ON REGULAR PREOPEN SETS AND \( p^\ast \)-CLOSED SPACES

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**Abstract.** We introduce the notions of regular preopen sets and \( p^\ast \)-closed spaces and investigate some of these properties. Also we give a characterization of \( p \)-closed spaces.

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1. Introduction

Mashhour et al[5] defined the notion of preopen set in a topological space and obtained its various properties. With the aid of preopen sets, they introduced and investigated modified continuous functions called precontinuous function and weak precontinuous function.

Dontchev et al[2] introduced and investigated \( p \)-closed spaces. Our primary goal is to introduce and investigate regular preopen sets and a new class of topological spaces called \( p^\ast \)-closed spaces and investigate \( p \)-closed spaces.

2. Preliminaries

Throughout the present paper, \( X \) and \( Y \) denote topological spaces. Let \( A \subset X \). We denote the interior and the closure of \( A \) by \( \text{int}(A) \) and \( cl(A) \), respectively. A subset \( A \) of \( X \) is said to be preopen[5] if \( A \subset \text{cl}(\text{int}(A)) \). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing \( A \) is called the preclosure of \( A \) and is denoted by \( pcl(A) \). The preinterior of \( A \) is defined by the union of all preopen sets contained in \( A \) and is denoted by \( pint(A) \). It is clear that \( A \) is preopen if and only if \( A = pint(A) \) and \( A \) is preclosed if and only if \( A = pcl(A) \).
From definitions, we can see that the following inclusion relations hold:

\[ \text{int}(A) \subset \text{pint}(A) \subset A \subset \text{pcl}(A) \subset \text{cl}(A). \]

Therefore, if \( A \) is closed (resp. open), then \( A \) is preclosed (resp. preopen).

It easy to see that for a subset \( A \) of \( X \), \( \text{pcl}(X - A) = X - \text{pint}(A) \) and \( \text{pint}(X - A) = X - \text{pcl}(A) \).

A subset \( A \) of \( X \) is said to be \( \alpha \)-open [6] if \( A \subset \text{int} (\text{cl}(\text{int}(A))) \). The complement of an \( \alpha \)-open set is called \( \alpha \)-closed. We can see that every \( \alpha \)-open (resp. \( \alpha \)-closed) set is a preopen (resp. preclosed) set.

3. Regular preopen sets

A subset \( A \) of \( X \) is said to be regular preopen (resp. regular preclosed) if \( A = \text{pint}(\text{pcl}(A)) \) (resp. \( A = \text{pcl}(\text{pint}(A)) \)).

It is clear that a regular preopen set is preopen.

A space \( X \) is called extremally predisconnected if for all preopen subset \( U \) of \( X \), \( \text{pcl}(U) \) is a preopen subset of \( X \).

**Proposition 3.1.** If \( A \) is a preclopen set in \( X \), then \( A \) is a regular preopen set. Moreover, if \( X \) is extremally predisconnected then the converse holds.

**Proof.** If \( A \) is a preclopen set, then \( A = \text{pcl}(A) \) and \( A = \text{pint}(A) \), and so we have \( A = \text{pint}(\text{pcl}(A)) \). Hence \( A \) is regular preopen.

Suppose that \( X \) is an extremally predisconnected space and \( A \) is a regular preopen set in \( X \). Then \( A \) is preopen and so \( \text{pcl}(A) \) is a preopen set. Hence \( A = \text{pint}(\text{pcl}(A)) = \text{pcl}(A) \) and hence \( A \) is a preclosed set. \( \square \)

**Example 3.1.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then \((X, \tau)\) is not extremally predisconnected, and \( \{a\} \) is a regular preopen set but not preclopen set.

**Theorem 3.2.** For a subset \( A \) of \( X \), consider the following statements.

1. \( A \) is preclopen.
2. \( A = \text{pcl}(\text{pint}(A)) \).
3. \( X - A \) is regular preopen.
4. \( A \) is regular preopen.

