GLOBAL EXISTENCE AND STABILITY FOR EULER-BERNOULLI BEAM EQUATION WITH MEMORY CONDITION AT THE BOUNDARY

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Abstract. In this article we prove the existence of the solution to the mixed problem for Euler-Bernoulli beam equation with memory condition at the boundary and we study the asymptotic behavior of the corresponding solutions. We proved that the energy decay with the same rate of decay of the relaxation function, that is, the energy decays exponentially when the relaxation function decay exponentially and polynomially when the relaxation function decay polynomially.

1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions of Euler-Bernoulli Beam Equation with boundary condition of memory type. For this, we consider the following initial boundary-value problem:

\begin{align*}
(1.1) & \quad u_{tt} + uu_{xxx} + f(u_t) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
(1.2) & \quad u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0, \quad \forall t > 0, \\
(1.3) & \quad -u(L,t) + \int_0^t g(t-\tau)u_{xxx}(L,\tau)d\tau = 0, \quad \forall t > 0, \\
(1.4) & \quad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega,
\end{align*}

where $\Omega = [0,L]$, $\| \cdot \|$ is the norm of $L^2(\Omega)$.

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The integral equation (1.3) describes the memory effect which can be caused, for example, by the interaction with another viscoelastic element. Frictional dissipative boundary condition for the wave equation was studied by several authors, see for example([1, 3, 4, 5, 6, 8, 12, 13, 14]) among others. In these works existence of solutions and exponential stabilization were proved for linear and for nonlinear equations. In contrast with the large literature for frictional dissipative, for boundary condition with memory, we have only a few works as for example([2, 3, 9, 10, 11]). The main result this paper is to show that the solutions of system (1.1)–(1.4) decays uniformly in time with the same rate of decay of the relaxation function. More precisely, denoting by $\kappa$ the resolvent kernel of $g'/g(0)$, we show that the solution decays exponentially to zero provided $\kappa$ decays exponentially to zero. When the resolvent kernel $\kappa$ decays polynomially, we show that the corresponding solution also decays polynomially to zero.

The method used here is based on the construction of a suitable Lyapunov functional $L$ satisfying

$$\frac{d}{dt}L(t) \leq -c_1L(t) + c_2e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}L(t) \leq -c_1L(t)^{1+1/\alpha} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants $c_1, c_2, \gamma, \alpha$.

Note that, because of condition (1.2) the solution of system (1.1)–(1.4) must belong to the following space:

$$V = \{u \in H^2(0,L) \mid u(0) = u_x(0) = 0\}$$

and

$$W = \{u \in V \cap H^4(0,L) \mid u_{xx}(L) = 0\}.$$
following binary operators:

\[(f \Box \phi)(t) = \int_0^t f(t-s) |\phi(t) - \phi(s)|^2 ds,\]

\[(f \ast \phi)(t) = \int_0^t f(t-s) \phi(s) ds,\]

where * is the convolution product.

Differentiating (1.3), we arrive to the Volterra integral equations:

\[u_{xxx}(L, t) + \frac{1}{g(0)} g' \ast u_{xxx}(L, t) = \frac{1}{g(0)} u_t(L, t).\]

Applying the Volterra’s inverse operator, we get

\[u_{xxx}(L, t) = \frac{1}{g(0)} \left\{u_t(L, t) + \kappa \ast u_t(L, t)\right\},\]

where the resolvent kernel satisfy

\[\kappa + \frac{1}{g(0)} g' \ast \kappa = -\frac{1}{g(0)} g'.\]

Denoting by \(\tau = \frac{1}{g(0)}\), we obtain

\[u_{xxx}(L, t) = \tau \left\{u_t(L, t) + \kappa(0) u(L, t) - \kappa(t) u(L, 0) + \kappa' \ast u(L, t)\right\}.\]

Since we are interested in relaxation function of exponential or polynomial type and identity (2.1) involve the resolvent kernel \(\kappa\), we want to know if \(\kappa\) has the same properties. The following lemma answers this question. Let \(h\) be a relaxation function and \(\kappa\) its resolvent kernel, that is

\[\kappa(t) - \kappa \ast h(t) = h(t).\]

**Lemma 2.1.** If \(h\) is a positive continuous function, then \(\kappa\) also is a positive continuous function. Moreover,

