GROUP ACTIONS IN A REGULAR RING

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Abstract. Let $R$ be a ring with identity, $X$ the set of all nonzero, nonunits of $R$ and $G$ the group of all units of $R$. We will consider two group actions on $X$ by $G$, the regular action and the conjugate action. In this paper, by investigating two group actions we can have some results as follows: First, if $G$ is a finitely generated abelian group, then the orbit $O(x)$ under the regular action on $X$ by $G$ is finite for all nilpotents $x \in X$. Secondly, if $F$ is a field in which $2$ is a unit and $F \setminus \{0\}$ is a finitely generated abelian group, then $F$ is finite. Finally, if $G$ in a unit-regular ring $R$ is a torsion group and $2$ is a unit in $R$, then the conjugate action on $X$ by $G$ is trivial if and only if $G$ is abelian if and only if $R$ is commutative.

1. Introduction and basic definitions

Let $R$ be a ring with identity $1$, $X$ the set of all nonzero, nonunits of $R$ and $G$ the group of all units of $R$. In this paper, we will consider two group actions of $G$ on $X$. We call the action, $((g, x) \rightarrow gx)$ from $G \times X$ to $X$, regular action and the action, $((g, x) \rightarrow gxg^{-1})$ from $G \times X$ to $X$, conjugate action. If $\phi : G \times X \rightarrow X$ is one of the above actions, then for each $x \in X$, we define the orbit of $x$ by $O(x) = \{\phi(g, x) : g \in G\}$. We also define the stabilizer of $x$ by $\text{Stab}(x) = \{g \in G : \phi(g, x) = x\}$. Recall that $G$ is transitive on $X$ (or $G$ acts transitively on $X$) if there is an $x \in X$ with $O(x) = X$ and the group action on $X$ by $G$ is trivial if $O(x) = \{x\}$ for all $x \in X$.

We define the index of a nilpotent $x \in R$ by the positive integer $n$ such that $x^n = 0$ and $x^{n-1} \neq 0$ and denote it by $\text{ind}(x)$. In particular, the additive zero $0$ in $R$ is nilpotent of index $1$. We define the index

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of $R$ by the supremum of the indices of all nilpotents of $R$ and denote it by $\text{ind}(R)$. If $\text{ind}(R)$ is finite, then we say that $R$ has a bounded index. A ring $R$ is von-Neumann regular (or simply regular) (resp. unit-regular) provided that for any $a \in R$ there exists an element $r \in R$ (resp. $u \in G$) such that $a = ara$ (resp. $a = lua$). A ring $R$ is strongly regular provided that for any $a \in R$ there exists an element $r \in R$ such that $a = ra^2$. Also a regular ring $R$ is abelian provided all idempotents in $R$ is central. It is known in [1] that $R$ is abelian regular ring if and only if $R$ is strongly regular and that an abelian regular ring is unit-regular. It is also known by [1, Corollary 7.1] that if $R$ is regular of bounded index, then $R$ is unit-regular.

Through this paper, unless stated otherwise, $R$ is a ring with identity $1$, $G$ is the group of all units of $R$ and $X$ is the set of all nonzero, nonunits in $R$. Also for each $x \in X$, $O(x)$ is considered as an orbit of $x$ under the given group action. Let $N(R)$ be the set of all nilpotents of $R$. In [3], If $R$ is a ring such that $X$ is a finite union of orbits under the regular action, then the Jacobson radical $J$ is a nil ideal of $R$, and so $J \subseteq N(R)$.

2. Regular action in regular rings

The following theorem has been proved in [3]:

**Theorem 2.1.** Let $R$ be a ring such that $X$ is a finite union of orbits under the regular action on $X$ by $G$. Then $X$ is the set of all right zero-divisors of $R$. Moreover, if $X$ is a nonempty finite set, then $R$ is a finite ring.

**Lemma 2.2.** Let $R$ be a ring such that $X$ is a union of $n$ orbits under the regular action on $X$ by $G$. Then $n + 1 \geq \text{ind}(R)$.

