A STUDY ON THE EFFECTIVE ALGORITHMS BASED ON THE WEGMANN’S METHOD

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ABSTRACT. Determinations of conformal map from the unit disk onto a Jordan region are reduced to solve the Theodorsen equation which is an integral equation for the boundary correspondence function. Among numerical conformal maps the Wegmann’s method is well known as a Newton efficient one for solving Theodorsen equation. However this method has not so wide class of convergence. We proposed as an improved method for convergence by applying a low frequency filter to the Wegmann’s method. In this paper, we investigate error analysis and propose an automatic algorithm based on this analysis.

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1. Introduction

Conformal mapping has been a familiar tool of science and engineering for generations. The methods of numerical conformal mapping are usually classified into those which construct the map from standard domain such as the unit disk onto the ‘problem domain’, and those which construct the map in the reverse direction. We treat numerical conformal mapping from the unit disk onto the Jordan regions as the problem domain in here. The traditional standard methods of this type are based on Theodorsen integral equation[1]. Wegmann’s method is well known as a Newton-like efficient one for solving Theodorsen equation[2]. The non-discretized Wegmann’s method is quadratic convergent. However, the discretized iteration method of Wegmann has not so wide class of convergence[3, 4]. We proposed an improved method for convergence by applying a low frequency pass filter to the Wegmann’s method[4]. In this paper, we
investigate error analysis and propose an automatic based on this analysis. The numerical conformal mapping can be automatically approximated by this automatic algorithm according to once given the problem domain and the required accuracy.

Results of some test calculations are reported.

2. Wegmann’s iterative method

We begin with defining functional space by some signs used in this paper. \( t \in T \) is used as a variable for \( 2\pi \) periodic function, where \( T \) is the quotient space \( T := R/2\pi Z \) (\( R \) is real number and \( Z \) is integer). We define the below notations as follows:

- \( C(T) : 2\pi \) periodic complex continuous function space;
- \( C_R(T) : 2\pi \) periodic real continuous function space;
- \( C^m(T)(m \geq 1) : 2\pi \) periodic complex continuous function space of \( m \) times differentiable;
- \( C^m_R(T)(m \geq 1) : 2\pi \) periodic real continuous function space of \( m \) times differentiable;
- \( A(D) : \) complex continuous function space being analytic in and continuous in \( D \). Where \( D \) is the interior of a unit circle;
- \( A(D)|_T : \) set of boundary function’s \( f, f(t) = h(e^{it}) \) for \( A(D) \).

Let \( \Phi \) be a conformal map of the unit disk \( D \) with boundary \( \gamma \) onto a given Jordan domain \( \Delta \) with boundary \( \Gamma \). We assume \( \Phi \) is normalized by \( \Phi(0) = 0 \) and \( \Phi'(0) > 0 \). Then the map \( \Phi \) can be extended to the closure \( \overline{D} \) of \( D \), inducing a conformal map \( \Phi : \gamma \rightarrow \Gamma \).

Let \( \Gamma \) be defined by \( \eta(s) \in C^1(T) \) as follows:

\[
\Gamma := \left\{ \eta(s) : s \in [0, 2\pi] \right\}.
\]

Since the conformal map is fully determined by its value on the boundary in terms of \( \Phi \in A(\overline{D}) \), one can reduce the problem of computing it to that of computing the boundary map \( \Phi : \gamma \rightarrow \Gamma \). Therefore the problem of computing \( \Phi : \gamma \rightarrow \Gamma \) becomes to the problem of computing \( s(s(t)) \) satisfying

\[
\eta(s) \in A(\overline{D})|_T \quad \text{and} \quad [Im \eta(s)]_0 = 0,
\]

where \( [Im \eta(s)]_0 \) is 0 dimension Fourier coefficient.