1. If there exist positive constants \(c_0\) and \(\gamma\) with \(c_0 < \gamma\) such that

   \[h(t) \leq c_0 e^{-\gamma t},\]

   then, the function \(\kappa\) satisfies

   \[\kappa(t) \leq \frac{c_0 (\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},\]

   for all \(0 < \epsilon < \gamma - c_0\).
(2) Given \( p > 1 \), let us denote by
\[
c_p = \sup_{t \in \mathbb{R}^+} \int_0^t (1 + t)^p(1 + t - s)^{-p}(1 + s)^{-p}ds.
\]
If there exists a positive constant \( c_0 \) with \( c_0c_p < 1 \) such that
\[
h(t) \leq c_0(1 + t)^{-p},
\]
then the function \( \kappa \) satisfies
\[
\kappa(t) \leq \frac{c_0}{1 - c_0c_p}(1 + t)^{-p}.
\]

Proof. Note that \( \kappa(0) = h(0) > 0 \). Now, we take \( t_0 = \inf\{t \in \mathbb{R}^+ : \kappa(t) = 0\} \), so \( \kappa(t) > 0 \) for all \( t \in [0, t_0) \). If \( t_0 \in \mathbb{R}^+ \), from (2.2) we get that \(-\kappa \ast h(t_0) = h(t_0)\) but this is contradictory. Therefore \( \kappa(t) > 0 \) for all \( t \in \mathbb{R}^+_0 \). Now, let us fix \( \epsilon \), such that \( 0 < \epsilon < \gamma - c_0 \) and denote by
\[
\kappa_\epsilon(t) = e^{\epsilon t} \kappa(t), \quad h_\epsilon(t) = e^{\epsilon t} h(t).
\]
Multiplying (2.2) by \( e^{\epsilon t} \), we get
\[
\kappa_\epsilon(t) = h_\epsilon(t) + \kappa_\epsilon \ast h_\epsilon(t),
\]
hence
\[
\sup_{s \in [0,t]} \kappa_\epsilon(s) \leq \sup_{s \in [0,t]} h_\epsilon(s) + c_0 \sup_{s \in [0,t]} \kappa_\epsilon(s).
\]
Therefore
\[
\kappa_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},
\]
which implies our first assertion. To show the second part let us consider the following notations
\[
\kappa_p(t) = (1 + t)^p\kappa(t), \quad h_p(t) = (1 + t)^p h(t).
\]
Multiplying (2.2) by \((1 + t)^p\), we get \( \kappa_p(t) = h_p(t) + \int_0^t \kappa_p(t - s)(1 + t - s)^{-p}(1 + t)^p h(s)ds \), hence
\[
\sup_{s \in [0,t]} \kappa_p(s) \leq \sup_{s \in [0,t]} h_p(s) + c_0c_p \sup_{s \in [0,t]} \kappa_p(s)
\]
\[
\leq c_0 + c_0c_p \sup_{s \in [0,t]} \kappa_p(s).
\]
Therefore
\[
\kappa_p(t) \leq \frac{c_0}{1 - c_0c_p},
\]
which proves our second assertion. \( \square \)
Due to this Lemma, in the remainder of this paper, we shall use (2.1) instead of (1.3). The following lemma state an important property of the convolution operator.

**Lemma 2.2.** For \( f, \phi \in \mathcal{C}^1([0, \infty); \mathbb{R}) \), we have

\[
\int_0^t f(t-s)\phi(s)ds \cdot \phi_t = -\frac{1}{2} f(t)|\phi(t)|^2 + \frac{1}{2} f' \Box \phi - \frac{1}{2} \frac{d}{dt} \left[ f \Box \phi - \left( \int_0^t f(s)ds \right) \phi^2 \right].
\]

The proof of this lemma follows by differentiating the term \( f \Box \phi \).

The first-order energy of system (1.1)–(1.4) is given by

\[
E(t, u) = \frac{1}{2} \left( \|u_t(t)\|^2 + \|u_{xx}(t)\|^2 - \tau \kappa' \Box u(L, t) + \tau \kappa(t) |u(L, t)|^2 \right).
\]

We summarize the well-posedness of (1.1)–(1.4) in the following theorem.

**Theorem 2.1.** Let \( \kappa \in \mathcal{C}^2(\mathbb{R}^+) \) be such that \( \kappa, -\kappa', \kappa'' \geq 0 \) and \( f: \mathbb{R} \to \mathbb{R} \) is continuously differentiable function there exist \( \rho \) a positive constant such that \( f(0) = 0 \) and \( (f(r) - f(s), r - s) \geq \rho |r - s|^2 \), \( \forall r, s \in \mathbb{R} \).

