INTEGRAL EVALUATION OF THE LINEARIZATION COEFFICIENTS OF THE PRODUCT OF TWO LEGENDRE POLYNOMIALS

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Abstract. In this paper, we derive an elementary formula for the linearization coefficients of the product of two Legendre polynomials. Our main purpose is to study the method which can be used to derive the formula, which is straightforward from the integration and their orthogonality.

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1. Introduction

An addition formula for trigonometric function given by
\[ \cos m\theta \cos n\theta = \frac{1}{2} \cos(m+n)\theta + \frac{1}{2} \cos(m-n)\theta \] (1.1)
pertains to Chebyshev polynomials of the first kind \( p_n^{(-1/2,-1/2)}(x) \), where \( x = \cos \theta \). It is called a linearization formula since it represents a product of two polynomials as a linear combination of other polynomials of the same kind.

Our concern, for a given sequence of polynomials \( \{p_n(x)\} \), is to know the coefficient \( a(k,m,n) \) in
\[ p_m(x)p_n(x) = \sum_{k=0}^{m+n} a(k,m,n)p_k(x). \] (1.2)
If \( p_n(x) \)'s are orthogonal with respect to a distribution \( d\alpha(x) \), then we have

\[
a(k, m, n) = \frac{1}{h_k} \int_I p_m(x)p_n(x)p_k(x) \, d\alpha(x),
\]

(1.3)

where \( h_k \) is the \( L^2 \)-norm of \( p_k(x) \) with respect to \( d\alpha(x) \).

Thus the problem of the evaluation of the integral of the product of three orthogonal polynomials of the same kind is equivalent to the linearization problem. One way of obtaining linearization formulae is to find polynomials for which the integral (1.3) can be computed. A simpler integral would involve only the product of two polynomials, but this should yield the orthogonality relation. As we know, using the generating function is one way of obtaining orthogonality in some cases, for example, Hermite polynomial. At this point the idea of using the generating function to obtain a linearization formula for Jacobi polynomials does not appear positively, because the product of these three polynomials does not seem tractable.

Dougall [7] first stated the linearization of Legendre polynomials but he did not give his proof. Adams [1] remarked that both he and Ferrers [8] found it by finding coefficients for small values of \( m \) (Adams derived these coefficients for \( m = 1, 2, 3, 4 \), and guessed that the result hold for arbitrary \( m \) and \( n \) by using induction. In fact, they used the three term recurrence relations of orthogonal polynomials multiplied by any polynomial what they need. This is a way to find the formula, but R. Askey suggested in his book [2] that a more systematic method should be found. Neumann [15] found such a method, and it is still known as the most powerful method. He computed a fourth order differential equation satisfied by \( p_n(x)p_m(x) \) and used it to set up a recurrence relation for the linearization coefficients. Then he solved this recurrence relation. His argument is used in Hobson [11] and Hylleraas [12] to set up a similar equation satisfied by the product of ultraspherical polynomials.

For Jacobi polynomials the situation was far from satisfaction. Gasper [9, 10] had studied but the results were very complicated partial relations.

In this work, we evaluate the formula for the linearization coefficients of the product of any two Legendre polynomials, as a special case of Jacobi polynomials, by using the integration and their orthogonality. This idea is due to Baily [5], he expressed Jacobi polynomials as a hypergeometric series.

### 2. Hypergeometric series and Jacobi polynomials

Let us first recall some well-known notations and definitions for hypergeometric series. The Pochhammer symbol \((a)_n\) is defined by

\[
(a)_n = \prod_{m=0}^{n-1} (a + m), \quad n = 1, 2, \cdots, \quad (a)_0 = 1.
\]

(2.1)
A hypergeometric series \( \sum c_n \) is a series where \( \frac{c_{n+1}}{c_n} \) is a rational function of \( n \).

By factorizing the polynomials in \( n \), we obtain

\[
\frac{c_{n+1}}{c_n} = \frac{(n + a_1) (n + a_2) \cdots (n + a_p) x}{(n + b_1) (n + b_2) \cdots (n + b_q) (n + 1)}.
\]

From (2.1) we have

\[
\sum_{n=0}^{\infty} c_n = c_0 \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} =: c_0 \, _pF_q \left( \begin{array}{c} a_1, \ldots, a_p \end{array} ; \begin{array}{c} b_1, \ldots, b_q \end{array}; x \right),
\]

where \( b_j \)'s are neither negative integers nor zeros. For typographical reasons, we will denote the sum on the right side of (2.2) by \( _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \) or simply \( pFq \).

