MARCINKIEWICZ-ZYGMUND LAW OF LARGE NUMBERS FOR BLOCKWISE ADAPTED SEQUENCES

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Abstract. In this paper we establish the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequences. Some related results are considered.

1. Introduction and notations

In [5] and [8] it was shown that some properties of independent sequences of random variables can be applied to the sequences consisting of independent blocks. Particularly, it was proved in [8] that if \((X_i)_{i=1}^\infty\), \(EX_i = 0\) is a sequence independent in blocks \([2^k, 2^{k+1})\), then it satisfies the Kolmogorov’s theorem: the condition \(\sum_{i=1}^\infty (EX_i^{2})i^{-2} < \infty\) implies the strong law large numbers (s.l.l.n.), i.e.,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n X_i = 0 \text{ a.s.}
\]

Strong law of large numbers for blockwise independent random variables was studied by V. F. Gaposhkin [4].

Marcinkiewicz-Zygmund type strong law of large numbers was studied by many authors. In 1981, N. Etemadi [3] proved that if \(\{X_n, n \geq 1\}\) is a sequence of pairwise i.i.d. random variables with \(E|X_1| < \infty\), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n (X_i - EX_1) = 0 \text{ a.s.}
\]

Later, in 1985, B. D. Choi and S. H. Sung [2] have shown that if \(\{X_n, n \geq 1\}\) are pairwise independent and are dominated in distribution

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by a random variable $X$ with $E|X|^p(\log^+ |X|)^2 < \infty$, $1 < p < 2$, then
\[
\lim_{n \to \infty} \frac{1}{n^p} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}
\]


In this paper we establish the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequences. Some related results are considered.

Let $\{\omega(n), n \geq 1\}$ be a strictly increasing sequence of positive integers with $\omega(1) = 1$. For each $k \geq 1$, we set $\Delta_k = [\omega(k), \omega(k + 1))$. We recall that the sequence $\{X_i, i \geq 1\}$ of random variables is blockwise independent with respect to blocks $\Delta_k$, if for any fixed $k$, the sequences $\{X_i\}_{i \in \Delta_k}$ are independent. Let $\{F_i, i \geq 1\}$ be a sequence of $\sigma$-fields such that for any fixed $k$, the sequences $\{F_i\}_{i \in \Delta_k}$ are increasing. The sequence $\{X_i, i \geq 1\}$ of random variables is said to be blockwise adapted to $\{F_i, i \geq 1\}$, if each $X_i$ is measurable with respect to $F_i$. The sequence $\{X_i, F_i, i \geq 1\}$ is said to be a block martingale difference with respect to blocks $\Delta_k$, if for any fixed $k$, the sequences $\{X_i, F_i\}_{i \in \Delta_k}$ are martingale differences. Denote

\[
N_m = \min\{n | \omega(n) \geq 2^m\},
\]
\[
s_m = N_{m+1} - N_m + 1,
\]
\[
\varphi(i) = \max_{k \leq m} s_k \text{ if } i \in [2^m, 2^{m+1}),
\]
\[
\Delta^{(m)} = [2^m, 2^{m+1}), m \geq 0,
\]
\[
\Delta_k^{(m)} = \Delta_k \cap \Delta^{(m)}, m \geq 0, k \geq 1,
\]
\[
p_m = \min\{k : \Delta_k^{(m)} \neq \emptyset\},
\]
\[
q_m = \max\{k : \Delta_k^{(m)} \neq \emptyset\}.
\]

Since $\omega(N_m - 1) < 2^m, \omega(N_m) \geq 2^m, \omega(N_{m+1}) \geq 2^{m+1}$ for each $m \geq 1$, the number of nonempty blocks $[\Delta_k^{(m)}]$ is not large than $s_m = N_{m+1} - N_m + 1$. Assume $\Delta_k^{(m)} \neq \emptyset$, let $r_k^{(m)} = \min\{r : r \in \Delta_k^{(m)}\}$.

The sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable $X$ if there exists a constant $C > 0$ such that

\[
P\{|X_n| > t\} \leq CP\{|X| > t\}
\]

for all nonnegative real numbers $t$ and for all $n \geq 1$. 
Finally, the symbol $C$ denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. Lemmas

In the sequel we will need the following lemmas.