If \( u_0 \in W, u_1 \in L^2(\Omega) \) satisfying the compatibility condition \( u_{xxx}(L, 0) = \tau u_t(L, 0) \), then there is only one solution \( u \) of system (1.1)–(1.4) satisfying

\[
(2.3) \quad u \in L^\infty(0, \infty; V), \ u' \in L^\infty(0, \infty; V), \ u'' \in L^\infty(0, \infty; L^2(\Omega)).
\]

**Proof.** The main idea is to use the Galerkin method. To do this let us take a basis \( \{w_j\}_{j \in \mathbb{N}} \) to \( V \) which is orthonormal in \( L^2(\Omega) \) and we represent by \( V_m \) the subspace of \( V \) generated by the first \( m \) vector. Standard results on ordinary differential equations guarantees that there exists only one local solution \( u^m(t) = \sum_{j=1}^m g_{j,m}(t)w_j \), of the approximate system,

\[
(2.4) \quad (u^m_t, w) + (u^m_{xx}, w_{xx}) + (f(u^m_t), w)
= -\left( \tau u^m_t(L, t) + \kappa(0)u^m(L, t) - \kappa(t)u^m(L, 0)
+ \kappa' * u^m(L, t), \ w(L, t) \right),
\]

for all \( w \in V_m \) with the initial data

\[
(u^m(0), u^m_t(0)) = (u^0, u^1).
\]
The extension of these solutions to the whole interval $[0, T], 0 < T < \infty$, is a consequence of the first estimate which we are going to prove below.

**A Priori Estimate I.**

Replacing $w$ by $u^m_t(t)$ in (2.4), using Lemma 2.2 and from hypothesis of $f$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \left( ||u^m_t(t)||^2 + ||u^m_{xx}(t)||^2 - \tau \kappa' \Box u^m(L, t) + \tau \kappa(t) |u^m(L, t)|^2 \right) + \rho ||u^m_t(t)||^2 \\
\leq -\frac{\tau}{2} |u^m_t(L, t)|^2 + \tau \kappa(t) (u^m(L, 0), u^m_t(L, t)) \\
+ \frac{\tau}{2} \kappa'(t) |u^m_t(L, t)|^2 - \frac{\tau}{2} \kappa'' \Box u^m(L, t) \\
\leq -\frac{\tau}{2} |u^m_t(L, t)|^2 + \tau \kappa^2(t)|u^m(L, 0)|^2 + \frac{\tau}{2} \kappa'(t) |u^m(L, t)|^2 \\
- \frac{\tau}{2} \kappa'' \Box u^m(L, t).
$$

Using $\kappa, -\kappa', \kappa'' \geq 0$, we get

$$
\frac{d}{dt} E(t, u^m) + \rho ||u^m_t(t)||^2 \leq c E(0, u^m).
$$

Integrating it over $[0, t]$ and taking into account the definition of the initial data of $u^m$, we conclude that

$$
(2.5) \quad ||u^m_t(t)||^2 + ||u^m_{xx}(t)||^2 + \rho \int_0^t ||u^m_t(s)||^2 ds \\
\leq c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}.
$$

**A Priori Estimate II.**

First of all, we are estimating $u^m_{tt}(0)$ in the $L^2$-norm. Considering $t = 0$ and $w = u^m_{tt}(0)$ in (2.4) and using the compatibility condition we obtain

$$
||u^m_{tt}(0)||^2 + \langle u^m_{xxxx}(0), u^m_{tt}(0) \rangle + \langle f(u^m_t(0)), u^m_{tt}(0) \rangle = 0.
$$

Since $u_0 \in W, u_1 \in L^2(\Omega)$, the growth hypothesis for the function $f$ imply that $f(u_t) \in L^2(\Omega)$. Hence $||u^m_{tt}(0)||^2 \leq C, \quad \forall m \in \mathbb{N}$.
Finally, differentiating (2.4) and multiplying the both sides of equation by \( \frac{1}{2} \frac{d}{dt} \left( ||u''_m(t)||^2 + ||u'''_m(t)||^2 \right) + (f'(u'_m)u''_m, u''_m) \)

\[
= -\tau |u''_m(L,t)|^2 - \tau \kappa'(0)(u'_m(L,t), u''_m(L,t)) \\
+ \tau \kappa'(t) \left( u'(L,0), u''_m(L,t) \right) - \tau \left( (\kappa' \star u''(L,t))_t, u''_m(L,t) \right).
\]