**Theorem 2.1.** The series \( _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \)

1. converges absolutely for all \( x \) if \( p \leq q \),
2. converges absolutely for \( |x| < 1 \) if \( p = q + 1 \), and
3. diverges for all \( x \neq 0 \) if \( p > q + 1 \) and the series does not terminate.

**Proof.** It is natural to apply the ratio test to determine the convergence of the series (2.2). Then we have

\[
\left| \frac{c_{n+1}}{c_n} \right| \leq |x| n^{p-q-1} \left( 1 + \frac{|a_1|}{n} \right) \cdots \left( 1 + \frac{|a_p|}{n} \right) \frac{1}{(1 + \frac{b_1}{n}) \cdots (1 + \frac{b_q}{n})}.
\]

From the above inequality, we obtain

\[
\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \begin{cases} 
0, & p < q \\
|x|, & p = q + 1 \\
\infty, & p > q + 1.
\end{cases}
\]

This completes our proof. \( \square \)

**Definition 2.1.** For all complex numbers \( x \neq 0, -1, -2, \ldots \), the gamma function \( \Gamma(x) \) is defined by

\[
\Gamma(x) = \lim_{k \to \infty} \frac{k! x^{-1}}{(x)_k}.
\]

**Definition 2.2.** The beta integral is defined by

\[
B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt
\]
for \( \Re x > 0, \ \Re y > 0 \).

The following theorem is known as Euler’s reflection theorem.

**Theorem 2.2.** **Euler’s reflection formula:**

\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}.
\]

**Lemma 2.1.** For \( a, \lambda \in \mathbb{R}, n \in \mathbb{N} \),

\[(a + \lambda)_n = \frac{(-1)^n}{(1 - \lambda - a)_n}.\]

**Proof.** From the definition of Pochhammer symbol and Theorem 2.2, we have

\[
(a + \lambda)_n = \frac{\Gamma(a + \lambda - n)}{\Gamma(a + \lambda)} \frac{\Gamma(n - a - \lambda)}{\Gamma(n + a + \lambda)} \sin \pi(a + \lambda) = \frac{(-1)^n}{(1 - \lambda - a)_n}.
\]

This completes our proof. \(\square\)

Next two identities are due to Whipple [20] and Waston [19], which play important roles in finding the linear coefficients of the product of any two Legendre polynomials in later. For the proof of the two following theorems see [4, 7, 19] and [20] respectively.

**Theorem 2.3.** (Whipple’s transformation) If \( a + b = 1 \) and \( d + e = 1 + 2c \), then

\[
\mathbf{3}_F_2\left(\begin{array}{ccc}
a, b, c; 1 \\
d, e, \end{array}\right) = \pi^{2 - 2c} \frac{\Gamma(d)\Gamma(e)}{\Gamma(\frac{a + d}{2})\Gamma(\frac{b + d}{2})\Gamma(\frac{b + e}{2})}(2.3)
\]

**Theorem 2.4.** (Watson’s transformation)

\[
\mathbf{3}_F_2\left(\begin{array}{ccc}
a, b, c; 1 \\
\frac{1 + a}{2}, 2c, \end{array}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)\Gamma\left(\frac{1 + a + b}{2}\right)\Gamma\left(\frac{1 - a + b + 2c}{2}\right)}{\Gamma\left(\frac{1 + a}{2}\right)\Gamma\left(\frac{1 + b}{2}\right)\Gamma\left(\frac{1 + b + 2c}{2}\right)}(2.4)
\]
The factorization
\[(a)_{2n} = 2^{2n} \left( \frac{a}{2} \right)_n \left( \frac{a + 1}{2} \right)_n\]
together with the definition of the gamma function leads immediately to Legendre's duplication formula contained in the next theorem, which also plays an important role in finding the linear coefficients in later.

**Theorem 2.5.** Legendre duplication formula:
\[
\Gamma(2a)\Gamma\left(\frac{1}{2}\right) = 2^{2a-1} \Gamma(a)\Gamma\left(a + \frac{1}{2}\right). \tag{2.5}
\]

**Proof.** The proof follows from
\[
B(x, x) = \int_0^1 [t(1 - t)]^{x - 1} dt = 2 \int_0^{\frac{1}{2}} [t(1 - t)]^{x - 1} dt.
\]
Substituting, then we see that
\[
B(x, x) = 2^{1-2x} B\left(x, \frac{1}{2}\right),
\]
which is equivalent with (2.5). This completes our proof. \(\square\)

The Jacobi polynomials \(p_n^{(\alpha,\beta)}(x)\), provide a class of functions which include all of orthogonal polynomials. The following (2.6) in Definition 2.3 is due to Baily [5].