**Proof.** Let $x \in R$ be any nilpotent such that $\text{ind}(x) = m$ ($m \geq 2$). Consider $O(x^i)$ and $O(x^j)$, where $i > j$. Then $O(x^i) \cap O(x^j) = \emptyset$. Indeed, if $O(x^i) \cap O(x^j) \neq \emptyset$, then $x^i = gx^j$ for some $g \in G$. Thus $0 = x^m = x^i x^{m-i} = gx^j x^{m-i} = gx^{m-i+j}$, and so $x^{m-i+j} = 0$, a contradiction. Hence $O(x^i) \cap O(x^j) = \emptyset$, where $i > j$, which implies that $O(x), O(x^2), \ldots, O(x^{m-1})$ are disjoint orbits. Therefore, $n \geq m - 1$, and then $n + 1 \geq \text{ind}(R)$. \hfill $\square$

In general, the converse of Lemma 3.2 is not true by taking an example of a ring $R = \mathbb{Z}_4 \oplus \mathbb{Z}$. Indeed, $\text{ind}(R) = 2$ but $X$ is not a union of finite number of orbits under the regular action on $X$ by $G$. 

Corollary 2.3. Let $R$ be a ring such that $X$ is a union of a finite number of orbits under the regular action on $X$ by $G$. Then $R$ is regular if and only if $R$ is unit-regular.

Proof. It follows from Lemma 2.2 and [1, Corollary 7.11]. □

Proposition 2.4. Let $R$ be a ring such that $X \neq \emptyset$. If the regular action on $X$ by $G$ is transitive, then for all $y, z \in X$, $yz = 0$.

Proof. By Lemma 2.2, $2 \geq \text{ind}(R)$. Since $X \neq \emptyset$, $2 = \text{ind}(R)$ and so there exists $x \in X$ with $x^2 = 0$. Since the regular action on $X$ by $G$ is transitive, $X = O(x)$. For all $y, z \in X$, $y = gx, z = hx$ for some $g, h \in G$, and then $yz = (gx)(hx) = g(xh)x = g(kx)x = (gk)x^2 = 0$ for some $k \in G$. □

Remark 1. (1) By Propositon 2.4, we can observe that if $R$ is a ring such that $X \neq \emptyset$ and the regular action on $X$ by $G$ is transitive, then $X \cup \{0\} = N(R)$ and for all $y, z \in X$, $y + z \in X$. Since for all $y, z \in X$, $yz = 0$, $R$ is a local ring, i.e., $J = X \cup \{0\}$.
(2) In general, we can observe that if $R$ is a ring such that $X$ is a union of $n$ orbits under the regular action on $X$ by $G$, then there exists $x \in X$ with $\text{ind}(x) = n + 1$ if and only if $R$ is a local ring.
(3) In any regular ring $R$ with $X \neq \emptyset$, there is no transitive regular action on $X$ by $G$.

Example 1. Let $p$ be any prime number, $k$ be any positive integer and let $\mathbb{Z}_{p^k} = \{0, 1, \ldots, p^k - 1\}$ be the ring of integers modulo $p^k$. Then $X = O(p) \cup O(p^2) \cup \cdots \cup O(p^{k-1}) = J \setminus \{0\}$ is a union of $k - 1$ orbits $O(p), \ldots, O(p^{k-1})$ under the regular action, and so $\mathbb{Z}_{p^k}$ is a local ring.

Example 2. Let $p$ be any prime number and let $\mathbb{Z}_{p^2} = \{0, 1, \ldots, p^2 - 1\}$ be the ring of integers modulo $p^2$. Then $N(\mathbb{Z}_{p^2}) = \{0, p, 2p, \ldots, (p - 1)p\} = X \cup \{0\}$ and so in $\mathbb{Z}_{p^2}$ there is a transitive regular action on $X$ by $G$, i.e., $X = O(p)$.

Proposition 2.5. Let $R$ be a ring. If $g \in G$ is of a finite order such that $O(1 - g) = \{1 - g\}$ under the regular action on $X$ by $G$, then $\text{ind}(1 - g) = 2$ and $g + g^{-1} = 2$.