Noting that

\[
(\kappa' \star u''(t)) = \kappa'(t)u''_0 + \int_0^t \kappa'(t-s)u''_m(s)ds
\]

and using Lemma 2.2, we obtain

\[
(2.6) \quad \frac{1}{2} \frac{d}{dt} \left( ||u''_m(t)||^2 + ||u'''_m(t)||^2 \right) + (f'(u'_m)u''_m, u''_m) \]

\[
= -\tau |u''_m(L,t)|^2 + \tau \kappa'(t) (u'(L,0), u''_m(L,t)) \\
+ \frac{\tau}{2} \kappa'(t)|u''(L,t)|^2 - \frac{\tau}{2} \kappa'' \square u''_m(L,t).
\]

We also note that from assumption on the function \( f \), we get

\[
(2.7) \quad \left| \left( f'(u'_m)u''_m, u''_m \right) \right| \leq c ||u''_m(t)||^2.
\]

Substitution of inequality (2.7) into (2.6), we arrive at

\[
(2.8) \quad \frac{1}{2} \frac{d}{dt} \left( ||u''_m(t)||^2 + ||u'''_m(t)||^2 \right) - \tau \kappa' \square u''_m(L,t) + \tau \kappa(t)u''_m(L,t) \leq c ||u''_m(t)||^2 + \tau \kappa'(t)|u'(L,0)|^2.
\]

Integrating with respect to the time and applying Gronwall’s inequality, we conclude that

\[
(2.8) \quad ||u''_m(t)||^2 + ||u'''_m(t)||^2 \leq c, \quad \forall m \in \mathbb{N}, \quad \forall t \in [0, T].
\]

By estimates (2.5) and (2.8), we obtain

\[
\begin{cases}
(u''_m) & \text{is bounded in } L^\infty(0, T; V), \\
(u'_m) & \text{is bounded in } L^\infty(0, T; V), \\
(u''_m) & \text{is bounded in } L^\infty(0, T; L^2(\Omega)).
\end{cases}
\]
Therefore, we can get subsequences, if necessary, denoted by \((u^m)\), such that
\[
\begin{align*}
    u^m &\to u \text{ weakly star in } L^\infty(0,T;V), \\
    u^m_t &\to u_t \text{ weakly star in } L^\infty(0,T;V), \\
    u^m_{tt} &\to u_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Omega)).
\end{align*}
\]

We can use Lions-Aubin Lemma to get the necessary compactness in order to pass (2.4) to the limit. Then it is matter of routine to conclude the existence of global solutions in \([0, T]\). The uniqueness is straightforward by standard methods and Gronwall’s inequality.

\[\square\]

### 3. Exponential decay

In this section, we shall study the asymptotic behavior of the solutions of system (1.1)–(1.4) when the resolvent kernel \(\kappa\) is exponentially decreasing, that is, there exist positive constants \(b_1, b_2\) such that

(3.1) \[\kappa(0) > 0, \quad \kappa'(t) \leq -b_1 \kappa(t), \quad \kappa''(t) \geq -b_2 \kappa'(t).\]

Note that this conditions implies that \(\kappa(t) \leq \kappa(0)e^{-b_1 t}\).

Our point of departure will be to establish some inequalities for the strong solution of system (1.1)–(1.4).

**Lemma 3.1.** Any strong solution \(u\) of system (1.1)–(1.4) satisfy

\[
\frac{d}{dt}E(t) \leq -\frac{\tau}{2}|u_t(L,t)|^2 + \frac{\tau}{2}\kappa^2(t)|u(L,t)|^2 + \frac{\tau}{2}\kappa'(t)|u(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u(L,t) - \rho\|u_t(t)\|^2.
\]

**Proof.** Multiplying (1.1) by \(u_t\) and integrating by parts over \(\Omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \left(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2\right) + \left(f(u_t(t)), u_t(t)\right) = -\left(u_{xxx}(L,t), u_t(t)\right).
\]
Substituting the boundary term, using Lemma 2.1 and assumption of $f$, we get

$$
\frac{1}{2} \frac{d}{dt} \left( \|u_t(t)\| + \|u_{xx}(t)\|^2 - \tau \kappa' \square u(L, t) + \tau \kappa(t) |u(L, t)|^2 \right) \\
+ \left( f(u_t(t)), u_t(t) \right)
= - \tau |u_t(L, t)|^2 + \frac{\tau}{2} \kappa'(t) |u(L, t)|^2 - \frac{\tau}{2} \kappa'' \square u(L, t) \\
+ \tau \kappa(t) \left( u(L, 0), u_t(L, t) \right) \\
\leq - \frac{\tau}{2} |u_t(L, t)|^2 + \frac{\tau}{2} \kappa'(t) |u(L, t)|^2 - \frac{\tau}{2} \kappa'' \square u(L, t) \\
+ \frac{\tau}{2} \kappa^2(t) |u(L, t)|^2.
$$

