ON C-BOCHNER CURVATURE TENSOR
OF A CONTACT METRIC MANIFOLD

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Abstract. We prove that a \((\kappa, \mu)\)-manifold with vanishing \(E\)-Bochner curvature tensor is a Sasakian manifold. Several interesting corollaries of this result are drawn. Non-Sasakian \((\kappa, \mu)\)-manifolds with \(C\)-Bochner curvature tensor \(B\) satisfying \(B(\xi, X) \cdot S = 0\), where \(S\) is the Ricci tensor, are classified. \(N(\kappa)\)-contact metric manifolds \(M^{2n+1}\), satisfying \(B(\xi, X) \cdot R = 0\) or \(B(\xi, X) \cdot B = 0\) are classified and studied.

1. Introduction

In [4], Blair, Koufogiorgos and Papantoniou introduced the class of contact metric manifolds \(M\) with contact metric structures \((\varphi, \xi, \eta, g)\), in which the curvature tensor \(R\) satisfies the equation

\[
R(X, Y) \xi = (\kappa I + \mu h) R_0(X, Y) \xi, \quad X, Y \in TM,
\]

where \((\kappa, \mu) \in \mathbb{R}^2\), \(2h\) is the Lie derivative of \(\varphi\) in the direction \(\xi\) and \(R_0\) is given by

\[
R_0(X, Y) Z = g(Y, Z) X - g(X, Z) Y, \quad X, Y, Z \in TM.
\]

A contact Riemannian manifold belonging to this class is called a \((\kappa, \mu)\)-manifold. Characteristic examples of non-Sasakian \((\kappa, \mu)\)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one.

On the other hand, S. Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [5]. A geometric
meaning of the Bochner curvature tensor is given by D. Blair in [1]. By using the Boothby-Wang’s fibration [6], M. Matsumoto and G. Chūman constructed $C$-Bochner curvature tensor [10] from the Bochner curvature tensor. In [7], H. Endo defined $E$-Bochner curvature tensor as an extended $C$-Bochner curvature tensor and showed that a $K$-contact manifold with vanishing $E$-Bochner curvature tensor is a Sasakian manifold.

A $K$-contact manifold is always a contact metric manifold, but the converse is not true in general. Thus, it is worthwhile to study $C$-Bochner curvature tensor and $E$-Bochner curvature tensor in contact metric manifolds. Extending the result of H. Endo[7] to a $(\kappa, \mu)$-manifold, we prove the following

**Theorem 1.1.** A $(\kappa, \mu)$-manifold with vanishing $E$-Bochner curvature tensor is a Sasakian manifold.

Then, we draw several corollaries of this result to $N(\kappa)$-contact metric manifolds [15], the unit tangent sphere bundles [4], $N(\kappa)$-contact space forms [8] and $(\kappa, \mu)$-space forms [9].

In [12] and [14], contact metric manifolds satisfying $R(X, \xi) \cdot S = 0$ are studied. Motivated by these studies, we classify non-Sasakian $(\kappa, \mu)$-manifolds with $C$-Bochner curvature tensor $B$ satisfying $B(\xi, X) \cdot S = 0$. In fact, we prove the following

**Theorem 1.2.** Let $M^{2n+1}$ be a non-Sasakian $(\kappa, \mu)$-manifold. If the $C$-Bochner curvature tensor $B$ satisfies $B(\xi, X) \cdot S = 0$, then we have one of the following:

(i) $M^{2n+1}$ is flat and 3-dimensional;
(ii) $M^{2n+1}$ is locally isometric to $E^{n+1}(0) \times S^n(4)$;
(iii) $M^{2n+1}$ is an $\eta$-Einstein manifold;
(iv) $M^{2n+1}$ is a 3-dimensional Einstein manifold.

In a recent paper [13], $N(\kappa)$-contact metric manifolds satisfying $R(X, Y) \cdot B = 0$ are studied. Hence, we classify $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$. In fact, we prove the following

**Theorem 1.3.** Let $M^{2n+1}$ ($n > 1$) be an $N(\kappa)$-contact metric manifold. If the $C$-Bochner curvature tensor $B$ satisfies $B(\xi, X) \cdot R = 0$, then either $M^{2n+1}$ is a Sasakian manifold or $M^{2n+1}$ is locally isometric to the sphere $S^{2n+1}(1)$.