Proof. Since $g \in G$ is of a finite order, $1 - g \in X$ for all $g(\neq 1) \in G$. If $O(1 - g) = \{1 - g\}$ for some $g \in G$ under the regular action on $X$ by $G$, then $g(1 - g) = 1 - g$, and so $(1 - g)^2 = 0$. Hence $\text{ind}(1 - g) = 2$, and then $g + g^{-1} = 2$. □
Example 3. Let $n$ be any positive integer and let $\mathbb{Z}_{4n} = \{0, 1, \ldots, 4n - 1\}$ be the ring of integers modulo $4n$. Then under the regular action on $X$ by $G$, $O(2n) = \{2n\}$ and $1 - 2n \in G$ is an involution. Thus $(1 - 2n) + (1 - 2n)^{-1} = (1 - 2n) + (1 - 2n) = 2$.

Proposition 2.6. A ring $R$ is strongly regular if and only if $O(x) = O(x^2)$ for all $x \in X$ under the regular action on $X$ by $G$.

Proof. Suppose that $R$ is strongly regular and let $x \in X$ be arbitrary. Since strongly regular ring is unit-regular, there exists $g \in G$ such that $x = xgx$. Since strongly regular ring is also abelian-regular, the idempotent $gx$ is central, and so $x = x(g)x = gx = gx^2$. Hence for all $x \in X$, $O(x) = O(x^2)$ under the regular action on $X$ by $G$. Conversely, suppose that $O(x) = O(x^2)$ for all $x \in X$ under the regular action on $X$ by $G$. Let $a \in R$ be an arbitrary element. If $a \in G$, then $a^{-1}a^2 = a$. If $a \in X$, then by assumption, $O(a^2) = O(a)$ and so $ga^2 = a$ for some $g \in G$. Hence $R$ is strongly regular. \hfill \□

Proposition 2.7. Let $R$ be a ring whose characteristic is not 2. If $x(\neq 0) \in N(R)$ with $\text{ind}(x) = n$ and $1 + x \in G$ is of finite order $k$, then $kx^n = 0$ and $2k^i x^n = 0$ for all $i = 1, \ldots, n - 2$.

Proof. Since $x(\neq 0) \in N(R)$ with $\text{ind}(x) = n$, $x^n = 0 \neq x^{n-1}$. Since $1 + x \in G$ has order $k$, $1 = (1 + x)^k = 1 + kx + \binom{k}{2} x^2 + \binom{k}{3} x^3 + \cdots + x^k$. Thus we have an equation $kx + \binom{k}{2} x^2 + \binom{k}{3} x^3 + \cdots + x^k = 0$ (1). By multiplying $x^{n-2}$ to both sides of (1), we can have $kx^{n-1} = 0$ since $x^n = 0 \neq x^{n-1}$. Next, by multiplying $2kx^{n-3}$ to both sides of (1), we can have $2kx^{n-2} = 0$ since $x^n = 0$ and $kx^{n-1} = 0$. By mathematical induction on $n$ and by multiplying $2k^{i-1}x^{n-i-2}$ to both sides of (1), we can have $2k^i x^{n-i-1} = 0$ for all $i = 1, \ldots, n - 2$. \hfill \□

Example 4. Let $\mathbb{Z}_{288} = \{0, 1, \ldots, 287\}$ be the ring of integers modulo 288. Let $6 \in N(\mathbb{Z}_{288})$ be an element of $\text{ind}(6) = 5$. Note that the order of $7$ ($= 1 + 6$) is 12. Then we can have $12 \cdot 6^4 = 2 \cdot 12 \cdot 6^3 = 2 \cdot 12^2 \cdot 6^2 = 2 \cdot 12^3 \cdot 6 = 0$.

Remark 2. (1) Let $R$ be a ring whose characteristic is 2. If $x(\neq 0) \in N(R)$ with $\text{ind}(x) = n$ and $1 + x \in G$ is of finite order $k$, then $k$ is even and $x^k = 0$. Indeed, if $k$ is odd, then we have $x + x^{k-1} + x^k = 0$ from an equation $(1 + x)^k = 1 + x + x^{k-1} + x^k = 1$, and so $x^{n-1} = 0$, a contradiction. Hence $k$ is even and then we have $x^k = 0$ from an equation $(1 + x)^k = 1 + x^k = 1$. 