Then we have that \( (1) \implies (2) \implies (3) \). Moreover, if \( X \) is an extremally predisconnected space, then \( (4) \implies (1) \) and \( (3) \implies (4) \), and hence the above statements are equivalent.
Proof. The implication (1) ⇒ (2) is obvious.

(2) ⇒ (3). Let \( A = pcl(pint(A)) \). Then \( X - A = X - pcl(pint(A)) = pint(X - pint(A)) = pint(pcl(X - A)) \), and hence \( X - A \) is a regular preopen set.

Suppose \( X \) is an extremally predisconnected space.

(3) ⇒ (4). From Proposition 3.1, \( X - A \) is a preopen and preclosed set, and hence \( A \) is a preopen and preclosed set. Thus \( A = pint(pcl(A)) \) and \( A \) is a regular preopen set.

(4) ⇒ (1). It follows from Proposition 3.1. □

**Theorem 3.3.** Let \( A \) be a subset of a space \( X \) and let \( cl(A) \) (resp. \( pcl(A) \)) be a regular preopen set. Then \( A \) is a preopen set in \( X \). Moreover, if \( X \) is extremally predisconnected then the converse holds.

Proof. Suppose that \( cl(A) \) is a regular preopen set. Then we have

\[
A \subseteq cl(A) \subseteq int(cl(cl(A))) = int(cl(A)).
\]

Hence we have \( A \) is a preopen set.

Assume that \( X \) is extremally predisconnected and \( A \) is a preopen set. Then \( cl(A) \) is a preopen set, and hence a preclopen set. Thus \( cl(A) \) is a regular preopen set. □

**Corollary 3.4.** Let \( X \) be an extremally predisconnected space. Then for each subset \( A \) of \( X \), the sets \( cl(int(A)) \), \( cl(pint(A)) \), \( pcl(int(A)) \) and \( pcl(pint(A)) \) are regular preopen sets.

**Proposition 3.5.** If a subset \( A \) of \( X \) is \( \alpha \)-open and \( \alpha \)-closed, then \( A \) is a regular preopen set.

Proof. Let \( A \) be an \( \alpha \)-open and \( \alpha \)-closed set. Then \( A \) is a preopen and preclosed set, and hence \( A \) is a regular preopen set. □

**Proposition 3.6.** If a subset \( A \) of \( X \) is \( \alpha \)-open and regular preopen, then \( A = int(cl(int(A))) \).

Proof. Let \( A \) be an \( \alpha \)-open and regular preopen set. Then \( A \subseteq int(cl(int(A))) \) and

\[
A = pint(pcl(A)) \supset pint(cl(int(pcl(A)))) \supset pint(cl(int(A))) \supset int(cl(int(A))).
\]

□
A point $x$ of $X$ is called a \textit{pre-$\theta$-cluster point}\cite{6} of a subset $A$ of $X$ if $\text{pcl}(U) \cap A \neq \phi$ for every preopen set $U$ containing $x$.

The set of all pre-$\theta$-cluster points of $A$ is called the \textit{pre-$\theta$-closure}\cite{6} of $A$ and is denoted by $\text{pcl}_\theta(A)$.

A subset $A$ of $X$ is said to be \textit{pre-$\theta$-closed}\cite{6} if $\text{pcl}_\theta(A) = A$. The complement of a pre-$\theta$-closed set is called \textit{pre-$\theta$-open}. Of course, a pre-$\theta$-closed(resp. pre-$\theta$-open) set is a preclosed(resp. preopen) set.

Proposition 3.7. Let $A$ and $B$ be subsets of a space $X$. Then the following properties hold:

(1) if $A \subset B$, then $\text{pcl}_\theta(A) \subset \text{pcl}_\theta(B)$,

(2) if $A_\alpha$ is pre-$\theta$-closed in $X$ for each $\alpha \in \Delta$, then $\bigcap_{\alpha \in \Delta} A_\alpha$ is pre-$\theta$-closed in $X$.

\textbf{Proof.} The proof of (1) is obvious.

