ON A COMPACT AND MINIMAL REAL HYPERSURFACE IN A QUATERNIONIC PROJECTIVE SPACE

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Abstract. For a compact and orientable minimal real hypersurface \( M \) in \( QP^n \), we prove that if the minimum of the sectional curvatures of \( M \) is \( \frac{3}{4n-1} \), then \( M \) is isometric to the geodesic minimal hypersphere \( M^0_{0,n-1} \).

1. Introduction

Let \( QP^n \) be a quaternionic projective space of real dimension \( 4n \), \( n \geq 2 \), with the Fubini-Study metric \( G \) of constant Q-sectional curvature 4 and let \( M \) be a connected \((4n-1)\)-dimensional real hypersurface of \( QP^n \).

Let \( N \) be a local unit normal vector field to \( M \). We denote by \( \{J_i\}_{i=1,2,3} \) is a local basis of the quaternionic Kähler structure of \( QP^n \). Then \( U_i = -J_i N, i = 1, 2, 3 \) are tangent to \( M \), which will be called structure vectors [10].

Now we put \( f_i(X) = g(X, U_i) \), for arbitrary \( X \in TM, i = 1, 2, 3 \), where \( TM \) is the tangent bundle of \( M \) and \( g \) denotes the Riemannian metric induced from the metric \( G \).

Now, let us consider the following conditions that the second fundamental tensor \( A \) of \( M \) in \( QP^n \) may satisfy

\[
(1.1) \quad (\nabla_X A)Y = \sum_{i=1}^{3} \{g(X, \phi_i Y) U_i - f_i(Y) \phi_i X\},
\]

\[
(1.2) \quad g((A \phi_i - \phi_i A)X, Y) = 0,
\]

for any \( i = 1, 2, 3 \) and any tangent vector fields \( X \) and \( Y \) to \( M \).

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Pak[10] investigated the above conditions and showed that they are equivalent to each other. Moreover he used the condition 1.1 to find a lower bound of \(|\nabla A|\) for real hypersurfaces in \(QP^n\). In fact, it was shown that \(|\nabla A|\geq 24(n-1)\) for any hypersurfaces and the equality holds if and only if the condition 1.1 holds. In this case it was also known that \(M\) is locally congruent to a real hypersurface of type \(A_1\) or type \(A_2\), which means a tube of radius \(r\) over \(QP^k\; (1 \leq k \leq n-1)\) in the notion of Berndt[1], and Martínez and Pérez[8].

Now the purpose of this paper is to give another new characterization of a minimal real hypersurface in \(QP^n\) by using Lemmas, to be stated in Section 3, which is a quaternionic version of result of Kon[5].

Now we prepare the following theorem [5] without proof in order to compare with our result:

**Theorem 1.1.** Let \(M\) be a compact orientable real minimal hypersurface of \(CP^n\). If the sectional curvature \(K\) of \(M\) satisfies \(K \geq 1/(2n-1)\), then \(M\) is the geodesic minimal hypersphere \(M^g_{0,n-1}\).

**2. Preliminaries**

A quaternionic Kähler manifold is a Riemannian manifold \((\tilde{M}, G)\) on which there exists a 3-dimensional vector bundle \(\tilde{V}\) of tensors of type \((1,1)\) with a local basis of almost Hermitian structures \(\{J_i\}_{i=1,2,3}\) satisfying the following conditions:

1. \(J_i^2 = -id, \ i = 1, 2, 3, \ J_iJ_j = -J_jJ_i = J_k,\) where \(id\) denotes the identity endomorphism on \(T\tilde{M}\) and \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

2. If \(\nabla\) denotes the Riemannian connection on \(\tilde{M}\), then there exist three local 1-forms \(P_i, i = 1, 2, 3\) on \(\tilde{M}\) such that

\[
\nabla_XJ_i = P_k(X)J_j - P_j(X)J_k
\]

for all vector field \(X\) on \(\tilde{M}\), where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

Let \(Q(X)\) be the 4-dimensional subspace spanned by vectors \(X, J_1X, J_2X\) and \(J_3X\) for any \(X \in T_p\tilde{M}, p \in \tilde{M}\). If the sectional curvature of any section for \(Q(X)\) depends only on \(X\), we call it \(Q\)-sectional curvature. A quaternionic space form of \(Q\)-sectional curvature \(c\) is a connected quaternionic Kähler manifold with constant \(Q\)-sectional curvature \(c\).