(2) Let $R$ be a ring with the characteristic 0 in which there is no left (or right) zero-divisors and let $M_m(R)$ ($m \geq 2$) be a full matrix ring of $m \times m$ over $R$. Then if $x(\neq 0) \in M_m(R)$ is nilpotent, then the order of $1 + x$ is not finite. In fact, assume that there exists a nilpotent $x(\neq 0) \in M_m(R)$ such that $\text{ind}(x) = n$ and $1 + x \in G$ is of finite order $k$. Since $x^{n-1} \neq 0$ and $kx^{n-1} = 0$ by Proposition 2.7, there exists $a(\neq 0) \in R$ such that $ka = 0$. Since $a \neq 0$ and $R$ has no left (or right) zero-divisors, we have $k \cdot 1 = 0$, which is a contradiction to the assumption that the characteristic of $R$ is 0.

**Corollary 2.8.** Let $R$ be a ring such that $X \neq \emptyset$ and the regular action on $X$ by $G$ is transitive. If the order of $1 + x \in G$ is finite for some $x \in X$, then the order of $1 + y \in G$ is equal to the order of $1 + x$ for all $y \in X$, i.e., $1 + J$ is a torsion group.

**Proof.** Let $k$ be the order of $1 + x$. Since the regular action on $X$ by $G$ is transitive, $X \cup \{0\} = J$, $y^2 = 0$ for all $y \in X$ and $y = gx$ for some $g \in G$ by Proposition 2.4. Since the order of $1 + x$ is $k$ and $x^2 = 0$, $kx = 0$ by Proposition 2.7. Note that $kx = 0$ if and only if $ky = 0$ for all $y \in X$ and also the order of $1 + x$ is equal to the one of $1 + y$ for all $y \in X$. Hence $1 + J$ is a torsion group.

**Example 5.** Let $p$ be any prime number. By Example 2, in a ring $\mathbb{Z}_p^2$, there is a transitive regular action on $X$ by $G$. Note that the order of $1 + y$ is $p$ for all $y \in X$.

**Theorem 2.9.** Let $R$ be a ring in which $G$ is a finitely generated abelian group. If $x(\neq 0) \in N(R)$, then the orbit $O(x)$ under the regular action on $X$ by $G$ is finite.

**Proof.** If $O(x) = \{x\}$ or $G = \{1\}$, then $O(x) = \{x\}$, and so $O(x)$ is finite. Thus suppose that $O(x) \neq \{x\}$ and $G \neq \{1\}$. Then $|O(x)| > 1$ and $\text{Stab}(x)$ is a proper subgroup of $G$. Let $H = \text{Stab}(x)$ and let $S = \{a_1, a_2, \ldots, a_k\}$ be the set of generators of $G$. Since $x(\neq 0) \in N(R)$, $x^n = 0$ and $x^{n-1} \neq 0$ for some positive integer $n$. Thus $(1 + x^{n-1})x = x$ implies that $1 + x^{n-1} \in H$ and so $H \neq \{1\}$. Since $H$ is a proper subgroup of $G$, $H$ is generated by $\{a_1^{s_1}, a_2^{s_2}, \ldots, a_k^{s_k}\}$ for some nonnegative integers $s_1, s_2, \ldots, s_k$ but not all $s_i = 1$. Let $g = \prod_{i=1}^k a_i^{s_i} \in G$ be arbitrary. Then $gx = \left(\prod_{i=1}^k a_i^{t_i}\right)x = \left(\prod_{s_i \geq 2} a_i^{t_i}\right)x$. For each $s_i \geq 2$, by the division algorithm for $\mathbb{Z}$, $t_i = r_i + q_i s_i$ for some $r_i, q_i \in \mathbb{Z}$ where $s_i - 1 \geq r_i \geq 0$. Thus for all $g \in G$, $gx = \left(\prod_{s_i \geq 2} a_i^{t_i}\right)x = \left(\prod_{s_i \geq 2} a_i^{r_i}\right)x$, and so $O(x)$ is finite.
Corollary 2.10. Let $R$ be a ring such that $X \neq \emptyset$ and the regular action on $X$ by $G$ is transitive. If $G$ is a finitely generated abelian group, then $R$ is finite. In addition, if $2$ is a unit in $G$, then $R$ is commutative.