**Lemma 2.1.** (Doob’s Inequality) If $\{X_i, \mathcal{F}_i\}_{i=1}^N$ is a martingale difference, $E|X_i|^p < \infty$ ($1 < p < \infty$), then

$$E\left| \max_{k \leq N} \sum_{i=1}^k X_i \right|^p \leq \left( \frac{p}{p-1} \right)^p E\left| \sum_{i=1}^N X_i \right|^p.$$

The next lemma is due to von Bahr and Esseen [1].

**Lemma 2.2.** (von Bahr and Esseen [1]) Let $\{X_i, \mathcal{F}_i\}_{i=1}^N$ be random variables such that $E\{X_{m+1}|S_m\} = 0$ for $0 \leq m \leq N-1$, where $S_0 = 0$ and $S_m = \sum_{i=1}^m X_i$ for $1 \leq m \leq N$, then

$$E|S_N|^p \leq C \sum_{i=1}^N E|X_i|^p$$

for all $1 \leq p \leq 2$, where $C$ is a constant independent of $N$.

By lemmas 2.1 and 2.2, we get the following lemma.

**Lemma 2.3.** If $\{X_i, \mathcal{F}_i\}_{i=1}^N$ is a martingale difference, $E|X_i|^p \leq \infty$ ($1 \leq p \leq 2$), then

$$E\left| \max_{k \leq N} \sum_{i=1}^k X_i \right|^p \leq C \sum_{i=1}^N E|X_i|^p,$$

where $C$ is a constant independent of $N$.

**Proof.** In the case $p = 1$, we have

$$E\left| \max_{k \leq N} \sum_{i=1}^k X_i \right| \leq E\left( \sum_{i=1}^N |X_i| \right) = \sum_{i=1}^N E|X_i|.$$

In the case $1 < p \leq 2$,

$$E\left| \max_{k \leq N} \sum_{i=1}^k X_i \right|^p \leq \left( \frac{p}{p-1} \right)^p E\left| \sum_{i=1}^N X_i \right|^p \text{ (By Lemma 2.1)}$$

$$\leq C \sum_{i=1}^N E|X_i|^p \text{ (By Lemma 2.2)}.$$
The proof of the lemma is completed.

**Lemma 2.4.** If $q > 1$ and $\{x_n, n \geq 0\}$ is a sequence of constants such that $\lim_{n \to \infty} x_n = 0$, then

$$\lim_{n \to \infty} q^{-n} \sum_{k=0}^{n} q^{k+1} x_k = 0.$$  

**Proof.** Let $s = q + \sum_{i=0}^{\infty} q^{-i}$. For any $\epsilon > 0$, there exists $k_0$ such that $|x_k| < \frac{\epsilon}{2s}$ for all $k \geq k_0$. Since $\lim_{n \to \infty} q^{-n} = 0$, so, there exists $n_0 \geq k_0$ such that $|q^{-n} \sum_{k=0}^{k_0} q^{k+1} x_k| < \frac{\epsilon}{2}$. It follows that, for all $n \geq n_0$, 

$$|q^{-n} \sum_{k=0}^{n} q^{k+1} x_k| \leq |q^{-n} \sum_{k=0}^{k_0} q^{k+1} x_k| + |q^{-n} \sum_{k=k_0+1}^{n} q^{k+1} x_k|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2s} (q + 1 + \frac{1}{q} + \cdots)$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which completes the proof. □

3. Main result

With the notations and lemmas accounted for, main results may now be established. Theorem 3.1 establishes the strong law of large numbers for block martingale differences.