The standard models of quaternionic space forms are the quaternionic projective space \(QP^n(c)(c > 0)\), the quaternionic space \(Q^n(c = 0)\) and the quaternionic hyperbolic space \(QH^n(c < 0)\) ([1]).
The curvature tensor $\hat{R}$ of $QP^n(c), n \geq 2$, is given by

$$
\hat{R}(X, Y)Z = \frac{c}{4}[G(Y, Z)X - G(X, Z)Y + \sum_{k=1}^{3}\{G(J_kX, Z)J_kX - G(J_kX, Z)J_kY - 2G(J_kX, Y)J_kZ\}]
$$

for any vector fields $X, Y$ and $Z$ on $QP^n(c)([2])$.

From now on we denote by $QP^n$ the quaternionic projective space of constant $Q$-sectional curvature 4.

Let $M$ be a connected $(4n-1)$-dimensional real hypersurface of $QP^n$ and let $N$ be a local unit normal vector field to $M$. The Riemannian connection $\hat{\nabla}$ in $QP^n$ and $\nabla$ in $M$ are related by the following formulas for arbitrary vector fields $X$ and $Y$ tangent to $M$:

\begin{align*}
(2.1) \quad & \hat{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \\
(2.2) \quad & \hat{\nabla}_X N = -AX,
\end{align*}

where $A$ is the second fundamental tensor of $M$ in $QP^n$. The mean curvature $h$ of $M$ is defined by $h = \frac{1}{4n-1}TrA$.

If $h = 0$, then $M$ is said to be minimal. Eigenvectors of the second fundamental tensor $A$ are called principal curvature vectors and called the corresponding eigenvalues principal curvatures. We put

\begin{align*}
(2.3) \quad & J_iX = \phi_iX + f_i(X)N, \quad J_iN = -U_i, \quad i = 1, 2, 3 \\
\end{align*}

for any vector field $X$ tangent to $M$, where $\phi_iX$ is the tangential parts of $J_iX$, $\phi_i$ are tensors of type (1,1) and $f_i$ are 1-forms for $i = 1, 2, 3$.

As $J_i^2 = -id, i = 1, 2, 3$, $id$ denoting the identity endomorphism on $TQP^n$, we get

\begin{align*}
(2.4) \quad & \phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1, 2, 3
\end{align*}

for any vector field $X$ tangent to $M$. As $J_iJ_j = -J_jJ_i = J_k, (i, j, k)$ being a cyclic permutation of $(1, 2, 3)$, we obtain

\begin{align*}
(2.5) \quad & f_i(U_i) = 1, \quad f_i(U_j) = f_i(U_k) = 0, \\
(2.6) \quad & \phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k
\end{align*}

and

\begin{align*}
(2.7) \quad & f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX),
\end{align*}

for any vector field $X$ tangent to $M$. 

It is also easy to see that for any \( X,Y \) tangent to \( M \),
\[
(2.8) \quad g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \quad g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y)
\]
and
\[
(2.9) \quad \phi_i U_j = -\phi_j U_i = U_k,
\]
where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

The covariant derivatives of \( J_i, i = 1, 2, 3 \), are given by
\[
(\tilde{\nabla}_X J_i = P_k(X)J_j - P_j(X)J_k)
\]
for any \( X \in TQP^n \), where \( P_i, i = 1, 2, 3 \), are local 1-forms on \( QP^n \).

Then from (2.1) and (2.2) we obtain
\[
(2.10) \quad \nabla_X U_i = -P_j(X)U_k + P_k(X)U_j + \phi_i AX
\]
and
\[
(2.11) \quad (\nabla_X \phi_i)Y = -P_j(X)\phi_k Y + P_k(X)\phi_j Y + f_i(Y)AX - g(AX, Y)U_i
\]
for any vector fields \( X,Y \) tangent to \( M \), where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

Since \( \phi_i \) is skew-symmetric and \( A \) is symmetric, (2.10) implies that
\[
(2.12) \quad \text{div} U_i = \sum_{a=1}^{4n-1} g(\nabla_a U_i, e_a) = -P_j(U_k) + P_k(U_j),
\]
where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

From the expression of the curvature tensor of \( QP^n, n \geq 2 \), the equations of Gauss and Codazzi are respectively given by
\[
(2.13) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{i=1}^{3} \{g(\phi_i Y, Z)\phi_i X \\
- g(\phi_i X, Z)\phi_i Y - 2g(\phi_i X, Y)\phi_i Z\} \\
+ g(AY, Z)AX - g(AX, Z)AY
\]
and
\[
(2.14) \quad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^{3} \{f_i(X)\phi_i Y - f_i(Y)\phi_i X \\
+ 2g(X, \phi_i Y)U_i\}
\]
for any \( X,Y,Z \) tangent to \( M \), where \( R \) denotes the curvature tensor of \( M \) ([8]).

