**BOUNDED MATRICES OVER REGULAR RINGS**

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Abstract. In this paper, we investigate bounded matrices over regular rings. We observe that every bounded matrix over a regular ring can be described by idempotent matrices and invertible matrices. Let $A, B \in M_n(R)$ be bounded matrices over a regular ring $R$. We prove that $(AB)^d = U(BA)^dU^{-1}$ for some $U \in GL_n(R)$.

Let $I$ be an ideal of a ring $R$. We say that $I$ is a bounded ideal of $R$ in case there exists a positive integer $m$ such that $x^m = 0$ for all nilpotent $x \in I$. We say that $A \in M_n(R)$ is a bounded matrix provided $M_n(R)AM_n(R)$ is a bounded ideal of $M_n(R)$.

Throughout, all rings are associative rings with identities. $U(R)$ denotes the set of units of $R$, $M_n(R)$ denotes the ring of $n \times n$ matrices over $R$ and $GL_n(R)$ stands for the $n$ dimensional general linear group of $R$.

**Lemma 1.** Let $A \in M_n(R)$ be a bounded matrix over a regular ring $R$. Then there exists a bounded ideal $I$ of $R$ such that $A \in M_n(I)$.

**Proof.** Since $A$ is a bounded matrix, $M_n(R)AM_n(R)$ is a bounded ideal of $M_n(R)$. Let $e_{ij}$ be a usual matrix unit (1 ≤ $i, j \leq n$), i.e., in the $(i, j)$-position its entry is 1; otherwise, its entries are 0. One easily checks that $e_{ij}M_n(R)AM_n(R)e_{ij} \cong Ra_{ij}R$ and $e_{ij}M_n(R)e_{ij} \cong R$. That is, $Ra_{ij}R$ is a bounded ideal of $R$. By [9, Corollary 6.7], the sum of two ideals with index at most $m$ must have index at most $m$; hence, we see

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that $I = \sum_{1 \leq i,j \leq n} Ra_{ij} R$ is a bounded ideal of $R$. Clearly, $A \in M_n(I)$. Therefore we complete the proof.

A square matrix $A$ over a ring $R$ is said to admit a diagonal reduction if there exist some invertible matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix. It is well known that every square matrix over unit-regular rings admits a diagonal reduction (cf. [10, Theorem 3]). P. Ara et al. have extended this result to separative exchange rings (cf. [2, Theorem 2.4]). On the other hand, Menal and Moncasi [11] showed that the diagonalizability for some rectangular matrix over some regular rings fails. Now we observe the following result.

**Theorem 2.** Every bounded matrix over a regular ring admits a diagonal reduction.

**Proof.** Let $A = (a_{ij}) \in M_n(R)$ be a bounded matrix over a regular ring $R$. By Lemma 1, there exists a bounded ideal $I$ of $R$ such that $A \in M_n(I)$. Using [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $a_{ij} \in eRe$, and so $A \in M_n(eRe)$. As $e \in I$, we deduce that $eRe$ is unit-regular. Applying [10, Theorem 3], there exist some $U', V' \in GL_n(eRe)$ such that $U'AV' = \text{diag}(r_1, \ldots, r_n)$ for some $r_1, \ldots, r_n \in eRe$. Set $U = U' + \text{diag}(1-e, \ldots, 1-e)$ and $V = V' + \text{diag}(1-e, \ldots, 1-e)$. Then $U, V \in GL_n(R)$. Furthermore, we have $UAV = U'AV' = \text{diag}(r_1, \ldots, r_n)$, as asserted.

**Corollary 3.** Every $n \times n (n \geq 2)$ bounded matrix over a regular ring is a sum of two invertible matrices.

**Proof.** Let $A = (a_{ij}) \in M_n(R)(n \geq 2)$ be a bounded matrix over a regular ring $R$. In view of Theorem 2, there exist $U, V \in GL_n(R)$ such that $UAV = \text{diag}(r_1, \ldots, r_n)$ for some $r_1, \ldots, r_n \in R$. Clearly, $\text{diag}(r_1, r_2, \ldots, r_n)$ is a sum of two invertible matrices, i.e., we have

$$\text{diag}(r_1, r_2, \ldots, r_n) = \begin{pmatrix}
    r_1 & 1 & \cdots & 0 & 0 \\
    0 & r_2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & r_{n-1} & 1 \\
    1 & 0 & \cdots & 0 & 0
\end{pmatrix}$$
Bounded matrices over regular rings

\[
\begin{bmatrix}
0 & -1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & r_n
\end{bmatrix}
\]