In the last, motivated by studies of contact metric manifolds satisfying $R(X, \xi) \cdot R = 0$ ([12, 14]), we consider $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$. In fact, we prove the following
Theorem 1.4. Let $M^{2n+1}$ be an $N(\kappa)$-contact metric manifold. Then the $C$-Bochner curvature tensor $B$ satisfies $B(\xi, X) \cdot B = 0$ if and only if $M^{2n+1}$ is a Sasakian manifold.

2. Contact metric manifolds

A $(2n+1)$-dimensional differentiable manifold $M$ is called an almost contact manifold if either its structural group can be reduced to $U(n) \times 1$ or equivalently, there is an almost contact structure $(\varphi, \xi, \eta)$ consisting of a $(1,1)$ tensor field $\varphi$, a vector field $\xi$, and a 1-form $\eta$ satisfying
\begin{align}
(1) \quad & \varphi^2 = -I + \eta \otimes \xi, \quad \text{and (one of)} \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0. \\
\end{align}

An almost contact structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M \times \mathbb{R}$ defined by
\begin{align}
(2) \quad J(X, \lambda \frac{d}{dt}) = (\varphi X - \lambda \xi, \eta(X) \frac{d}{dt})
\end{align}
is integrable, where $X$ is tangent to $M$, $t$ the coordinate of $\mathbb{R}$ and $\lambda$ a smooth function on $M \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of $\varphi$. Let $g$ be a compatible Riemannian metric with $(\varphi, \xi, \eta)$, that is,
\begin{align}
(3) \quad & g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \\
\end{align}
or equivalently,
\begin{align}
(4) \quad & g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X)
\end{align}
for all $X, Y \in TM$. Then, $M$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\varphi, \xi, \eta, g)$.

An almost contact metric structure becomes a contact metric structure if
\begin{align}
(5) \quad g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in TM.
\end{align}

In a contact metric manifold, the $(1,1)$-tensor field $h$ is symmetric and satisfies
\begin{align}
(6) \quad & h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla \xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0,
\end{align}
where $\nabla$ is the Levi-Civita connection.

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if
\begin{align}
(7) \quad & \nabla_X \varphi = R_0(\xi, X), \quad X \in TM,
\end{align}
while a contact metric manifold $M$ is Sasakian if and only if
\begin{equation}
R(X,Y)\xi = R_0(X,Y)\xi, \quad X,Y \in TM.
\end{equation}
A contact metric manifold is called a $K$-contact manifold if $\xi$ is a Killing vector field. An almost contact metric manifold is $K$-contact if and only if $\nabla \xi = -\varphi$. A $K$-contact manifold is a contact metric manifold, while converse is true if $h = 0$. A Sasakian manifold is always a $K$-contact manifold. A 3-dimensional $K$-contact manifold is a Sasakian manifold. Thus, a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.

The $(\kappa, \mu)$-nullity distribution $N(\kappa, \mu)$ ([4, 12]) of a contact metric manifold $M$ is defined by
\begin{equation}
N(\kappa, \mu) : p \rightarrow N_p(\kappa, \mu) = \{U \in T_pM \mid R(X,Y)U = (\kappa I + \mu h)R_0(X,Y)U\}
\end{equation}
for all $X,Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold with $\xi \in N(\kappa, \mu)$ is called a $(\kappa, \mu)$-manifold. For a $(\kappa, \mu)$-manifold, it follows that $h^2 = (\kappa - 1)\varphi^2$. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. In fact, for a $(\kappa, \mu)$-manifold, the conditions of being Sasakian manifold, $K$-contact manifold, $\kappa = 1$ and $h = 0$ are all equivalent. If $\mu = 0$, the $(\kappa, \mu)$-nullity distribution $N(\kappa, \mu)$ is reduced to the $\kappa$-nullity distribution $N(\kappa)$ ([15]). If $\xi \in N(\kappa)$, then we call a contact metric manifold $M$ an $N(\kappa)$-contact metric manifold. For more details we refer to [3].