We now put
\[
T := \nabla_{U_i}U_i + \nabla_{U_j}U_j + \nabla_{U_k}U_k + (\text{div} U_i)U_i + (\text{div} U_j)U_j + (\text{div} U_k)U_k
\]
and take an orthonormal basis \( \{ e_a \}_{a=1,\ldots,4n-1} \) of tangent vectors to \( M \) such that
\[
\begin{align*}
e_n &:= \phi_i e_1, \ldots, e_{2(n-1)} := \phi_i e_{n-1}, \\
e_{2n-1} &:= \phi_j e_1, \ldots, e_{3(n-1)} := \phi_j e_n, \\
e_{3n-2} &:= \phi_k e_1, \ldots, e_{4(n-1)} := \phi_k e_n, \\
e_{4n-3} &:= U_i, e_{4n-2} := U_j, e_{4n-1} := U_k.
\end{align*}
\]
Then it follows from (2.10) and (2.12) that
\[
T = \phi_i A U_i + \phi_j A U_j + \phi_k A U_k
\]
We note that \( T \) is a global vector field defined on \( M \). For later use we compute
\[
\text{div}(T) = \sum_{i=1}^{4n-1} g(\nabla e_a T, e_a).
\]
Differentiating (2.15) covariantly and using (2.4), (2.6), (2.9)–(2.11), and (2.14), we have
\[
\text{div}(T) = (\text{tr} A)(\sum_{i=1}^{3} g(A U_i, U_i)) - \sum_{i=1}^{3} g(A^2 U_i, U_i) + \sum_{i=1}^{3} \text{tr}(A \phi_i)^2 \\
- \sum_{i=1}^{n-1} \{ g((\nabla e_i A) \phi_i e_i - (\nabla \phi_i e_i A) e_i + (\nabla \phi_i e_i A) \phi_i e_i \\
- (\nabla \phi_i e_i A) \phi_j e_j, U_j) + g((\nabla e_i A) \phi_j e_j - (\nabla \phi_j e_j A) e_j + (\nabla \phi_j e_j A) \phi_j e_j \\
- (\nabla \phi_j e_j A) e_j + (\nabla \phi_j e_j A) \phi_k e_k) \\
- g((\nabla U_i A) U_k - (\nabla U_k A) U_i, U_j) - g((\nabla U_k A) U_i, U_j) \\
- (\nabla U_i A) U_k, U_j) - g((\nabla U_i A) U_j - (\nabla U_j A) U_i, U_k)\},
\]
or equivalently
\[
\text{div}(T) = (\text{tr} A)(\sum_{i=1}^{3} g(A U_i, U_i)) - \sum_{i=1}^{3} g(A^2 U_i, U_i) \\
+ \sum_{i=1}^{3} \text{tr}(A \phi_i)^2 + 12(n-1)
\]
Moreover we should explain model subspaces which will appear in our Theorem 3.3. We consider the Hopf fibration \( \tilde{\pi} : \)
\[
S^3 \longrightarrow S^{4n+3} \longrightarrow QP^n,
\]
where $S^k$ denotes the Euclidean sphere of curvature 1. In $S^{4n+3}$ we have the family of generalized Clifford surfaces whose spheres lie in quaternionic subspaces (cf. [7]):

$$M_{4p+3,4q+3} := S^{4p+3} \left( \sqrt{\frac{4p+3}{2(2n+1)}} \right) \times S^{4q+3} \left( \sqrt{\frac{4q+3}{2(2n+1)}} \right),$$

where $p + q = n - 1$. Then we have a fibration $\pi$:

$$S^3 \longrightarrow M_{4p+3,4q+3} \longrightarrow M^Q_{p,q},$$

compatible with $\tilde{\pi}$. In the special case $p = 0$, $M_{0,n-1}^Q$ is called the geodesic minimal hypersphere of $QP^n$, and is a homogeneous, positively curved manifold diffeomorphic to the sphere (for details, see [1, 7, 10]).

3. Main results

In order to prove our theorem, we need the following result.

**Lemma 3.1.** Let $M$ be a minimal real hypersurface of $QP^n$. Then

$$g(\nabla^2 A, A) = \sum_{a,b} g((R(e_b,e_a)A)e_b, Ae_a) - 9TrA^2$$

$$+ \frac{3}{2} \sum_i \| [\phi_i, A] \|^2,$$

where $[\phi_i, A]$ denotes $\phi_i A - A\phi_i$.