(2) Let $A_\alpha$ be pre-$\theta$-closed in $X$ for each $\alpha \in \Delta$. Then $A_\alpha = \text{pcl}_\theta(A_\alpha)$ for each $\alpha \in \Delta$. Thus we have

$$\text{pcl}_\theta\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \subset \bigcap_{\alpha \in \Delta} \text{pcl}_\theta(A_\alpha) = \bigcap_{\alpha \in \Delta} A_\alpha \subset \text{pcl}_\theta\left(\bigcap_{\alpha \in \Delta} A_\alpha\right).$$

Therefore, we have $\text{pcl}_\theta\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) = \bigcap_{\alpha \in \Delta} A_\alpha$ and hence $\bigcap_{\alpha \in \Delta} A_\alpha$ is pre-$\theta$-closed. \qed

Theorem 3.8. For any subset $A$ of an extremally predisconnected space $X$, the following hold:

$$\text{pcl}_\theta(A) = \bigcap\left\{V : A \subset V \text{ and } V \text{ is pre-$\theta$-closed}\right\}$$

$$= \bigcap\left\{V : A \subset V \text{ and } V \text{ is regular preopen}\right\}$$

\textbf{Proof.} We prove only the first equation since the other is similarly proved.

First, let $x \notin \text{pcl}_\theta(A)$. Then there is a preopen set $V$ with $x \in V$ such that $\text{pcl}(V) \cap A = \phi$. From Theorem 3.3, $X - \text{pcl}(V)$ is regular preopen, and hence
X − pcl(V) is a pre-θ-closed set containing A and x ̸∈ X − pclθ(V). Thus we have
\[ x ̸∈ \cap\{V : A ⊂ V \text{ and } V \text{ is pre-θ-closed}\}. \]
Conversely, suppose that \( x ̸∈ \cap\{V : A ⊂ V \text{ and } V \text{ is pre-θ-closed}\} \). Then there exists a pre-θ-closed set \( V \) such that \( A ⊂ V \) and \( x ̸∈ V \), and so there exists a preopen set \( U \) with \( x ∈ U \) such that \( U ⊂ pcl(U) ⊂ X − V \). Thus we have
\[ pcl(U) ∩ A ⊂ pcl(U) ∩ V = φ, \]
and hence \( x ̸∈ pclθ(A) \). □

**Theorem 3.9.** For any subset \( A \) of an extremally predis-connected space \( X \), the followings hold:
1. \( x ∈ pclθ(A) \) if and only if \( V ∩ A ≠ φ \) for each regular preopen set \( V \) with \( x ∈ V \);
2. \( A \) is pre-θ-open if and only if for each \( x ∈ A \) there exists a regular preopen set \( V \) with \( x ∈ V \) such that \( V ⊂ A \);
3. \( A \) is a regular preopen set if and only if \( A \) is pre-θ-clopen.

**Proof.** From theorem 3.2 and 3.3, (1) and (2) are obvious.

(3) Let \( A \) be a regular preopen set. Then \( A \) is a preopen set and so \( A = pcl(A) = pclθ(A) \), and hence \( A \) is pre-θ-closed. Since \( X − A \) is a regular preopen set, by the argument above, \( X − A \) is pre-θ-closed and \( A \) is pre-θ-open. The converse is obvious. □

It is obvious that regular preopen ⇒ pre-θ-open ⇒ preopen. But the converses are not necessarily true as the following examples show.

**Example 3.2.** Let \( X = \{a, b, c\} \) and \( τ = \{φ, X, \{a, b\}\} \). Then the subset \( \{a, b\} \) is a pre-θ-open set which is not regular preopen.

**Example 3.3.** Let \( X = \{a, b, c\} \) and \( τ = \{φ, X, \{a, b\}\} \). Then the subset \( \{a\} \) is a preopen set which is not pre-θ-open.