**Definition 2.3.** The Jacobi polynomial of degree \(n\) is defined by
\[
p_n^{(\alpha,\beta)}(x) := \frac{(\alpha + 1)_n}{n!} \binom{-n}{\frac{1}{2}}_{2F1} \left( n + \alpha + \beta + 1; \frac{1-x}{2} \right), \tag{2.6}
\]
or by
\[
(1 - x)^\alpha (1 + x)^\beta p_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^nn!} \frac{d^n}{dx^n} \left[ (1 - x)^{n+\alpha}(1 + x)^{n+\beta} \right]. \tag{2.7}
\]

Jacobi polynomials are orthogonal on \([-1, 1]\) with respect to \((1 - x)^\alpha (1 + x)^\beta\) when \(\alpha, \beta > -1\), and usually this restriction is assumed. However, many of the formulae hold without this restriction, so occasionally it will be dropped if interesting results are thus obtained. In such cases \(p_n^{(\alpha,\beta)}(x)\) will be defined by
(2.7), since (2.6) has to be interpreted as a limit when $\alpha$ is a negative integer. The equation (2.7) is often called the Rodrigues formula for Jacobi polynomials. The particular case for Legendre polynomials, when $\alpha = \beta = 0$, was published by O. Rodrigues in 1816.

Orthogonal polynomials have very nice properties, three term recurrence, real zero property, generating function; for references see [4, 6, 13, 14, 16, 18].

The orthogonality of Jacobi polynomials is that

$$\int_{-1}^{1} p_n^{(\alpha,\beta)}(x)p_m^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx $$

$$= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{mn}$$

$$:= h_n^{(\alpha,\beta)}.$$ 

To use later, we introduce the formulas following from Definition 2.3 of Jacobi polynomials.

**Lemma 2.2.** Given Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ we have,

$$p_n(x) = p_n^{(0,0)}(x)$$

$$= (-1)^n \frac{d^n}{dx^n}[(1-x)^n(1+x)^n],$$

(2.9)

$$\frac{d}{dx}p_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} p_{n-1}^{(\alpha+1,\beta+1)}(x),$$

(2.10)

3. The coefficients $a(k, m, n)$ of Linearization

Now we consider the linearization of products of Jacobi polynomials. Many mathematicians had been interested in this problem. For examples, see Askey and Gasper [3], Askey and Wilson [4], Roy [17], Gasper [9] and Wilson [21]. It is called a linearization formula to determine, or at least to give a clue to determine the coefficients $a(k, m, n)$ in

$$p_m(x)p_n(x) = \sum_{k=0}^{m+n} a(k, m, n)p_k(x).$$

Here, in case of Jacobi polynomials, we investigate the coefficients $a(k, m, n)$ in

$$p_n^{(\alpha,\beta)}(x)p_m^{(\gamma,\delta)}(x) = \sum_{k=0}^{m+n} \frac{a(k, m, n)}{h_k^{(\alpha+\gamma,\beta+\delta)}} p_k^{(\alpha+\gamma,\beta+\delta)}(x)$$

(3.1)
where $h_k^{(\alpha+\gamma,\beta+\delta)}$ is the $L^2$-norm of Jacobi polynomials $p_k^{(\alpha+\gamma,\beta+\delta)}(x)$. From the equation (3.1), we have

$$a(k,m,n) = \int_{-1}^{1} p_n^{(\alpha,\beta)}(x)p_m^{(\gamma,\delta)}(x)p_k^{(\alpha+\gamma,\beta+\delta)}(x)(1-x)^{\alpha+\gamma}(1+x)^{\beta+\delta} \, dx. \quad (3.2)$$

Thus the problem of the evaluation of the integral of the product of three orthogonal polynomials of the same kind is equivalent to the linearization problem. So, we must calculate the integral in (3.2) to find the coefficients $a(k,m,n)$.

We started to find the coefficients $a(k,m,n)$ in (3.1) with the general Jacobi polynomials, but it is difficult to calculate the integral of products of three Jacobi polynomials that is the product of unknown two sum of $\Phi_2$’s. But we can say about the coefficients of linearization for Legendre polynomials, as a special case of general Jacobi polynomials.