**Definition 1.** If a continuous function \( u(t) \) is expanded by Fourier series as follows:

\[
u(t) = a_0/2 + \sum_{\mu=1}^{\infty} (a_\mu \cos \mu t + b_\mu \sin \mu t)\]
We define $K$ as the conjugate operator in the below definition,

$$K u(t) = \sum_{\mu=1}^{\infty} (a_\mu \sin \mu t - b_\mu \cos \mu t).$$

Here is another theorem concerning boundary function and conjugate operator[2].

**Theorem 1.**

$$f(t) \in A(D)_{|T} \quad \iff \quad \text{Im} f(t) - \text{Im} f_0 = K \text{Re} f(t),$$

where $f(t)$ is a boundary function, $\text{Im} f(t)$ is $f(t)$'s imaginary part, $\text{Re} f(t)$ is $f(t)$'s real part and $f_0$ is $f$'s 0 dimensional Fourier coefficient.

The normalized condition (1) makes $\text{Im} \hat{\eta}_0 = 0$ possible.

Using Theorem 1, we derive a boundary function $\eta(s)$ as follows:

$$\eta(s) \in A(D)_{|T} \quad \iff \quad \text{Im} \eta(s) = K \text{Re} \eta(s). \quad (2)$$

Formula (2) leads to the below equation by which we can get $s$:

$$\Psi(s) := \text{Im} \eta(s) - K \text{Re} \eta(s) = 0. \quad (3)$$

We call this Theodorsen equation.

Among the various solutions for this equation, we discuss Wegmann's method, known as the most effective. Wegmann solved the original nonlinear equation (3) with a Newton's method like this:

$$\Psi_{s_k}(t) + \Psi_{s_k} \delta_k(t) = 0$$

$$s_{k+1}(t) = s_k(t) + \delta_k(t) \quad (4)$$

$\Psi_{s_k}$: differential of $\Psi$ with respect to $s_k$, $k = 0, 1, 2, \cdots$,

$\delta_k$: correction in the $k$th step.

Wegmann reduced calculation and memory space by rendering it a Riemann-Hilbert problem on the unit circle[6].

The iteration scheme can be performed numerically in the following way.

Let $n$ be a natural number as $N = 2n$. Choose equidistant points $t_\nu = 2\pi \nu / N$, $\nu = 0, 1, 2, \cdots, N - 1$ in the interval $[0, 2\pi]$. The conjugate function $Ku$ defined by Definition 1 is approximated by $K_N u$ as follows[9]:

$$K_N u(t) = \sum_{\mu=1}^{n-1} (a_\mu \sin \mu t - b_\mu \cos \mu t).$$

If initial value is determined we can calculate an approximate value $s_k(k = 1, 2, \cdots)$ with the following formulas:

$$v_k(t) := \theta(s_k(t) - t); \quad (5)$$

$$w_k(t) := K_N v_k(t); \quad (6)$$
\( q_k(t) := Im(\eta(s_k(t)) \exp(w_k(t) - i\theta(s_k(t)))) \); \hspace{1cm} (7)

\[ \hat{\dot{v}}_k : = \frac{1}{N} \sum_{\nu=0}^{N-1} v_k(t_{\nu}), \quad \hat{q}_k := \frac{1}{N} \sum_{\nu=0}^{N-1} q_k(t_{\nu}); \]

\[ \hat{\lambda}_k : = \hat{q}_k \cot \hat{\dot{v}}_k; \] \hspace{1cm} (8)

\[ \delta_k(t) : = Re\left( \frac{\eta(s_k(t))}{\eta'(s_k(t))} \right) - \frac{\hat{\lambda}_k + K q_k(t)}{|\eta'(s_k(t))| \exp(w_k(t))}; \] \hspace{1cm} (9)

\[ s_{k+1} := s_k + \delta_k, \] \hspace{1cm} (10)

where \( \eta'(s_k) \) is the differential of \( \eta \) with respect to \( s_k \).