**4. \( p \)-closed spaces and \( p^* \)-closed spaces**

A filterbase \( F \) in \( X \) \( pθ^* \)-converges (resp. \( rp \)-converges) to \( x_0 ∈ X \) if for each preopen (rep. regular preopen) set \( A \) with \( x_0 ∈ A \), there exists \( F ∈ F \) such that \( F ⊂ pcl(A) \) (resp. \( F ⊂ A \)).
A filterbase $\mathcal{F}$ in $X$ $p\theta^*$-accumulates (resp. $rp$-accumulates) to $x_0 \in X$ if for each preopen (resp. regular preopen) set $A$ with $x_0 \in A$ and each $F \in \mathcal{F}$, $F \cap \text{pcl}(A) \neq \emptyset$ (resp. $F \cap A \neq \emptyset$).

The following theorems are easy consequences of the above definitions.

**Theorem 4.1.** If a filterbase $\mathcal{F}$ in $X$ $p\theta^*$-converges (resp. $rp$-converges) to $x_0 \in X$, then $\mathcal{F}$ $p\theta^*$-accumulates (resp. $rp$-accumulates) to $x_0$.

**Theorem 4.2.** If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filterbases in $X$ such that $\mathcal{F}_2$ subordinate to $\mathcal{F}_1$ and $\mathcal{F}_2$ $p\theta^*$-accumulates (resp. $rp$-accumulates) to $x_0$, then $\mathcal{F}_1$ $p\theta^*$-accumulates (resp. $rp$-accumulates) to $x_0$.

**Theorem 4.3.** If $\mathcal{F}$ is a maximal filterbase in $X$, then $\mathcal{F}$ $p\theta^*$-accumulates (resp. $rp$-accumulates) to $x_0$ if and only if $\mathcal{F}$ $p\theta^*$-converges (resp. $rp$-converges) to $x_0$.

**Theorem 4.4.** Let $X$ be an extremally predisconnected space. Then we have that a filterbase $\mathcal{F}$ in $X$ $p\theta^*$-converges to $x_0$ if and only if $\mathcal{F}$ $rp$-converges to $x_0$.

**Theorem 4.5.** Let $X$ be an extremally predisconnected space. Then a filterbase $\mathcal{F}$ in $X$ $p\theta^*$-accumulates to $x_0$ if and only if $\mathcal{F}$ $rp$-accumulates to $x_0$.

A space $X$ is said to be $p$-closed [2] if every cover of $X$ by preopen sets has a finite subcover whose preclosures cover $X$.

A space $X$ is said to be $p^*$-closed if every cover of $X$ by regular preopen sets has a finite subcover.

From theorem 3.1 and 3.3, we have the following result.

**Proposition 4.6.** An extremally predisconnected space $X$ is $p$-closed if and only if it is $p^*$-closed.

**Theorem 4.7.** For a space $X$, the following are equivalent.

1. $X$ is $p$-closed.

2. For each family $\{F_\alpha : \alpha \in \Delta\}$ of preclosed subset of $X$ such that $\bigcap_{\alpha \in \Delta} F_\alpha = \emptyset$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $\bigcap_{\alpha \in \Delta_0} \text{pint}(F\alpha) = \emptyset$.

3. For each family $\{F_\alpha : \alpha \in \Delta\}$ of preclosed subsets of $X$, if $\bigcap_{\alpha \in \Delta_0} \text{pint}(F\alpha) \neq \emptyset$ for every finite subset $\Delta_0$ of $\Delta$, then $\bigcap_{\alpha \in \Delta} F\alpha \neq \emptyset$. 

(4) Every filterbase \( \mathcal{F} \) in \( X \) \( \phi^* \)-accumulates to \( x_0 \in X \).

(5) Every maximal filterbase \( \mathcal{F} \) in \( X \) \( \phi^* \)-converges to \( x_0 \in X \).