3. $(\kappa, \mu)$-manifolds with vanishing $E$-Bochner curvature tensor

In [10], Matsumoto and Chūman defined the $C$-Bochner curvature tensor in an almost contact metric manifold as follows:
\begin{equation}
B(X,Y) = R(X,Y) - \frac{m - 4}{2n + 4} R_0(Y,X) + \frac{1}{2n + 4} \left\{ R_0(QY,X) - R_0(QX,Y) + R_0(Q\varphi Y, \varphi X) - R_0(Q\varphi X, \varphi Y) + 2g(Q\varphi X, Y) \varphi + 2g(\varphi X, Y) \varphi + \eta(Y) R_0(QX,\xi) + \eta(X) R_0(QY,\xi) \right\} + \frac{m + 2n}{2n + 4} \{ R_0(\varphi Y, \varphi X) + 2g(\varphi X, Y) \varphi + \frac{m}{2n + 4} \{ \eta(Y) R_0(\xi, X) + \eta(X) R_0(Y, \xi) \},
\end{equation}
where $Q$ is the Ricci operator, $r$ is the scalar curvature and $m = \frac{2n+r}{2n+2}$.

For a $(\kappa, \mu)$-manifold $M^{2n+1}$, we have

$$R(X,Y)\xi = (\kappa I + \mu h) R_0(X,Y) \xi,$$

which is equivalent to

$$R(\xi, X) = R_0(\xi, (\kappa I + \mu h) X) = -R(X, \xi).$$

In particular, we get

$$R(\xi, X) \xi = \kappa(\eta(X)\xi - X) - \mu h X = -R(X, \xi)\xi.$$

From (9), (10), and (11), it follows that

$$B(X, Y)\xi = \left(2 \left(\frac{\kappa - 1}{n+2}\right) I + \mu h\right) R_0(X,Y)\xi,$$

$$B(\xi, X) = R_0 \left(\xi, \left(\frac{2(\kappa - 1)}{n+2} I + \mu h\right) X\right) = -B(X, \xi).$$

Consequently, we have

$$B(\xi, X)\xi = \frac{2(\kappa - 1)}{n+2} (\eta(X)\xi - X) - \mu h X = -B(X, \xi)\xi,$$

$$\eta(B(X,Y)\xi) = 0,$$

$$\eta(B(\xi, X)Y) = \frac{2(\kappa - 1)}{n+2} (g(X,Y) - \eta(X)\eta(Y)) + \mu g(hX,Y).$$

In [7], H. Endo extended the concept of $C$-Bochner curvature tensor to $E$-Bochner curvature tensor as follows:

$$B^e(X,Y)Z = B(X,Y)Z - \eta(X)B(\xi,Y)Z - \eta(Y)B(X,\xi)Z - \eta(Z)B(X,Y)\xi.$$ 

Then, he showed that a $K$-contact manifold with vanishing $E$-Bochner curvature tensor is a Sasakian manifold. Now, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $M^{2n+1}$ be a $(\kappa, \mu)$-manifold. If $E$-Bochner curvature tensor of $M^{2n+1}$ vanishes, then from (15) and (18), we have

$$0 = B^e(X,\xi)\xi = -B(X,\xi)\xi = \frac{2(\kappa - 1)}{n+2} (\eta(X)\xi - X) - \mu h X,$$

which, in view of $h^2 = (\kappa - 1) \phi^2$, implies that

$$h^2 = \frac{(n+2) \mu}{2} h.$$
Taking the trace of (20), we obtain
\[ 0 = \text{trace} (h^2) = 2n (1 - \kappa). \]
From (21), we have \( \kappa = 1 \). Thus, \( M^{2n+1} \) becomes Sasakian.

**Corollary 3.1.** An \( N(\kappa) \)-contact metric manifold with vanishing \( E \)-Bochner curvature tensor is a Sasakian manifold.

The unit tangent sphere bundle \( T_1 M \) equipped with the standard contact metric structure is a \( (\kappa, \mu) \)-manifold if and only if the base manifold \( M \) is of constant curvature \( c \) with \( \kappa = c (2 - c) \) and \( \mu = -2c \) ([4]). In case of \( c \neq 1 \), the unit tangent sphere bundle is non-Sasakian [16]. We denote by \( T_1 M(c) \) the unit tangent sphere bundle of a space of constant curvature \( c \) with standard contact metric structure. Then, applying Theorem 1.1 to \( T_1 M(c) \), we have