The following theorem is the main result of this paper, which is evaluated directly from the integration (3.2).

**Theorem 3.1.** (Linearization of Legendre polynomials) For Legendre polynomials $p_n(x) = p_n^{(0,0)}(x)$,

$$p_m(x)p_n(x) = \sum_{k=0}^{\min(m,n)} \frac{2m+2n+1-4k}{2m+2n+1-2k} \times \frac{\left(\frac{1}{2}\right)_k (\frac{1}{2})_{m-k} (\frac{1}{2})_{n-k} (m+n-k)!}{k! (m-k)! (n-k)! (\frac{1}{2})_{m+n-k}} p_{m+n-2k}(x). \quad (3.3)$$

**Proof.** We write the problem for general Jacobi polynomials as follow

$$p_n^{(\alpha,\beta)}(x)p_m^{(\gamma,\delta)}(x) = \sum_{k=0}^{m+n} \frac{a(k,m,n)}{h_k^{(\alpha+\gamma,\beta+\delta)}} p_k^{(\alpha+\gamma,\beta+\delta)}(x) \quad (3.4)$$

where $h_k^{(\alpha+\gamma,\beta+\delta)}$ is the $L^2$-norm of Jacobi polynomials $p_k^{(\alpha+\gamma,\beta+\delta)}(x)$. Now, let’s find the coefficients $a(k,m,n)$: By using the distribution

$$d\alpha(x) = (1-x)^{\alpha+\gamma}(1+x)^{\beta+\delta} \, dx,$$

we have

$$a(k,m,n) = \int_{-1}^{1} p_n^{(\alpha,\beta)}(x)p_m^{(\gamma,\delta)}(x)p_k^{(\alpha+\gamma,\beta+\delta)}(x)(1-x)^{\alpha+\gamma}(1+x)^{\beta+\delta} \, dx.$$
The formula (2.7) in Definition 2.3 gives the following

$$a(k, m, n) = \int_{-1}^{1} p_n^{(\alpha, \beta)}(x)p_m^{(\gamma, \delta)}(x) \left(1 - x\right)^{k+\alpha+\gamma}(1 + x)^{k+\beta+\delta} \, dx.$$  \hfill (3.5)

By using integration by parts, differential formula (2.10) in Lemma 2.2 and Leibniz rule for differentiation, the equation (3.5) can be calculated as follows:

$$\int_{-1}^{1} \frac{1}{2^k k!} \frac{d^k}{dx^k} \left[p_n^{(\alpha, \beta)}(x)p_m^{(\gamma, \delta)}(x)\right] (1 - x)^{k+\alpha+\gamma}(1 + x)^{k+\beta+\delta} \, dx$$

$$= \frac{1}{2^k k!} \sum_{j=0}^{k} \binom{k}{j} \int_{-1}^{1} \frac{d^j}{dx^j} \left[p_n^{(\alpha, \beta)}(x)\right] \frac{d^{k-j}}{dx^{k-j}} \left[p_m^{(\gamma, \delta)}(x)\right] (1 - x)^{k+\alpha+\gamma}(1 + x)^{k+\beta+\delta} \, dx$$

$$= \frac{1}{2^k k!} \sum_{j=0}^{k} \binom{k}{j} (n + \alpha + \beta + 1)_j (m + \gamma + \delta + 1)_{k-j} \int_{-1}^{1} (1 - x)^{n+j} (1 + x)^{k+\gamma-j} (1 - x)^{k+\beta-j} p_n^{(\alpha+\gamma-j, \beta-j)}(x) \, dx$$

$$= \frac{1}{2^k k!} \sum_{j=0}^{k} \binom{k}{j} (n + \alpha + \beta + 1)_j (m + \gamma + \delta + 1)_{k-j} \times \left[\frac{(-1)^{n-j}}{2^{n-j}(n-j)!} \frac{(-1)^{m-k+j}}{2^{m-k+j}(m-k+j)!} \int_{-1}^{1} (1 - x)^{n+j}(1 + x)^{n+\beta} \frac{d^{m-k+j}}{dx^{m-k+j}} \left[(1 - x)^{m+\gamma}(1 + x)^{m+\delta}\right] \, dx\right]$$

