) \leq h(D^2 f) \leq h(f) + \sup_{x \in M} h(D^2 f, \pi^{-1}(x)).$$

Since $M$ is compact, $f$ is a $C^1$ diffeomorphism, then we can take

$$M = \max_{x \in M} \| Df(x) \|, \quad m = \min_{x \in M} \| Df(x) \|.$$

For any $x \in M$, $u, v \in S_x M$, we have

$$d(D^2 f(x)u, D^2 f(x)v)$$

$$= |D^2 f(x)u - D^2 f(x)v|$$

$$= \left| \frac{Df(x)u}{|Df(x)u|} - \frac{Df(x)v}{|Df(x)v|} \right|$$

$$= \frac{1}{|Df(x)u| \cdot |Df(x)v|} \left| |Df(x)v| \cdot Df(x)u - |Df(x)u| \cdot Df(x)v \right|$$

$$\leq \frac{1}{m^2} \left| |Df(x)v| \cdot [Df(x)(u - v)] - [Df(x)u] \cdot Df(x)v \right|$$

$$\leq \frac{1}{m^2} [M^2 (u - v) + M |Df(x)(u - v)|]$$

$$\leq \frac{2M^2}{m^2} |u - v|.$$

This shows that $\{D^2 f(x) \mid x \in M\}$ are equi-continuous with respect to $d$.

Since $D^2 f(x) : S_x M \rightarrow S_{f(x)} M$ is a homeomorphism, then it is monotone and $|\deg D^2 f(x)| = 1$. Hence, from Theorem 2.1 and Corollary 2.6, we have $h(D^2 f, \pi^{-1}(x)) = 0$ for any $x \in M$. Therefore, from (3) we have

$$h(f) = h(D^2 f).$$

$\square$
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