**Proof.** Let $\{e_a\}$ be an orthonormal frame for $M$. Then (2.14) implies

$$\sum_a (\nabla_{e_a} A)e_a = 0.$$  

Thus, from (2.10), (2.11), (2.14), and (3.2) we obtain

$$g(\nabla^2 A, A)$$

$$= \sum_{a,b} g((\nabla_{e_b} \nabla_{e_a} A)e_a, Ae_a)$$

$$= \sum_{a,b} g((R(e_b,e_a)A)e_b - \sum_i \{g(\nabla_{e_b} U_i, e_a)\phi_i e_b$$

$$+ f_i(e_a)(\nabla_{e_b} \phi_i)e_b - g(\nabla_{e_b} U_i, e_b)\phi_i e_a - f_i(e_b)(\nabla_{e_a} \phi_i)e_a$$

$$+ 2g(e_a, (\nabla_{e_b} \phi_i)e_b)U_i + 2g(e_a, \phi_i e_b)\nabla_{e_b} U_i, Ae_a)$$
= \sum_{a,b} g((R(e_b, e_a)A)e_b, Ae_a) - 3 \sum_i g(A^2 U_i, U_i) + 3 \sum_i \text{Tr}(A\phi_i)^2.

Since \(\text{Tr}(A\phi_i)^2 = -\text{Tr}A^2 + g(A^2 U_i, U_i) + \frac{1}{2}||\phi_i, A||^2\), we obtain

\begin{equation}
-3 \sum_i g(A^2 U_i, U_i) + 3 \sum_i \text{Tr}(A\phi_i)^2 = -9 \text{Tr}A^2 + \frac{3}{2} \sum_i ||\phi_i, A||^2.
\end{equation}

Substituting (3.4) into (3.3), we have our assertion. \(\square\)

**Lemma 3.2.** Let \(M\) be a compact and orientable minimal real hypersurface in \(Q^n\). If the minimum of the sectional curvatures of \(M\) is \(3/(4n-1)\), then \(\|\nabla A\|^2 = 24(n-1)\) and \(g((A\phi_i - \phi_i A)X, Y) = 0\), \(i = 1, 2, 3\).

**Proof.** We choose an orthonormal frame \(\{e_a\}\) of \(M\) such that \(Ae_a = \lambda_a e_a\), \(a = 1, 2, \cdots, 4n-1\).

We denote by \(K_{ab}\) the sectional curvature of \(M\) spanned by \(e_a\) and \(e_b\).

Then we have

\[
\sum_{a,b} g((R(e_a, e_b)A)e_a, Ae_b) = \sum_{a,b} \{g(R(e_a, e_b)Ae_a, Ae_b) - g(AR(e_a, e_b)e_a, Ae_b)\} = \frac{1}{2} \sum_{a,b} (\lambda_a - \lambda_b)^2 K_{ab} \geq \frac{3}{2(4n-1)} \sum_{a,b} (\lambda_a - \lambda_b)^2 = 3 \text{Tr}A^2.
\]

Consequently, we see

\begin{equation}
3 \text{Tr}A^2 - \sum_{a,b} g((R(e_a, e_b)A)e_a, Ae_b) \leq 0.
\end{equation}

Since we have \(\frac{1}{2} \Delta \text{Tr}A^2 = \|\nabla A\|^2 + g(\nabla^2 A, A)\), we obtain

\begin{equation}
\int_M \|\nabla A\|^2 * 1 = -\int_M g(\nabla^2 A, A) * 1.
\end{equation}
From Lemma 3.1, (3.6) and (2.16) we have

\[
0 \leq \int_M \left[ \|\nabla A\|^2 - 24(n-1) + \frac{1}{2} \sum_i \|\phi_i, A\|^2 \right] \ast 1
\]

\[
= \int \left[ 9 \text{Tr} A^2 - \sum_{a,b} g((R(e_a,e_b)A)e_a, Ae_b) - 24(n-1) \right.
\]

\[
- \sum_i \|\phi_i, A\|^2 \left] \ast 1
\]

\[
= \int \left[ 3 \text{Tr} A^2 - \sum_{a,b} g((R(e_a,e_b)A)e_a, Ae_b) \right] \ast 1.
\]

From this and (3.5) we complete the proof. \(\square\)

Combining Lemma 3.2 and the result of Kwon and Pak[6], we see that \(M\) is \(M^Q_{p,q}\).

On the other hand if \(p, q \geq 1\), then the sectional curvature \(K\) of \(M^Q_{p,q}\) takes values 0 for some plane section [10]. But the sectional curvature \(K\) of \(M^Q_{0,n-1}\) satisfies \(K \geq 3/(4n-1)\).

Consequently, \(M\) is the geodesic minimal hypersphere \(M^Q_{0,n-1}\).

**Theorem 3.3.** Let \(M\) be a compact and orientable minimal real hypersurface in \(QP^n\). If the minimum of the sectional curvatures of \(M\) is \(3/(4n-1)\), then \(M\) is isometric to the geodesic minimal hypersphere \(M^Q_{0,n-1}\).

**References**


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