3. Implementation of Wegmann’s iteration

We define low frequency filter \( L_l \) as

\[ L_l(e^{imt}) = \begin{cases} 
    e^{imt} : & 0 \leq |m| \leq n-l, \\
    0 : & n-l < |m| \leq n,
\end{cases} \]

where \( l \) is the parameter for excluding several high frequency from the back. By applying \( L_l \) to the approximate value (10) in terms of Wegmann’s iteration as follows:

\[ s_{k+1}^* := L_l(s_k(t) - t) + t. \] \hspace{1cm} (11)

We can make a class of convergence more wide.

Let \( \Pi_n \) be a subspace of \( C_\mathbb{R}(T) \) as like following trigonometric polynomials

\[ \Pi_n = \left\{ a_0/2 + \sum_{\mu=1}^{n-1} (a_\mu \cos \mu t + b_\mu \sin \mu t) + a_n \cos nt/2 \right\}. \]

Then we can obtain the coefficients of \( s_{k+1} - t \) by FFT(Fast Fourier Transform) because \( s_{k+1} - t \in \Pi_n \).

The filter parameter \( l \) can be determined as follows:

Assume the boundary function of Jordan region (problem domain) is given by

\[ \eta(t) = (1 + \xi(t))e^{it} ; \quad \xi(t) \in C^2_\mathbb{R}(T). \]

Let \( \xi \) and \( \xi' \) can be represented Fourier series by

\[ \xi(t) = \sum_{\nu=0}^{\infty} c_\nu e^{i\nu t} \text{ and } \xi'(t) = \sum_{\nu=0}^{\infty} d_\nu e^{i\nu t} = \sum_{\nu=0}^{\infty} \nu c_\nu e^{i\nu t}. \] \hspace{1cm} (12)
Using $c_\nu$ and $d_\nu$ ($\nu = 1, 2, \cdots$) in (12) amounts such as $D_0$ and $D_\nu$ are defined as
\[
D_0 := |c_0| + 4 \sum_{k=1}^{\infty} |d_k|, \quad D_\mu := 2|d_\mu| + 4 \sum_{k=\mu+1}^{\infty} |d_k|, \quad 1 \leq \mu \leq \infty.
\]
(13)

If we determine $l$ satisfying $D_l < 1$ then the iteration from (5) to (11) is convergent by [4] and numerical experiments for this improved method are reported in [4]. The Sobolev space $W$ is defined by
\[
W := \{ f \in C(T) : f' \in L^2(0, 2\pi) \}.
\]
The norm on $W$ is defined by
\[
\|f\|_W := \max(\|f\|_\infty, \|f'\|_2),
\]
where $f'$ is a differential of $f$,
\[
\|f\|_\infty := \max_{t \in T} |f(t)| \quad \text{and} \quad \|f'\|_2 := \left( \int_0^{2\pi} |f'(t)|^2 dt \right)^{1/2}.
\]
Let $A_r$ be the space of analytic functions in $G_r$ defined by
\[
G_r := \{ z : |Imz| < r \} \quad (r > 0).
\]
The norm on $A_r$ is defined by
\[
\|f\|_r := \sup_{z \in G_r} |f(z)|.
\]
By Wegmann[5] and a formula $\Phi_{k+1}$ such as
\[
\Phi_{k+1}(e^{it}) := -(\lambda_k + K_N q_k + u_k^n \cos nt - iq_k) \exp(it - w_k + iv_k),
\]
we can obtain $C_m$ satisfying
\[
\|s_{k+1} - s\|_W \leq C_m \|\eta(s_{k+1}(e^{it})) - \Phi_{k+1}(e^{it})\|_r,
\]
(14)
where $s_{k+1}$ is approximate value and $s$ is real value. That means we can estimate errors by right hand of (14) even if we don’t know a real value.