Proof. $R$ is finite by Theorem 2.1 and Theorem 2.9. Also if $2$ is a unit in $G$, then $R$ is commutative by [2, Theorem 2.11].

Theorem 2.11. Let $R$ be a ring such that $G$ is a finitely generated abelian group and $2$ is a unit in $G$. If $e \in X$ is idempotent, then $O(e)$ under the regular action is finite.

Proof. The proof is similar to the one of Theorem 2.9. If $O(e) = O(e^2) = \{e\}$ or $G = \{1\}$ for idempotent $e \in X$, then $O(e) = \{e\}$, and so $O(e)$ is finite. Suppose that $O(e) \neq \{e\}$ and $G \neq \{1\}$. Then $|O(e)| > 1$ and Stab($e$) is a proper subgroup of $G$. Let $H = \text{Stab}(e)$ and let $S = \{a_1, a_2, \ldots, a_k\}$ be the set of generators of $G$. Since $e \in X$ is idempotent and $2$ is a unit in $G$, $2e - 1(\neq 1) \in G$. Thus $(2e - 1)e = e$ implies that $2e - 1 \in H$ and so $H \neq \{1\}$. Since $H$ is a proper subgroup of $G$, $H$ is generated by $\{a_1^{s_1}, a_2^{s_2}, \ldots, a_k^{s_k}\}$ for some nonnegative integers $s_1, s_2, \ldots, s_k \geq 0$ but not all $s_i = 1$. Let $g = \prod_{i=1}^{k} a_i^{t_i} \in G$ be arbitrary. Then $ge = \left(\prod_{i=1}^{k} a_i^{t_i}\right) e = \left(\prod_{s_i \geq 2} a_i^{r_i}\right) e$. For each $s_i \geq 2$, by the division algorithm for $\mathbb{Z}$, $t_i = r_i + q_i s_i$ for some $r_i, q_i \in \mathbb{Z}$, where $s_i - 1 \geq r_i \geq 0$. Thus for all $g \in G$, $gx = \left(\prod_{s_i \geq 2} a_i^{r_i}\right) e = \left(\prod_{s_i \geq 2} a_i^{r_i}\right) e$, and so $O(e)$ is finite.

Corollary 2.12. Let $R$ be a unit-regular ring. If $G$ is a finitely generated abelian group and $2$ is a unit in $G$, then every orbit under the regular action is a finite set.

Proof. By [4, Lemma 2.3], every orbit is $O(e)$ for some idempotent $e \in X$ and so is $O(e)$ is a finite set by Theorem 2.11.

Corollary 2.13. Let $R$ be a regular ring such that $X \neq \emptyset$ and $G$ is a finitely generated abelian group and $2$ is a unit in $G$. Then the following are equivalent:

1. $X$ is a union of finite number of orbits under the regular action on $X$ by $G$;
2. $X$ is finite;
3. $R$ is finite commutative.
Proof. (1) \(\Rightarrow\) (2). Suppose that \(X\) is a union of finite number of orbits under the regular action on \(X\) by \(G\). Since \(\text{ind}(R)\) is finite, a regular ring \(R\) is unit-regular by Corollary 2.3. Since \(G\) is abelian group, \(R\) is commutative by [4, Theorem 3.2]. By Corollary 2.12, every orbit under the regular action on \(X\) by \(G\) is finite. Since there exists a finite number of orbits under the regular action on \(X\) by \(G\), \(X\) is finite.

(2) \(\Rightarrow\) (3). It follows from Theorem 2.1.