**Theorem 3.1.** Let $\{X_i, \mathcal{F}_i\}_{i=1}^{\infty}$ be a block martingale difference with respect to blocks $\Delta_k$, $(1 \leq p \leq 2)$. If

$$\sum_{i=1}^{\infty} \frac{E|X_i|^2}{i^p} < \infty,$$

then

$$\sum_{i=1}^{n} \frac{X_i}{n^{1/p}\phi^n(n)} \to 0 \quad a.s. \text{ as } n \to \infty.$$  

**Proof.** Let

$$\gamma_k^{(m)} = \max_{n \in \Delta_k^{(m)}} \left| \sum_{i=r_k^{(m)}}^{n} X_i \right|, \quad m \geq 0, k \geq 1.$$
\[ \gamma_m = 2^{-m-1} \varphi^{-1}(2^m) \sum_{p_m \leq k \leq q_m} \gamma_k^{(m)}, \quad m \geq 0. \]

Using Lemma 2.3 for martingale differences \( \{X_i, \mathcal{F}_i, i \in \Delta_k^{(m)}\} \), we have
\[ E|\gamma_k^{(m)}|^2 \leq C \sum_{i \in \Delta_k^{(m)}} EX_i^2, \text{ for all } m \geq 0, k \geq 1. \]

It implies
\[ E|\gamma_m|^2 \leq 2^{-2m-2} \varphi^{-1}(2^m) s_m \sum_{k=p_m}^{q_m} E|\gamma_k^{(m)}|^2 \]
\[ \leq 2^{-2m-2} \sum_{k=p_m}^{q_m} E|\gamma_k^{(m)}|^2 \]
\[ \leq C 2^{-2m-2} \sum_{i=2^m}^{2^{m+1}-1} EX_i^2 \]
\[ \leq C \sum_{i=2^m}^{2^{m+1}-1} \frac{X_i^2}{i^p}. \]

Thus
\[ \sum_{m=0}^{\infty} E|\gamma_m|^2 \leq C \sum_{i=1}^{\infty} \frac{X_i^2}{i^p} < \infty. \]

By the Markov inequality and the Borel-Cantelli lemma, we get
\[ \lim_{m \to \infty} \gamma_m = 0 \quad a.s. \quad (3.1) \]

On the other hand
\[ 0 \leq 2^{-m} \varphi^{-1}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma_i^{(k)} \leq 2^{-m} \sum_{k=0}^{m} 2^{k+1} \gamma_k. \quad (3.2) \]

By \( (3.1), (3.2) \) and Lemma 2.4, we get
\[ \lim_{m \to \infty} 2^{-m} \varphi^{-1}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma_i^{(k)} = 0 \quad a.s. \quad (3.3) \]
Assume $n \in \Delta_k^{(m)}$, we have

$$0 \leq n^{-\frac{1}{2}} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} X_i$$

(3.4)

$$\leq 2^{-m} \varphi^{-\frac{1}{2}}(2^m) \sum_{k=0}^{m} \sum_{i=p_k}^{q_k} \gamma^{(k)}_i.$$  

By (3.3) and (3.4), we get

$$n^{-\frac{1}{2}} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} X_i \to 0 \text{ a.s. (as } n \to \infty \text{).}$$

The proof is completed. □

In the next theorem, we set up the Marcinkiewicz-Zygmund law of large numbers for blockwise adapted sequences which are stochastically dominated by a random variable $X$.

**Theorem 3.2.** Let $\{\mathcal{F}_i, i \geq 1\}$ be a sequence of $\sigma$-fields such that for any fixed $k$, the sequences $\{\mathcal{F}_i, i \in \Delta_k\}$ are increasing and $\{X_i, i \geq 1\}$ is blockwise adapted to $\{\mathcal{F}_i, i \geq 1\}$. If $\{X_i, i \geq 1\}$ is stochastically dominated by a random variable $X$ such that either

$$E|X| \log^+ |X| < \infty \text{ if } p = 1,$$

or

$$E|X|^p < \infty \text{ if } 1 < p < 2,$$

then

$$\lim_{n \to \infty} \frac{1}{n^{\frac{1}{2}} \varphi(n)} \sum_{i=1}^{n} (X_i - a_i) = 0 \text{ a.s.,}$$

where $a_i = EX_i$ if $i = r^{(m)}_k$ and $a_i = E(X_i|\mathcal{F}_{i-1})$ if $i \neq r^{(m)}_k$ for $k \geq 1$ and $m \geq 0$.