Using assumption of $f$, we obtain

$$
\frac{d}{dt} E(t) \leq - \frac{\tau}{2} |u_t(L, t)|^2 + \frac{\tau}{2} \kappa^2(t) |u(L, t)|^2 + \frac{\tau}{2} \kappa'(t) |u(L, t)|^2 \\
- \frac{\tau}{2} \kappa'' \square u(L, t) - \rho \|u_t(t)\|^2.
$$

Let us consider the following binary operator:

$$(\kappa \diamond \varphi)(t) = \int_0^t \kappa(t - s)(\varphi(t) - \varphi(s))ds.$$

Then applying the Hölder’s inequality for $0 \leq \mu \leq 1$, we have

$$(3.2) \quad |(\kappa \diamond \varphi)(t)|^2 \leq \left[ \int_0^t |\kappa(s)|^{2(1-\mu)}ds \right] |\kappa|^{2\mu} |\square \varphi|(t).$$

Let us introduce the following functional:

$$\Psi(t) = \delta(u_t(t), u(t)) \quad \text{with} \quad 0 < \delta < \frac{1}{2}.$$  

The following lemma plays an important role for the construction of the Lyapunov functional.

**Lemma 3.2.** For any strong solution of system (1.1)–(1.4), we get

$$
\frac{d}{dt} \Psi(t) \leq - \delta |u_{xx}(t)|^2 + c |u_t(t)|^2 + c |u(t)|^2 + c |u_t(L, t)|^2 + c |u(L, t)|^2 \\
+ c \kappa(t) |u(L, 0)|^2 + c \kappa(0) |\kappa'| \square u(L, t) + c \kappa(t) |u(L, t)|^2.
$$
Proof. From (1.1), it follows that
\[
\frac{d}{dt} \Psi(t) = \delta(u(t), u_t(t)) + \delta \|u_t(t)\|^2
\]
(3.3)
\[
= \delta \|u_t(t)\|^2 - \delta \|u_{xx}(t)\|^2 - \delta \left( f(u(t)), u(t) \right)
- \delta \tau u_t(L, t), u(L, t) + \delta \tau \kappa(t)u(L, 0), u(L, t))
- \delta \tau (\kappa(0)u(L, t) + \kappa u(L, t), u(L, t)).
\]
Note that
\[-\kappa(0)u(L, t) - \kappa u(L, t)
\]
(3.4)
\[
= -\int_0^t \kappa'(t-s) [u(L, s) - u(L, t)] ds - \kappa(t)u(L, t)
\]
\[
\leq \left( \int_0^t |\kappa'(s)| ds \right)^{1/2} \left[ |\kappa'| \Box u(L, t) \right]^{1/2} + \kappa(t)|u(L, t)|
\]
\[
\leq |\kappa(t) - \kappa(0)|^{1/2} \left[ |\kappa'| \Box u(L, t) \right]^{1/2} + \kappa(t)|u(L, t)|.
\]
Using (3.3), (3.4), and Young’s inequality, it follows that
\[
\frac{d}{dt} \Psi(t) \leq \delta \|u_t(t)\|^2 - \delta \|u_{xx}(t)\|^2 - \delta \left( f(u(t)), u(t) \right)
- \delta \tau u_t(L, t), u(L, t) + \delta \tau \kappa(t)u(L, 0), u(L, t))
+ \delta \tau (|\kappa(t) - \kappa(0)|^{1/2} \left[ |\kappa'| \Box u(L, t) \right]^{1/2}, u(L, t)) + \delta \tau \kappa(t)|u(L, t)|^2
\leq - \delta \|u_{xx}(t)\|^2 + c\|u_t(t)\|^2 + c|u_t(L, t)|^2 + c |u(L, t)|^2
+ c\kappa(t)|u(L, 0)|^2 + c\kappa(0)|\kappa'| \Box u(L, t) + c\kappa(t)|u(L, t)|^2.
\]
\[
\square
\]
Let us introduce the Lyapunov functional
(3.5)
\[
\mathcal{L}(t) = NE(t) + \Psi(t), \text{ with } N > 0.
\]
Using Young’s inequality and taking N large enough we find that
(3.6) \[ q_0 E(t) \leq \mathcal{L}(t) \leq q_1 E(t), \text{ for some positive constants } q_0 \text{ and } q_1. \]
We will show later that the functional \( \mathcal{L} \) satisfies the inequality of the following Lemma.