$$= \frac{(-1)^{n+m+k}}{2^{n+m+k} k!} \sum_{j=0}^{k} \binom{k}{j} (n + \alpha + \beta + 1)_j (m + \gamma + \delta + 1)_{k-j} \times \left[\frac{(-1)^{n-j}}{(n-j)!} \frac{(-1)^{m-k+j}}{(m-k+j)!} \int_{-1}^{1} (1 - x)^{n+j}(1 + x)^{n+\beta} \frac{d^{m+n-k}}{dx^{m+n-k}} \left[(1 - x)^{m+\gamma}(1 + x)^{m+\delta}\right] \, dx\right]$$
\[
\int_{-1}^{1} (1-x)^{\gamma-n+k}(1+x)^{\delta-n+k} p_{n+m-k}(x) dx.
\]

From (2.7) in Definition 2.3, we can write the last part of the above equation as follow

\[
\left(\frac{\gamma - n + k + 1}{2^k k!}\right)_{n+m-k} \sum_{j=0}^{k} \binom{k}{j} (n + \alpha + \beta + 1)_j (m + \gamma + \delta + 1)_{k-j} \times \frac{(-1)^{n-j}}{(n-j)!(m-k+j)!} \int_{-1}^{1} (1-x)^{\gamma-n+k}(1+x)^{\delta-n+k} dx
\]

By using the value of \(h_n^{(\alpha,\beta)}\) in (2.8), the above equation can be written as follow

\[
\left(\frac{\gamma - n + k + 1}{2^k k!}\right)_{n+m-k} \sum_{j=0}^{k} \binom{k}{j} (n + \alpha + \beta + 1)_j (m + \gamma + \delta + 1)_{k-j} \times \frac{(-1)^{n-j}}{(n-j)!(m-k+j)!} \times \frac{2^{(\alpha+\gamma+k+1)} \Gamma(\alpha + \gamma + k + 1) \Gamma(\beta + \delta + k + 1)}{\Gamma(\alpha + \beta + \gamma + \delta + 2k + 2)} \times \sum_{l=0}^{m+n-k} \frac{(-m-n+k)_l (m-n+\gamma+\delta+k+1)_l}{(\gamma-n+k+1)_l 2^l!}
\]
\[ \gamma = a \]

To obtain the coefficient \( a \), we use the relation (632 S. B. Park and J. H. Kim)

As a result of the above computation, we have

\begin{align*}
\frac{a(k, m, n)}{\Gamma(\alpha + \beta + \gamma + \delta + 1)} &= \frac{\Gamma(\alpha + \beta + \gamma + \delta + k + 1)}{\Gamma(\alpha + \beta + \gamma + \delta + 2k + 2)} \times \sum_{j=0}^{k} \frac{(-n)_j (n + \alpha + \beta + 1)_j (-k)_j}{(m - k + 1)_j (-m - \gamma - \delta - k)_j j!} \\
&\times \sum_{l=0}^{m+n-k} \frac{(-n - m + k)_l (m - n + \gamma + \delta + k + 1)_l (\alpha + \gamma + k + 1)_l}{(\gamma - n + k + 1)_l (\alpha + \beta + \gamma + \delta + 2k + 2)_l l!}.
\end{align*}

As a result of the above computation, we have

\[ a(k, m, n) = (-1)^n 2^{\alpha + 1} (\gamma - n + k + 1)_{m + n - k} (m + \gamma + \delta + 1)_k \]

\begin{align*}
&\times \frac{\Gamma(\alpha + \gamma + k + 1) \Gamma(\beta + \delta + k + 1)}{\Gamma(\alpha + \beta + \gamma + \delta + 2k + 2)} \\
&\times _3 F_2 \left( \begin{array}{c}
-n, \\
-m - \gamma - \delta - k,
\end{array} \begin{array}{c}
\alpha + \beta + 1, -k; 1
\end{array} \right) \\
&\times _3 F_2 \left( \begin{array}{c}
-n - m + k, m - n + \gamma + \delta + k + 1, \\
\gamma - n + k + 1, \alpha + \beta + \gamma + \delta + 2k + 2
\end{array} \begin{array}{c}
\alpha + \gamma + k + 1; 1
\end{array} \right). 
\end{align*}