**Corollary 3.2.** If \( T_1 M(c) \) is of vanishing \( E \)-Bochner curvature tensor, then \( c = 1 \).

In an almost contact metric manifold, if \( X \) is a unit vector which is orthogonal to \( \xi \), we say that \( X \) and \( \phi X \) span a \( \phi \)-section. If the sectional curvature \( c(X) \) of all \( \phi \)-sections is independent of \( X \), we say that \( M \) is of pointwise constant \( \phi \)-sectional curvature. If an \( N(\kappa) \)-contact metric manifold \( M \) is of pointwise constant \( \phi \)-sectional curvature \( c \), then we say it an \( N(\kappa) \)-contact space form \( M(c) \). The curvature tensor of \( M(c) \) is given by [8].

\[
4 R(X, Y)Z = (c + 3) \{ g(Y, Z) X - g(X, Z) Y \}
+ (c - 1) \{ \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \\
+ \eta(Y) g(X, Z) \xi - \eta(X) g(Y, Z) \xi \\
+ g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y - 2(\phi X, Y) \phi Z \}
+ 4 (\kappa - 1) \{ \eta(Y) \eta(Z) X - \eta(X) \eta(Z) Y \\
+ \eta(X) g(Y, Z) \xi - \eta(Y) g(X, Z) \xi \}
+ 4 \{ g(hY, Z) X - g(hX, Z) Y \\
+ g(Y, Z) hX - g(X, Z) hY \\
+ \eta(X) \eta(Z) hY - \eta(Y) \eta(Z) hX \\
+ \eta(Y) g(hX, Z) \xi - \eta(X) g(hY, Z) \xi \}
+ 2 \{ g(hY, Z) hX - g(hX, Z) hY \\
+ g(\phi hX, Z) \phi hY - g(\phi hY, Z) \phi hX \} \tag{22}
\]
for all \( X, Y, Z \in TM \), where \( c \) is a constant on \( M \) if \( \dim(M) > 3 \).
Now, applying Theorem 1.1 to an $N(\kappa)$-contact space form, we are able to state the following

**Corollary 3.3.** An $N(\kappa)$-contact space form with vanishing $E$-Bochner curvature tensor is a Sasakian space form.

Let $M$ be a $(2n+1)$-dimensional $(\kappa, \mu)$-manifold $(n > 1)$. If $M$ has constant $\varphi$-sectional curvature $c$ then it is called a $(\kappa, \mu)$-space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [9]

$$R(X, Y)Z = \frac{c+3}{4} \{g(Y, Z)X - g(X, Z)Y\}$$

$$+ \frac{c-1}{4} \{2(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X\}$$

$$+ \frac{c+3-4\kappa}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}$$

$$+ \frac{1}{2} \{g(hY, Z)hX - g(hX, Z)hY$$

$$+ g(\varphi hX, Z)\varphi hY - g(\varphi hY, Z)\varphi hX\}$$

$$+ g(\varphi Y, \varphi Z)hX - g(\varphi X, \varphi Z)hY$$

$$+ g(hX, Z)\varphi^2 Y - g(hY, Z)\varphi^2 X$$

$$+ \mu \{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY$$

$$+ g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi\}$$

for all $X, Y, Z \in TM$, where $c + 2\kappa = -1 = \kappa - \mu$ if $\kappa < 1$.

Now, applying Theorem 1.1 to a $(\kappa, \mu)$-contact space form, we are able to state the following

**Corollary 3.4.** A $(\kappa, \mu)$-contact space form with vanishing $E$-Bochner curvature tensor is a Sasakian space form.

**Remark 3.5.** Theorem 1.1, Corollary 3.1, Corollary 3.2, Corollary 3.3, and Corollary 3.4 are valid for vanishing of $C$-Bochner curvature tensor also.

4. $(\kappa, \mu)$-manifolds satisfying $B(\xi, X) \cdot S = 0$

Before proving Theorem 1.2, we give some results and a brief introduction to $\eta$-Einstein $(\kappa, \mu)$-manifold.
Lemma 4.1. [4] In a non-Sasakian \((\kappa, \mu)\)-manifold \(M^{2n+1}\), the Ricci operator \(Q\) is given by
\[
Q = (2(n-1) - n\mu)I + (2(n-1) + \mu)h + (2(1-n) + n(2\kappa + \mu))\eta \otimes \xi.
\]
Consequently, the Ricci tensor \(S\) is given by
\[
S(X, Y) = (2(n-1) - n\mu)g(X, Y) + (2(n-1) + \mu)g(hX, Y) + (2(1-n) + n(2\kappa + \mu))\eta(X)\eta(Y).
\]

We also recall the following theorem due to D. Blair.