LEMMA 3.3. Let \( f \) be a real positive function of class \( C^1 \). If there exists positive constants \( \gamma_0, \gamma_1 \) and \( c_0 \) such that \( f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma t} \), then there exist positive constants \( \gamma \) and \( c \) such that \( f(t) \leq (f(0) + c)e^{-\gamma t} \).
Proof. First, let us suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by

$$F(t) = f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$ 

Then

$$F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 F(t).$$

Integrating from 0 to $t$, we arrive to

$$F(t) \leq F(0) e^{-\gamma_0 t} \Rightarrow f(t) \leq (f(0) + c_0 \gamma_1 - c_0) e^{-\gamma_0 t}.$$ 

Now, we shall assume that $\gamma_0 \geq \gamma_1$. In this conditions, we get

$$f'(t) \leq -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \Rightarrow (e^{\gamma_1 t} f(t))' \leq c_0.$$ 

Integrating from 0 and $t$, we obtain

$$f(t) \leq (f(0) + c_0 (\gamma_1 - \epsilon)) e^{-\epsilon t}.$$ 

Since $t \leq (\gamma_1 - \epsilon) e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$, we conclude that

$$f(t) \leq (f(0) + c_0 (\gamma_1 - \epsilon)) e^{-\epsilon t}.$$ 

This completes the proof.

Finally, we shall show the main result of this section.

**Theorem 3.1.** Let us suppose that the initial data $(u_0, u_1) \in W \times W$ and that the resolvent $\kappa$ satisfies the conditions (3.1). Then there exist positive constants $\alpha_1$ and $\gamma_1$ such that

$$E(t) \leq \alpha_1 e^{-\gamma_1 t} E(0) \quad \text{for all} \quad t \geq 0.$$ 

**Proof.** We will suppose that $(u_0, u_1) \in (H^4(\Omega) \cap W) \times W$ and satisfies the compatibility conditions $u_{xxx}(L, 0) = \tau u_t(L, 0)$; our conclusion will follow by standard density arguments. Using Lemmas 3.1 and Lemma 3.2, we get

$$\frac{d}{dt} \mathcal{L}(t) \leq -\left(\frac{\tau}{2} N - c\right) |u_t(L, t)|^2 + \frac{\tau}{2} N \kappa^2(t) |u(L, t)|^2 + \frac{\tau}{2} N \kappa'(t) |u(L, t)|^2 - \frac{\tau}{2} N \kappa'' \Box u(L, t) - (N \rho - \delta) |u_t(t)|^2 - c|u_{xx}(t)|^2 + c|u(L, t)|^2 + c\kappa(t) |u(L, 0)|^2 + c\kappa(0) |\kappa'| \Box u(L, t) + c\kappa(t) |u(L, t)|^2.$$ 

Then, choosing $N$ large enough and $N \rho > \delta$, $\frac{\tau}{2} N > c$, we obtain

$$\frac{d}{dt} \mathcal{L}(t) \leq -q_2 E(t) + c\kappa^2(t) E(0), \quad \text{where} \quad q_2 > 0 \quad \text{is a small constant.}$$
Here we use (3.1) to conclude the following estimates for the corresponding two terms appearing in Lemma 3.1.

\[ -\frac{\tau}{2} \kappa'' \Box u(L, t) \leq c_1 \kappa' \Box u(L, t), \]
\[ -\frac{\tau}{2} \kappa' |u(L, t)|^2 \leq -c_1 \kappa |u(L, t)|^2. \]

Finally, from (3.1) and (3.6), we conclude that

\[ \frac{d}{dt} \mathcal{L}(t) \leq -\frac{q_2}{q_1} \mathcal{L}(t) + cE(0) \exp(-2b_1 t). \]

Using the exponential decay of the resolvent kernel \( \kappa \) and Lemma 3.3, we conclude

\[ \mathcal{L}(t) \leq \{ \mathcal{L}(0) + c \} e^{-\gamma_1 t} \text{ for all } t \geq 0. \]

Use of (3.6) now completes the proof. \( \square \)