Therefore we get

\[
A = U^{-1}
\begin{bmatrix}
 r_1 & 1 & \cdots & 0 & 0 \\
 0 & r_2 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & r_{n-1} & 1 \\
 1 & 0 & \cdots & 0 & 0
\end{bmatrix}
V^{-1}
\]

\[
+ U^{-1}
\begin{bmatrix}
0 & -1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & r_n
\end{bmatrix}
V^{-1},
\]

as desired.

\begin{proof}
 If \(n \geq 2\), then the result holds by Corollary 3. We now assume that \(n = 1\). Let \(x \in R\) such that \(RxR\) is a bounded ideal of \(R\). In view of [11, Lemma 1.1], we have an idempotent \(e \in RxR\) such that \(x \in eRe\). As \(eRe \subseteq RxR\), we deduce that \(eRe\) is a regular ring of bounded index; hence, it is unit-regular. Thus we have an idempotent \(f \in eRe\) and a unit \(v \in eRe\) such that \(x = fv\). Let \(u = v + (1 - e)\). Then we have \(x = f(v + (1 - e)) = (1/2 + (2f - 1)/2)u = u/2 + (2f - 1)u/2\). Clearly, \(u/2, (2f - 1)u/2 = (u^{-1}(4f - 2))^{-1} \in U(R)\). Therefore we get the result.
\end{proof}

Corollary 4. Let \(R\) be a regular ring with \(1/2 \in R\). Then every \(n \times n\) bounded matrix over \(R\) is a sum of two invertible matrices.

A ring \(R\) is said to be a clean ring in case every element in \(R\) is a sum of an idempotent and a unit. We know that every strongly \(\pi\)-regular ring is a clean ring (cf. [15, Theorem 1]). That author proved that every exchange ring with artinian primitive factors is a clean ring (see [5, Theorem 1]). A natural problem is how to extend this fact to matrices over a ring which is not a clean ring.
Theorem 5. Every bounded matrix over a regular ring is a sum of an idempotent matrix and an invertible matrix.

Proof. Let \( A = (a_{ij}) \in M_n(R) \) be a bounded matrix over a regular ring \( R \). In view of Lemma 1, there exists a bounded ideal \( I \) of \( R \) such that \( A \in M_n(I) \). Since all \( a_{ij} \in I \), by [11, Lemma 1.1], we have an idempotent \( e \in I \) such that all \( a_{ij} \in eRe \); hence, \( A \in M_n(eRe) \). Clearly, \( eRe \) is a regular ring of bounded index. It follows by [9, Theorem 7.12] that \( M_n(eRe) \) is a regular ring of bounded index. Using [9, Theorem 7.15], we know that \( M_n(eRe) \) is a strongly \( \pi \)-regular ring; hence, it is a clean ring by [15, Theorem 1]. Thus we have an idempotent matrix \( E' \in M_n(eRe) \) and an invertible \( U' \in M_n(eRe) \) such that \( A = E + U \). Therefore
\[
A = E' + \text{diag}(1-e, \ldots, 1-e) + (U' - \text{diag}(1-e, \ldots, 1-e)).
\]
Let \( E = E' + \text{diag}(1-e, \ldots, 1-e) \) and \( U = U' - \text{diag}(1-e, \ldots, 1-e) \). Then \( E = E^2 \in M_n(R) \) and \( U \in GL_n(R) \). In addition, we have \( A = E + U \). Thus the result follows.

Analogously, we deduce that every bounded matrix over a regular ring is a product of an idempotent matrix and an invertible matrix. We denote the set of all lower triangular matrices by \( \mathcal{L} \), i.e., \( \mathcal{L} = \{ (a_{ij}) \mid a_{ij} = 0 \text{ whenever } i < j \} \), and denote the set of all upper triangular matrices by \( \mathcal{U} \), i.e., \( \mathcal{U} = \{ (a_{ij}) \mid a_{ij} = 0 \text{ whenever } i > j \} \).