**Proof.** Let $X'_i = X_i I\{|X_i| \leq \frac{1}{i^p}\}$, $b_i = EX'_i$ if $i = r^{(m)}_k$ and $b_i = E(X'_i|\mathcal{F}_{i-1})$ if $i \neq r^{(m)}_k$ for $k \geq 1$ and $m \geq 0$. We have

$$E(X'_i - b_i)^2 \leq E|X'_i|^2 = \int_0^{i^p} P(|X_i|^2 > t)dt$$
\[
\leq C \int_{0}^{\frac{1}{i^{1/p}}} P(|X|^2 > t)dt
\]

\[
= C \int_{0}^{\frac{1}{i^{1/p}}} (P(t < |X|^2 < \frac{2}{i^{1/p}}) + P(i^{1/p} \leq |X|^2)) dt
\]

\[
= C\left( \int_{0}^{\frac{1}{i^{1/p}}} x^2 dF(x) + i^{\frac{2}{p}} P(i^{\frac{2}{p}} \leq |X|^2) \right),
\]

where \( F(x) \) is the distribution function of \( X \).

\[
\sum_{i=1}^{\infty} \frac{1}{i^{2/p}} \int_{0}^{\frac{1}{i^{1/p}}} x^2 dF(x) \leq C \sum_{i=1}^{\infty} \frac{1}{i^{2/p}} \sum_{k=1}^{i} \int_{(k-1)^{1/p}}^{k^{1/p}} x^2 dF(x)
\]

\[
\leq C \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \frac{1}{i^{2/p}} \int_{(k-1)^{1/p}}^{k^{1/p}} x^2 dF(x)
\]

\[
\leq C \sum_{k=1}^{\infty} k^{\frac{p-2}{p}} \int_{(k-1)^{1/p}}^{k^{1/p}} x^2 dF(x)
\]

\[
\leq C \sum_{k=1}^{\infty} \int_{(k-1)^{1/p}}^{k^{1/p}} x^p dF(x)
\]

\[
\leq CE|X|^p < \infty,
\]

and

\[
\sum_{i=1}^{\infty} P(i^{\frac{2}{p}} \leq |X|^2) = \sum_{i=1}^{\infty} P(i \leq |X|^p) \leq CE|X|^p < \infty.
\]

Hence

\[
\sum_{i=1}^{\infty} \frac{E(X'_i - b_i)^2}{i^{1/p}} < \infty.
\]

For each \( k > 1 \) and \( m \geq 0 \), sequence \( \{X'_i - b_i, \mathcal{F}_i, i \in \Delta_k^{(m)}\} \) is a martingale difference. By using the proof of Theorem 3.1, we get

\[
(3.5) \quad n^{-\frac{1}{p}} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} (X'_i - b_i) \to 0 \text{ a.s. (as } n \to \infty). \]
Next,
\[ \sum_{i=1}^{\infty} P(X_i \neq X'_i) = \sum_{i=1}^{\infty} P(|X_i| > \frac{i^p}{i^p}) \]
\[ \leq C \sum_{i=1}^{\infty} P(|X| > \frac{i^p}{i^p}) \]
(3.6)
\[ \leq C \sum_{i=1}^{\infty} P(|X|^p > i) \leq CE[X]^p < \infty. \]

Finally, we prove that
(3.7) \[ \lim_{n \to \infty} \frac{1}{n^{p'}} \sum_{i=1}^{n} (a_i - b_i) = 0 \text{ a.s.} \]

In the case \( p=1 \),
\[ \sum_{n=1}^{\infty} n^{-1} E[|X_n|I(|X_n| > n)] = \sum_{n=1}^{\infty} n^{-1} \int_{n}^{\infty} P(|X_n| > x) \, dx \]
\[ \leq C \sum_{n=1}^{\infty} n^{-1} \int_{n}^{\infty} P(|X| > x) \, dx \]
\[ = C \sum_{n=1}^{\infty} n^{-1} \sum_{i=n}^{\infty} \int_{i}^{i+1} P(|X| > x) \, dx \]
\[ \leq C \sum_{i=1}^{\infty} P(|X| > i) \sum_{n=1}^{i} n^{-1} \]
\[ \leq C \sum_{i=1}^{\infty} (1 + \log i) P(|X| > i) < \infty. \]

This implies that \( \sum_{n=1}^{\infty} n^{-1}(a_n - b_n) < \infty \) a.s. By using Kronecker’s lemma, we get (3.7).