4. Polynomial rate of decay

The proof of the existence of global solutions for (1.1)–(1.4) with resolvent kernel \( \kappa \) decaying polynomially is essentially the same as in Section 3. Here our attention will be focused on the uniform rate of decay when the resolvent \( \kappa \) decays polynomially such as \((1 + t)^{-p}\). In this case, we will show that the solution also decays polynomially with the same rate. We shall use the following hypotheses:

\[ 0 < \kappa(t) \leq b_0 (1 + t)^{-p}, \]
\[ -b_1 \kappa^\frac{p+1}{p} \leq \kappa'(t) \leq -b_2 \kappa^\frac{p+1}{p}, \]
\[ b_3 (-\kappa')^\frac{p+2}{p+1} \leq \kappa''(t) \leq b_4 (-\kappa')^\frac{p+2}{p+1}, \]

where \( p > 1 \) and \( b_i, \ i = 0, 1, \cdots, 4, \) are positive constants.

Also we assume that

\[ \int_0^\infty |\kappa'(t)|^r dt < \infty \text{ if } r > \frac{1}{p+1}. \]

The following lemmas will play an important role in the sequel.
Lemma 4.1. Let $m$ and $h$ be integrable functions, $0 \leq r < 1$ and $q > 0$. Then, for $t \geq 0$:
\[
\int_0^t |m(t-s)h(s)|ds \leq \left( \int_0^t |m(t-s)|^{1+\frac{1-q}{q}}h(s)|ds \right)^{\frac{q}{q+r}} \left( \int_0^t |m(t-s)|^r|h(s)|ds \right)^{\frac{1}{q+r}}.
\]

Proof. Let $v(s) = |m(t-s)|^{1-\frac{r}{q+r}}h(s)|^{\frac{q+r}{q}}$, $w(s) = |m(t-s)|^{\frac{q+r}{q}}|h(s)|^{\frac{1}{q+r}}$. Then using Hölder’s inequality with $\delta = \frac{q}{q+r}$ for $v$ and $\delta^* = q+1$ for $w$, we arrive to the conclusion.

Lemma 4.2. Let $p > 1$, $0 \leq r < 1$ and $r \geq 0$. Then for $r > 0$,
\[
|\kappa' \square u(t,L)|^{\frac{1-(1-r)(p+1)}{1-(r+1)(p+1)}} \leq 2\left( \int_0^t |\kappa'(s)|^r ds \|u\|_{L^\infty(0,T;L^2(0,L))} \right)^{\frac{1}{(1-r)(p+1)}} \left( |\kappa'|^{1+\frac{1}{r+1}} \square u(t,L) \right),
\]
and for $r = 0$,
\[
|\kappa' \square u(t,L)|^{\frac{p+2}{r+1}} \leq 2 \left( \int_0^t \|u(s)\|_{L^2(0,L)}^2 ds + t\|u(s)\|_{L^2(0,L)}^2 \right)^{\frac{1}{r+1}} \left( |\kappa'|^{1+\frac{1}{r+1}} \square u(t,L) \right).
\]

Proof. The above inequality is a immediate consequence of Lemma 4.1 with $m(s) = |\kappa'(s)|$, $h(s) = |u(x,t) - u(x,s)|^2$, $q = (1-r)(p+1)$, and $t$ fixed.

Lemma 4.3. Let $\alpha > 0$, $\beta \geq \alpha + 1$, and $f \geq 0$ be differentiable function satisfying $f'(t) \leq \frac{\bar{c}_1}{f(0)} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)\beta} f(0)$ for $t \geq 0$ and some positive constants $\bar{c}_1$, $\bar{c}_2$. Then there exists a constant $\bar{c}_3 > 0$ such that for $t \geq 0$, $f(t) \leq \frac{\bar{c}_3}{(1+t)^{\beta}} f(0)$.

Proof. Let $t \geq 0$ and
\[
F(t) = f(t) + \frac{2\bar{c}_2}{\alpha} (1 + t)^{-\alpha} f(0).
\]
Then
\[
F' = f' - 2\bar{c}_2(1 + t)^{-(\alpha+1)} f(0) \leq \frac{\bar{c}_1}{f(0)^{\alpha}} f^{1+\frac{1}{\alpha}} - \bar{c}_2(1 + t)^{-(\alpha+1)} f(0),
\]
where we used $\beta \geq \alpha + 1$. Hence

$$F' \leq -\frac{c}{f(0)\pi^\frac{1}{\alpha}} \left( f^{1+\frac{1}{\alpha}} + (1 + t)^{-(\alpha + 1)} f(0)^{1+\frac{1}{\alpha}} \right) \leq -\frac{c}{F(0)^\frac{1}{\alpha}} F^{1+\frac{1}{\alpha}}.$$

Integration yields $F(t) \leq \frac{F(0)}{(1+ct)^{\frac{1}{\alpha}}} \leq \frac{c}{(1+t)^{\frac{1}{\alpha}}} f(0)$, hence $f(t) \leq \frac{c_3}{(1+t)^{\frac{1}{\alpha}}} f(0)$ for some $c_3$, which proves Lemma 4.3.