Lemma 6. Let \( A \in M_n(R) \) be a matrix over a unit-regular ring \( R \). Then \( A \) can be written as \( A = LUM \), \( L \in \mathcal{L}, U \in \mathcal{U}, M \in \mathcal{L} \) and in \( U \) and \( M \) all the diagonal entries are equal to 1.

Proof. Obviously, \( R \) is a Hermite ring. On the other hand, \( R \) has stable range one. Therefore we get the result by [14, Theorem 3.1].

Theorem 7. Every bounded matrix over a regular ring is a product of at most three triangular matrices.

Proof. Let \( A = (a_{ij}) \in M_n(R) \). According to Lemma 1, there exists a bounded ideal \( I \) of \( R \) such that all \( A \in M_n(I) \). By [11, Lemma 1.1], we have an idempotent \( e \in I \) such that all \( a_{ij} \in eRe \); hence, \( A \in M_n(eRe) \). As \( eRe \) is a regular ring of bounded index, it follows from [9, Corollary 7.11] that \( eRe \) is unit-regular. Thus, by Lemma 6, \( A \) can be written as \( A = LUM \), \( L \in \mathcal{L}, U \in \mathcal{U}, M \in \mathcal{L} \) and in \( U \) and \( M \) all the diagonal entries are equal to \( e \). One directly checks that
\[
A = (L + \text{diag}(1-e, \ldots, 1-e))(U + \text{diag}(1-e, \ldots, 1-e))M \quad \text{and in } L + \text{diag}(1-e, \ldots, 1-e) \quad \text{and } U + \text{diag}(1-e, \ldots, 1-e) \quad \text{all the diagonal entries are equal to } e. \]

\( \square \)
COROLLARY 8. Let $A = (a_{ij}) \in M_n(R)$. If all $Ra_{ij}R$ are bounded ideals of $R$, then $A$ is a product of at most three triangular matrices.

Proof. Let $I = \sum_{1 \leq i,j \leq n} Ra_{ij}R$. By [9, Corollary 7.8], $I$ is a bounded ideal of $R$. It follows by [11, Lemma 1.1] that $A \in M_n(eRe)$ for some idempotent $e \in I$. Clearly, $M_n(eRe)$ is a regular ring of bounded index from [9, Theorem 7.12]. As a result, $A$ is a bounded matrix over $eRe$. Using Theorem 7, $A$ can be written as $A = LUM$, $L \in \mathcal{L}$, $U \in \mathcal{U}$, $M \in \mathcal{L}$ and in $U$ and $M$ all the diagonal entries are equal to $e$. Similarly to Theorem 7, we get $A = (L + \text{diag}(1-e, \ldots, 1-e)) (U + \text{diag}(1-e, \ldots, 1-e)) M$ and in $L + \text{diag}(1-e, \ldots, 1-e)$ and $U + \text{diag}(1-e, \ldots, 1-e)$ all the diagonal entries are equal to 1.

Let $A = (a_{ij}) \in M_n(R)$. If all $Ra_{ij}R$ are nil ideals of bounded index, by Corollary 8, we see that $A$ is a product of at most three triangular matrices.

Recall that a matrix $A \in M_n(R)$ has the Drazin inverse in case there exist a positive integer $m$ and a matrix $X \in M_n(R)$ such that $A^m = A^{m+1}X$, $AX = XA$ and $X = XAX$. Clearly, the solution $X$ is unique, and we say that $X$ is the Drazin inverse $A^d$ of $A$.

Theorem 9. Let $A, B \in M_n(R)$ be bounded matrices over a regular ring $R$. Then there exists an invertible matrix $U$ such that $(AB)^d = U(AB)^dU^{-1}$.

Proof. Assume that $AB = (c_{ij})$, $BA = (d_{ij}) \in M_n(R)$. Set $I = \sum_{1 \leq i,j \leq n} Re_{ij}R + \sum_{1 \leq i,j \leq n} Rd_{ij}R$. Since $A$ and $B$ are both bounded matrices, so are $AB$ and $BA$. Similarly to Lemma 1, we show that all $Re_{ij}R$ and all $Rd_{ij}R$ are bounded ideal of $R$. It follows by [9, Corollary 7.8] that $I$ is a bounded ideal of $R$. Using [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $c_{ij} \in eRe$ and all $d_{ij} \in eRe$. Clearly, $eRe$ is a regular ring of bounded index; hence, so is $M_n(eRe)$ by [9, Theorem 7.12]. It follows by [9, Theorem 7.15] that $M_n(eRe)$ is strongly $\pi$-regular. That is, $AB, BA \in M_n(eRe)$ have the Drazin inverses. In addition, $M_n(eRe)$ has stable range one by [1, Theorem 4]. Therefore there exists some $V \in GL_n(eRe)$ such that $(AB)^d = V(AB)^dV^{-1}$ by [7, Theorem 1.2]. Set $U = V + \text{diag}(1-e, \ldots, 1-e)$. As $AB, BA \in M_n(eRe)$, by the uniqueness of the Drazin inverses of $AB$ and $BA$, we deduce that $(AB)^d, (BA)^d \in M_n(eRe)$. Therefore $(AB)^d = U(AB)^dU^{-1}$, as asserted.