### 4. Automatic algorithm

Now we propose the automatic algorithm to obtain $s_{k+1}$ for conformal map that can determine $s_{k+1}$ according to the given problem domain and required accuracy. We usually get the approximate value which is much more close to the true value as the discrete number $s$ increased. It is possible to derive the automatic algorithm satisfying required accuracy because we can estimate error with (14) whether the discrete number has to be increased from the initial discrete number.

The detailed automatic algorithm is as follows:
Step 1. Obtain Fourier coefficient $\dot{\xi}$ from the formula of (12) and decide the parameter of low frequency filter from the formula (13) by using of Fourier coefficient for $\dot{\xi}$.

Step 2. Initial discrete number $N$ is to be the number of the coefficient term required at Step 1. Decide the proper initial value and the required accuracy $\varepsilon$.

Step 3. Perform the iterations from (5) to (11) until $\delta_k < \varepsilon$.

Step 4. Estimate the error of the approximate value which is obtained at Step 3 by means of the right side of formula (14). If the estimate error is smaller than $\varepsilon$, the iteration method is terminated. Otherwise go to the next step.

Step 5. The discrete number is to be double. That is, $N$ is to be $2N$. Let the approximate value $s_k$ which is obtained at Step 4 be discrete on $2N$ and be initial value and go to Step 3.

Step 1 means the parameter of low frequency filter is determined by $\dot{\xi}$ that determine the boundary function of the given problem domain. This proposed algorithm have some properties as follows:

1. The parameter of low frequency filter and the initial discrete number are automatically decided by the boundary function of the given problem domain.
2. The degree of difficulty for a given problem could be estimated according to the size of low frequency filter parameter.
3. We can get much more fast speed in this algorithm by means of approximate value obtained at the previous step as the initial value for the next iteration.
4. It is possible to decide automatically the discrete number $N$ with sufficient accuracy required.

5. Numerical experiences

The eccentric circle which the real value is known is treated for the example of the numerical experiment. Let us fix notations of numerical estimate error $ER$ and real error $E$ as follows:

$$ER := \max_{\nu=0,1,\ldots,N-1} |s(t_\nu) - s_{k+1}(t_\nu)|$$

$$E := \max_{\nu=0,1,\ldots,N-1} |\eta(s_{k+1}(t_\nu)) - \Phi_{k+1}(e^{it_\nu})|, \ t_\nu = 2\pi\nu/N.$$ 

The eccentric circle has the following function on boundary of the eccentric circle.
\[ \eta(s) = \rho(s) e^{is}, \quad \rho(s) = \frac{R \cos s + \sqrt{1 - R^2 \sin^2 s}}{R + 1}, \quad 0 \leq R < 1. \]

Real value:

\[ s(t) = \arctan \frac{R \sin t}{1 - R \cos t} + t. \]

This is an example of problem which is getting difficult because the transformation is more serious as configuration parameter \( R \) is larger toward 1. Table 1 shows results which we had experiments with the proposed algorithm from Step 1 to Step 5. In Table 1, we show the initial discrete number is fixed with \( N = 64 \) when \( R = 0.6 \) is given and it is automatically increased into \( N = 128 \) so that approximate value can be obtained satisfying required accuracy. The number of iteration \( k \) has been reduced steeply by use of the approximate value which is obtained from the previous discrete number as initial value in next iteration step.

Table 1. The result of iteration by automatic algorithm when \( R = 0.6 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( ER )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.649E+00</td>
<td>0.185E+00</td>
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<td>2</td>
<td>0.251E-01</td>
<td>0.240E-01</td>
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<tr>
<td>3</td>
<td>0.365E-03</td>
<td>0.444E-03</td>
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<td>4</td>
<td>0.110E-05</td>
<td>0.492E-05</td>
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<td>0.191E-06</td>
</tr>
<tr>
<td>6</td>
<td>0.944E-08</td>
<td>0.108E-07</td>
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<tr>
<td>( N = 128 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.323E-14</td>
<td>0.402E-13</td>
</tr>
</tbody>
</table>

References


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