**Theorem 4.1.** Assume that $(u_0, u_1) \in W \times L^2(0, L)$ and that the resolvent $\kappa$ satisfies (4.1). Then there exists a positive constant $c$ for which

$$E(t) \leq \frac{c}{(1+t)^{p+1}} E(0).$$

**Proof.** We will suppose that $(u_0, u_1) \in H^2(0, L) \cap W \times W$ and satisfies the compatibility condition; our conclusion will follows by standard density arguments. We define the functional $L$ as in (3.5) and we have the equivalence to the energy term $E$ as given in (3.6) again.

From the Lemmas 3.1 and 3.2, we conclude that

$$\frac{d}{dt} L(t) \leq -c_1 \left( |u_t(L, t)|^2 + \kappa'' \Box u(L, t) \right. + \left. \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right) + c_2 \kappa^2(t) E(0). \quad (4.3)$$

From hypothesis (4.1), we obtain

$$\frac{d}{dt} L(t) \leq -c_1 \left( |u_t(L, t)|^2 + (-\kappa')^{1+\frac{1}{p+1}} \Box u(L, t) \right. + \left. \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right) + c_2 \kappa^2(t) E(0). \quad (4.4)$$

From Lemma 4.2, we get

$$(-\kappa')^{1+\frac{1}{p+1}} \Box u(L, t) \geq \frac{c}{(\int_0^t |\kappa'|^{\cdot} ds)^{\frac{1}{1+(1-r)(p+1)}} E(0)^{\frac{1}{(1-r)(p+1)}}} \left( (-\kappa') \Box u(L, t) \right)^{1+\frac{1}{(1-r)(p+1)}}. \quad (4.5)$$

On the other hand, since the energy is bounded, we have

$$\left( |u_t(L, t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right)^{1+\frac{1}{(1-r)(p+1)}} \leq c E(0)^{\frac{1}{1+(1-r)(p+1)}} \left( |u_t(L, t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right). \quad (4.6)$$
Substituting (4.5) and (4.6) into (4.4), we arrive at
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\frac{c}{E(0)^{1/(p+1)}} \left[ \left( |u_t(L, t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right)^{1+\frac{1}{(1-r)(p+1)}} 
\right. \\
+ \left. \left( |\kappa' \Box u(L, t)| \right)^{1+\frac{1}{(1-r)(p+1)}} \right] + c\kappa^2(t)E(0).
\]

Taking into account (3.6), we conclude that
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{1/(p+1)}} \mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + c\kappa^2(t)E(0).
\]

Applying the Lemma 4.3 with \( f = \mathcal{L} \) and \( \beta = 2p \), we have:
\[
\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}} \mathcal{L}(0).
\]

Since \((1 - r)(p + 1) > 1\),
\[
\int_0^\infty E(s)ds \leq c \int_0^\infty \mathcal{L}(s)ds \leq c\mathcal{L}(0),
\]
\[
t\|u(t)\|^2_{L^2(0,L)} \leq c\mathcal{L}(t) \leq c\mathcal{L}(0),
\]
\[
\int_0^t \|u(s)\|^2_{L^2(0,L)}ds \leq c \int_0^\infty \mathcal{L}(t)dt \leq c\mathcal{L}(0).
\]

In this conditions applying Lemma 4.2 for \( r = 0 \), we get
\[
(-\kappa')^{1+\frac{1}{p+1}} \Box u(L, t) \geq \frac{c}{E(0)^{1/(p+1)}} \left( (-\kappa) \Box u(L, t) \right)^{1+\frac{1}{p+1}}.
\]

Using the same arguments as in the derivation of (4.7), we have
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{1/(p+1)}} \mathcal{L}(t)^{1+\frac{1}{p+1}} + c\kappa^2(t)E(0).
\]

Applying the Lemma 4.3, we obtain
\[
\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}} \mathcal{L}(0).
\]

Finally, from (3.6) we obtain \( E(t) \leq \frac{c}{(1+t)^{p+1}} E(0) \), which complete the present proof.
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