To obtain the coefficient \( a(k, m, n) \) for Legendre polynomials, we set \( \alpha = \beta = \gamma = \delta = 0 \). Then we have

\[ a(k, m, n) = (-1)^n 2^{\alpha + 1} (\gamma - n + k + 1)_{m + n - k} (m + \gamma + \delta + 1)_k \]

\begin{align*}
&\times \frac{\Gamma(\alpha + \gamma + k + 1) \Gamma(\beta + \delta + k + 1)}{\Gamma(\alpha + \beta + \gamma + \delta + 2k + 2)} \\
&\times \sum_{j=0}^{k} \frac{(-n)_j (n + \alpha + \beta + 1)_j (-k)_j}{(m - k + 1)_j (-m - \gamma - \delta - k)_j j!} \\
&\times \sum_{l=0}^{m+n-k} \frac{(-n - m + k)_l (m - n + \gamma + \delta + k + 1)_l (\alpha + \gamma + k + 1)_l}{(\gamma - n + k + 1)_l (\alpha + \beta + \gamma + \delta + 2k + 2)_l l!}.
\end{align*}

Because of the orthogonality and symmetricity of Legendre polynomials, the expression of (3.1) becomes

\[ p_n(x)p_m(x) = \sum_{k=0}^{\min(m, n)} \frac{a(n + m - 2k, m, n)}{h_{n + m - 2k}} p_{n+m-2k}(x). \]
Then we have the coefficients

\[ a(m+n-2k,m,n) \]

Substituting \( n + m - 2k \) for \( k \) in (3.7), we have

\[ a(m+n-2k,m,n) = \frac{(-1)^n 2(\text{m}+1)_{m+n-2k} \text{G}(m+n-2k+1)}{(-n+2k)! n! \text{G}(2m+2n-4k+2)} \]

\times \sum_{j=0}^{n+m-2k} \frac{(-n)_j (n+1)_j (-m-n+2k)_j}{(-n+2k+1)_j (-2m-n+2k)_j j!} \]

\times \prod_{l=0}^{2k} \frac{(-2k)_l (2m-2k+1)_l (m+n-2k+1)_l}{(m-2k+1)_l (2m+2n-4k+2)_l l!},

that is,

\[ a(m+n-2k,m,n) = \frac{(-1)^n 2(\text{m}+1)_{m+n-2k} \text{G}(m+n-2k+1)}{(-n+2k)! n! \text{G}(2m+2n-4k+2)} \]

\times {}_3F_2 \left( \begin{array}{l} -n, \quad n+1, \quad -m-n+2k; \quad 1 \\ -2m-n+2k, \quad -n+2k+1 \end{array} \right) \]

\times {}_3F_2 \left( \begin{array}{l} -2k, \quad 2m-2k-1, \quad m+n-2k+1; \quad 1 \\ -2k+1, \quad 2(m+n-2k+1) \end{array} \right).

Here, we first apply Whipple’s transformation (2.3) to first \(_3F_2\) series and Watson’s transformation (2.4) to second \(_3F_2\), after that, also apply Legendre duplication formula (2.5) and Euler’s reflection formula in Theorem 2.2, then (3.9) can be simplified as follow

\[ a(m+n-2k,m,n) = \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{-k} (\frac{1}{2})_{n-k} (m+n-k)!}{k!(m-k)!(n-k)!} \]

\times \frac{\Gamma(1+m+n-k)2^{2m+2n-2k+1}}{\Gamma(2m+2n-2k+2)}.

Then we have the coefficients

\[ a(m+n-2k,m,n) = \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{m-k} (\frac{1}{2})_{n-k} (m+n-k)!}{k!(m-k)!(n-k)!} \]

\times \frac{\Gamma(1+m+n-k)2^{2m+2n-2k}}{\Gamma(2m+2n-2k+1)} \frac{(2m+2n-4k+1)}{(2m+2n-2k+1)}.

Since we can simplify the ratio of two gamma functions in (3.11) as follow

\[ \frac{\Gamma(1+m+n-k)}{\Gamma(2m+2n-2k+1)} = \frac{1}{2^{2m+2n-2k} \left(\frac{1}{2}\right)_{m+n-k}},
we have the result
\[
\frac{a(m + n - 2k, m, n)}{h_{m+n-2k}} = \frac{2m + 2n + 1 - 4k}{2m + 2n + 1 - 2k} \times \frac{(\frac{1}{2})_k (\frac{1}{2})_{m-k} (\frac{1}{2})_{n-k} (m + n - k)!}{k! (m - k)! (n - k)! (\frac{1}{2})_{m+n-k}}.
\]
This completes our proof. \(\square\)

References


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