In the case \( 1 < p < 2 \), since
\[ \sum_{n=1}^{\infty} n^{-\frac{3}{2}} E[|X_n|I(|X_n| > n^{\frac{3}{2}})] \]
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\[ \mathbb{E}\left|X\right|^p < \infty \quad (1 \leq p < 2) \]

It follows that

\[ \sum_{n=1}^{\infty} n^{-\frac{1}{p}} (a_n - b_n) < \infty \text{ a.s..} \]

By Kronecker’s lemma, we get (3.7).

Combining (3.5), (3.6) and (3.7) we obtain

\[ \lim_{n \to \infty} \frac{1}{n^\frac{1}{p} \varphi(n)} \sum_{i=1}^{n} (X_i - a_i) = 0 \text{ a.s.} \]

This completes the proof of theorem.

The following corollaries extend the classical Marcinkiewicz-Zygmund strong law of large numbers.

**Corollary 3.3.** Let \( \{X_i, i \geq 1\} \) be a sequence of blockwise independent random variables with respect to blocks \( \Delta_k \). If \( \{X_i, i \geq 1\} \) is stochastically dominated by a random variable \( X \), \( \mathbb{E}|X|^p < \infty \) \((1 \leq p < 2)\), then

\[ \lim_{n \to \infty} \frac{1}{n^\frac{1}{p} \varphi(n)} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.} \]

**Proof.** Let \( F_i = \sigma(X_{r_k}^{(m)}, \ldots, X_i) \) (the \( \sigma \)-field generated by \( X_{r_k}^{(m)}, \ldots, X_i \)) if \( i \in \Delta_k^{(m)} \). Then \( \{X_i, i \geq 1\} \) is blockwise adapted to \( \{F_i, i \geq 1\} \).

From the independence of sequence \( \{X_i, i \in \Delta_k^{(m)}\} \) we get for all \( k \) and \( m \)

\[ E(X_i|F_{i-1}) = EX_i \text{ if } i \neq r_k^{(m)}. \]
By the proof of Theorem 3.2, we only need prove for the case $p = 1$.

In the case $p = 1$, also using the proof of Theorem 3.2, we get
\begin{equation}
(3.8) \quad n^{-1} \varphi^{-\frac{1}{2}}(n) \sum_{i=1}^{n} (X_i - EX'_i) \to 0 \text{ (as } n \to \infty),
\end{equation}
where $X'_i = X_i I(|X_i| \leq i)$. On the other hand
\[
E[|X_i|I(|X_i| > i)] = \int_{i}^{\infty} P(|X_i| > x)dx \\
\leq C \int_{i}^{\infty} P(|X| > x)dx \to 0 \text{ as } i \to \infty.
\]
Thus
\begin{equation}
(3.9) \quad |n^{-1} \sum_{i=1}^{n} (EX_i - EX'_i)| \leq n^{-1} \sum_{i=1}^{n} E[|X_i|I(|X_i| > i)] \to 0 \text{ as } n \to \infty.
\end{equation}
Combining (3.8) and (3.9) we obtain
\[
\lim_{n \to \infty} \frac{1}{n \varphi^\frac{1}{2}(n)} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}
\]
\[
\square
\]

**Corollary 3.4.** If $\omega(k) = 2^k$ (or $\omega(k) = \lfloor qk \rfloor, q > 1$) and $\{X_i, i \geq 1\}$ is $\Delta_k$-independent, $P\{|X_i| \geq t\} \leq CP\{|X| \geq t\}$ for all nonnegative real numbers $t$, $E|X|^p < \infty$, $(1 \leq p < 2)$, then
\[
\lim_{n \to \infty} \frac{1}{n^p} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}
\]

**Proof.** Really, in that case $\varphi(i) = O(1)$, so, from Corollary 3.3, we obtain
\[
\lim_{n \to \infty} \frac{1}{n^p} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}
\]
\[
\square
\]

**References**


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