Theorem 4.2. [2, 3] A contact metric manifold \(M^{2n+1}\) satisfying \(R(X, Y)\eta = 0\) is locally isometric to \(E^{n+1}(0) \times S^n(4)\) for \(n > 1\) and flat for \(n = 1\).

Theorem 4.3. [15] If an \(N(k)\)-contact metric manifold of dimension \(\geq 5\) is Einstein, then it is necessarily Sasakian.

We also need the following definition.

Definition 4.4. [11] A contact metric manifold \(M\) is said to be \(\eta\)-Einstein if the Ricci operator \(Q\) satisfies
\[
Q = aI + b\eta \otimes \xi,
\]
where \(a\) and \(b\) are smooth functions on the manifold. In particular, if \(b = 0\), then \(M\) is an Einstein manifold.

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let \(M^{2n+1}\) be a non-Sasakian \((\kappa, \mu)\)-manifold. If \(\kappa = 0 = \mu\), then from (10), we have \(R(X, Y)\eta = 0\), which in view of Theorem 4.2 implies that \(M^{2n+1}\) satisfies one of the statements (i) and (ii). Now, we assume that \(\kappa\) and \(\mu\) are not simultaneously zero. From (25), we have
\[
S(hX, Y) = (2(n-1) - n\mu)g(hX, Y) - (\kappa - 1)(2(n-1) + \mu)g(X, Y)
+ (\kappa - 1)(2(n-1) + \mu)\eta(X)\eta(Y),
\]
where we have used \(\eta \circ h = 0\), \(h^2 = (\kappa - 1)\varphi^2\) and (3). The condition \(B(\xi, X) \cdot S = 0\) gives
\[
S(B(\xi, X)Y, \xi) + S(Y, B(\xi, X)\xi) = 0.
\]
In view of \(Q\xi = 2n\kappa\xi\), we get
\[
S(X, \xi) = 2n\kappa\eta(X),
\]
On C-Bochner curvature tensor of a contact metric manifold

which implies that

\[(29) \quad S(B(\xi, X)Y, \xi) = 2n\kappa\eta(B(\xi, X)Y),\]

Using (17) in the above equation, we get

\[(30) \quad S(B(\xi, X)Y, \xi) = 2n\kappa\mu g(hX, Y) + \frac{4\kappa(\kappa - 1)}{n + 2} (g(X, Y) - \eta(X)\eta(Y)).\]

In view of (15) and (28), we have

\[(31) \quad S(B(\xi, X)\xi, Y) = \frac{4\kappa(\kappa - 1)}{n + 2} \eta(X)\eta(Y) - \frac{2(\kappa - 1)}{n + 2} S(X, Y) - \mu S(hX, Y).\]

From (27), (30), and (31), we have

\[(32) \quad \frac{2(\kappa - 1)}{n + 2} S(X, Y) = \frac{4\kappa(\kappa - 1)}{n + 2} g(X, Y) + 2n\mu g(hX, Y) - \mu S(hX, Y).\]

Finally, from (25), (26), and (32), we have

\[(33) \quad S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y),\]

where

\[a = \frac{(\kappa - 1)(4\kappa + (n + 2)(2(n - 1) + \mu)\mu) - \mu(2 + n\mu)(2(n - 1) - n\mu)}{2(n - 1) + \mu},\]

and

\[b = \frac{\mu((\kappa - 1)(2(n - 1) + \mu)^2 + (2 + n\mu)(n(2\kappa + \mu) - 2(n - 1)))}{2(n - 1) + \mu}.\]

Thus, if \(\mu \neq 0\), then \(M^{2n+1}\) is an \(\eta\)-Einstein manifold, which is the statement (iii). If \(\mu = 0\) and \(\kappa \neq 0\), then in view of (34), we get \(b = 0\) in (33); thus \(M^{2n+1}\) becomes an Einstein manifold. Moreover, if \(n > 1\), in view of Theorem 4.3, we conclude that \(M^{2n+1}\) is Sasakian, which is a contradiction. Therefore we have the statement (iv). This completes the proof. \(\square\)

In view of Theorem 1.2, we are able to state the following two corollaries.
Corollary 4.5. Let $M^{2n+1}$ be a non-Sasakian $N(\kappa)$-contact metric manifold of dimension $\geq 5$. If the $C$-Bochner curvature tensor $B$ satisfies $B(\xi, X) \cdot S = 0$, then $M^{2n+1}$ is locally isometric to $E^{n+1}(0) \times S^n(4)$.