\[\Box\]
Corollary 10. Let \( A, B \in M_n(R) \) be bounded matrices over a regular ring \( R \). Then the following are equivalent:

(1) \( AM_n(R) \cong BM_n(R) \).
(2) There exist some \( U, V \in GL_n(R) \) such that \( A = UBV \).

Proof. (2) \( \Rightarrow \) (1) is clear.

(1) \( \Rightarrow \) (2) Since \( R \) is a regular ring, so is \( M_n(R) \). Since \( A = (a_{ij}) \) is a bounded matrix, by Lemma 1, there exists a bounded ideal \( I \) of \( R \) such that \( A \in M_n(I) \). According to [11, Lemma 1.1], we have an idempotent \( e \in I \) such that \( A \in M_n(eRe) \). Clearly, \( eRe \) is a regular ring of bounded index, and so it is unit-regular. Using [9, Corollary 4.7], \( M_n(eRe) \) is unit-regular. Thus we have some \( C' \in GL_n(eRe) \) such that \( A = AC'A \). Set \( C = C' + \text{diag}(1 - e, \ldots, 1 - e) \). Then \( A = ACA \) with \( C \in GL_n(R) \). Similarly, we have some \( D \in GL_n(R) \) such that \( B = BDB \). Set \( E = AC \) and \( F = BD \). Then \( E, F \in M_n(R) \) are idempotent matrices and \( EM_n(R) \cong FM_n(R) \). Thus we get \( G \in EM_n(R)F \) and \( H \in FM_n(R)E \) such that \( E = GH \) and \( F = HG \). One easily checks that \( M_n(R)GM_n(R) \subseteq M_n(R)EM_n(R) \subseteq M_n(R)AM_n(R) \); hence, \( G \) is a bounded matrix. Likewise, \( H \) is a bounded index. By virtue of Theorem 9, we have \( U, V' \in GL_n(R) \) such that \( E^d = UFV' \). That is, \( E = UFV' \). Set \( V = D V' C^{-1} \). Therefore \( A = E C^{-1} = UFV' C^{-1} = UBDV'C^{-1} = UBV \), as asserted.

Corollary 11. Let \( A \) and \( B \) be \( n \times n \) matrices over a bounded ideal of a regular ring \( R \). Then the following are equivalent:

(1) \( AM_n(R) \cong BM_n(R) \).
(2) There exist some \( U, V \in GL_n(R) \) such that \( A = UBV \).

Proof. (2) \( \Rightarrow \) (1) is trivial.

(1) \( \Rightarrow \) (2) Suppose that \( A = (a_{ij}), B = (b_{ij}) \in M_n(I) \) and \( I \) is a bounded ideal of a regular ring \( R \). By [11, Lemma 1.1], there exists an idempotent \( e \in I \) such that all \( a_{ij}, b_{ij} \in eRe \); hence, \( A, B \in M_n(eRe) \). Clearly, \( eRe \) is a regular ring of bounded index, so is \( M_n(eRe) \). Thus we see that \( A \) and \( B \) are both bounded matrices over \( eRe \). It follows from \( AM_n(R) \cong BM_n(R) \) that \( AM_n(eRe) \cong BM_n(eRe) \). Using Corollary 10, we have \( U', V' \in GL_n(eRe) \) such that \( A = U' BV' \). Set \( U = U' + \text{diag}(1 - e, \ldots, 1 - e) \) and \( V = V' + \text{diag}(1 - e, \ldots, 1 - e) \). Then \( A = UBV \) and \( U, V \in GL_n(R) \), as asserted.
References


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