(3) \(\Rightarrow\) (1). It is clear. \(\square\)

It is well-known in the field theory that if \(F\) is a finite field, then \(F \setminus \{0\}\) is a cyclic group. But in [4, Theorem 3.7] the converse could be true in case that 2 is a unit in \(F\). In general, we have the following Theorem:

**Theorem 2.14.** Let \(F\) be a field in which 2 is a unit. If \(F \setminus \{0\}\) is a finitely generated abelian group, then \(F\) is a finite field.

**Proof.** Consider a ring \(R = F \times F\). Then \(R\) is a unit-regular ring and \((2, 2)\) is a unit in \(R\). Since \(F \setminus \{0\}\) is a finitely generated abelian group, the group of units of \(R\) is a finitely generated abelian group. Take an idempotent \((1, 0)\) \(\in R\). Then the orbit \(O((1, 0))\) is equal to \(\{(g, 0) : g \in F \setminus \{0\}\}\) under the regular action on \(X\) by \(G\). By Theorem 2.11, \(O((1, 0))\) is a finite set, i.e., \(|O((1, 0))| = |F \setminus \{0\}|\). Hence \(F\) is a finite field. \(\square\)

**3. Conjugate action in regular rings**

We begin with the following Lemma:

**Lemma 3.1.** Let \(R\) be a ring such that \(G\) is a torsion group. If the conjugate action on \(X\) by \(G\) is trivial, then \(G\) is abelian.

**Proof.** Let \(g, h \in G\) be arbitrary. Since the order of \(g\) is finite, \(1 - g \in X\). Since the conjugate action on \(X\) by \(G\) is trivial, the orbit \(O(1 - g) = \{1 - g\}\), i.e., \(h(1 - g)h^{-1} = 1 - g\) and so \(gh = hg\). Hence \(G\) is abelian. \(\square\)

Note that the converse of Lemma 3.1 is not true by the following example:
Example 6. Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\} \). Then \( R \) is a non-commutative ring but \( G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \) is an abelian group. The orbit of \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) is equal to \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \not= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \), and so the conjugate action on \( X \) by \( G \) is not trivial.

Proposition 3.2. Let \( R \) be a unit-regualr ring such that \( G \) is a torsion group and 2 is a unit. Then the following are equivalent:

1. The conjugate action on \( X \) by \( G \) is trivial;
2. \( G \) is abelian;
3. \( R \) is commutative.

Proof. (1) \( \Rightarrow \) (2). It follows from the Lemma 3.1.
(2) \( \Rightarrow \) (3). It follows from [4, Theorem 3.2].
(3) \( \Rightarrow \) (1). It is clear. \( \square \)

Proposition 3.3. Let \( R \) be a ring such that the conjugate action on \( X \) by \( G \) is transitive. If \( N(R) \not= \{0\} \), then for all \( y, z \in X \), \( yz = 0 \).

Proof. Since \( N(R) \not= \{0\} \), there exists a nonzero \( x \in N(R) \) such that \( \text{ind}(x) = n \) for some positive integer \( n \). Then \( O(x^2) = 0 \). Indeed, if \( O(x^2) \not= 0 \), then \( O(x) = O(x^2) = X \) since the conjugate action on \( X \) by \( G \) is transitive, and so \( x^2 = gxg^{-1} \) for some \( g \in G \), which implies that \( \text{ind}(x) < n \), a contradiction. Next, for all \( y, z \in X \), \( y = hxh^{-1}, z = kzk^{-1} \) for some \( h, k \in G \), and so \( y^2 = hx^2h^{-1} = 0, z^2 = kx^2k^{-1} = 0 \). Note that for all \( y, z \in X \), \( y + z \in X \) and then \( 0 = (y + z)^2 = yz + zy \), and so \( yz = -zy \). Hence \( yz = (hxh^{-1})(kzk^{-1}) = hx(h^{-1}kx)k^{-1} = -h(h^{-1}kx)xk^{-1} = -kx^2k^{-1} = 0 \).

Remark 3. By Proposition 3.3, if \( R \) is a ring such that the conjugate action on \( X \) by \( G \) is transitive and \( N(R) \not= \{0\} \), then \( R \) is a local ring and \( J^2 = (0) \).

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