Corollary 4.6. Let $M^{2n+1}$ be a non-Sasakian $N(\kappa)$-contact metric manifold with $n > 1$ and $\kappa \neq 0$. Then, the $C$-Bochner curvature tensor $B$ never satisfies $B(\xi, X) \cdot S = 0$.

5. $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$

In an $N(\kappa)$-contact metric manifold $M^{2n+1}$, we have

\[ R(X,Y)\xi = \kappa R_0(X,Y)\xi, \]
\[ R(\xi, X) = \kappa R_0(\xi, X) = -R(X, \xi). \]

Consequently,

\[ B(X,Y)\xi = \frac{2(\kappa - 1)}{n+2} R_0(X,Y)\xi, \]
\[ B(\xi, X) = \frac{2(\kappa - 1)}{n+2} R_0(\xi, X) = -B(X,\xi). \]

Proof of Theorem 1.3. The condition $B(\xi, X) \cdot R = 0$ gives

\[ 0 = [B(\xi, X), R(Y,Z)]\xi - R(B(\xi, X)Y, Z)\xi - R(Y, B(\xi, X)Z)\xi, \]

which in view of (38) provides

\[ 0 = \frac{2(\kappa - 1)}{n+2} \{ g(X, R(Y,Z)\xi) \xi - \eta(R(Y,Z)\xi) Y - g(X,Y) R(\xi,Z)\xi \\
 + \eta(Y) R(X,Z)\xi - g(X,Z) R(Y,\xi) \xi + \eta(Z) R(Y,X)\xi \\
 - \eta(X) R(Y,Z)\xi + R(Y,Z) X \} \]

Using (36), we get

\[ \frac{2(\kappa - 1)}{n+2} (R(Y,Z) - \kappa R_0(Y,Z)) = 0. \]

Therefore, either $\kappa = 1$ or

\[ R(Y,Z)X = \kappa (g(X,Z)Y - g(X,Y)Z). \]

It is well known that, except for the flat 3-dimensional case, a contact metric manifold of constant curvature is Sasakian and of constant curvature $+1$. Therefore, in view of (39), $M$ is locally isometric to the sphere $S^{2n+1}(1)$. Thus, the proof is complete. \qed
Remark 5.1. If the $N(\kappa)$-contact metric manifold is assumed to be complete and simply connected, then in the Theorem 1.3, local isometry is replaced by global isometry.

6. $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$

This section is devoted to the proof of Theorem 1.4.

Proof of Theorem 1.4. The condition $B(\xi, X) \cdot B = 0$ gives

$$0 = [B(\xi, X), B(Y, Z)] \xi - B(B(\xi, X) Y, Z) \xi - B(Y, B(\xi, X) Z) \xi,$$

which in view of (38) provides

$$0 = \frac{2(\kappa - 1)}{n + 2} \left[ g(X, B(Y, Z) \xi) \xi - \eta(B(Y, Z) \xi) X - g(X, Y) B(\xi, Z) \xi + \eta(Y) B(X, Z) \xi - g(X, \xi) B(Y, Z) \xiight].$$

Using (37), we get

$$\frac{2(\kappa - 1)}{n + 2} \left( R(Y, Z) - \frac{2(\kappa - 1)}{(n + 2)} R_0(Y, Z) \right) = 0.$$

Therefore, either $\kappa = 1$ or

$$B(Y, Z) X = \frac{2(\kappa - 1)}{(n + 2)} (g(X, Z) Y - g(X, Y) Z).$$

Contracting $Y$ in the above equation, we conclude that

$$0 = \frac{2(\kappa - 1)}{(n + 2)} (2ng(X, Z)),$$

which gives $\kappa = 1$. Thus in the both cases $M^{2n+1}$ is a Sasakian manifold. Conversely, if $M^{2n+1}$ is a Sasakian manifold, then in view of (38) we have $B(\xi, X) \cdot B = 0$. □

References


B. J. Papantoniou, *Contact Riemannian manifolds satisfying $R(\xi, X)\cdot R = 0$ and $\xi \in (k, \mu)$-nullity distribution*, Yokohama Math. J. 40 (1993), no. 2, 149–161.


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