**Proof.** The equivalence \((2) \iff (3)\) is obvious.

\((2) \Rightarrow (1)\). Let \( \{ A_\alpha : \alpha \in \Delta \} \) be a family of preopen subsets of \( X \) such that \( X = \bigcup_{\alpha \in \Delta} A_\alpha \). Then each \( X - A_\alpha \) is a preclosed subset of \( X \) and \( \bigcap_{\alpha \in \Delta} (X - A_\alpha) = \phi \), and so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( \bigcap_{\alpha \in \Delta_0} \text{pint}(X - A_\alpha) = \phi \), and hence

\[
X = \bigcup_{\alpha \in \Delta_0} \left( X - \text{pint}(X - A_\alpha) \right) = \bigcup_{\alpha \in \Delta_0} \text{pcl}(A_\alpha).
\]

\((4) \Rightarrow (2)\). Let \( \{ A_\alpha : \alpha \in \Delta \} \) be a family of preclosed subsets of \( X \) such that \( \bigcap_{\alpha \in \Delta} A_\alpha = \phi \). Suppose that for every finite subfamily \( \{ A_{\alpha_i} : i = 1, 2, \ldots, n \} \),

\[
\bigcap_{i=1}^{n} \text{pint}(A_{\alpha_i}) \neq \phi.
\]

Then \( \bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi \) and

\[
\mathcal{F} = \left\{ \bigcap_{i=1}^{n} A_{\alpha_i} : n \in N, \alpha_i \in \Delta \right\}
\]

forms a filterbase in \( X \). By (4), \( \mathcal{F} \) \( \phi^* \)-accumulates to some \( x_0 \in X \). Thus for every preopen set \( A \) with \( x_0 \in A \) and every \( F \in \mathcal{F} \), \( F \cap \text{pcl}(A) \neq \phi \). Since \( \bigcap_{F \in \mathcal{F}} F = \phi \), there exists a \( F \in \mathcal{F} \) such that \( x_0 \notin F \), and so there exists \( \alpha_0 \in \Delta \) such that \( x_0 \notin A_{\alpha_0} \) and hence \( x_0 \in X - A_{\alpha_0} \) and \( X - A_{\alpha_0} \) is a preopen set. Thus \( x_0 \notin \text{pint}(A_{\alpha_0}) \) and \( x_0 \in X - \text{pint}(A_{\alpha_0}) \), and hence

\[
F_0 \cap \left( X - \text{pint}(A_{\alpha_0}) \right) = F_0 \cap \text{pcl}(X - A_{\alpha_0}) = \phi,
\]

which is a contradiction.

\((5) \Rightarrow (4)\). Let \( \mathcal{F} \) be a filterbase in \( X \). Then there exists a maximal filterbase \( \mathcal{G} \) in \( X \) such that \( \mathcal{G} \) subordinate to \( \mathcal{F} \). Since \( \mathcal{G} \) \( \phi^* \)-converges to \( x_0 \), \( \mathcal{F} \) \( \phi^* \)-accumulate to \( x_0 \) by theorem 4.2 and 4.3.

\((1) \Rightarrow (5)\). Suppose that \( \mathcal{F} = \{ F_\alpha : \alpha \in \Delta \} \) is a maximal filterbase in \( X \) which does not \( \phi^* \)-converge to any point in \( X \). From theorem 4.3, \( \mathcal{F} \) does not \( \phi^* \)-accumulate to any point in \( X \). Thus for every \( x \in X \), there exist preopen \( A_x \) containing \( x \) and \( F_{\alpha_x} \in \mathcal{F} \) such that \( F_{\alpha_x} \cap \text{pcl}(A_x) = \phi \). Since \( \{ A_x : x \in X \} \) is a preopen cover of \( X \), there exists a finite subfamily \( \{ A_{x_i} : i = 1, 2, \ldots, n \} \) such that \( X = \bigcup_{i=1}^{n} \text{pcl}(A_{x_i}) \). Because \( \mathcal{F} \) is filterbase in \( X \), there exists \( F_0 \in \mathcal{F} \)
such that $F_0 \subset \bigcap_{i=1}^{n} F_{a_{x_i}}$, and hence $F_0 \bigcap pcl(A_{x_i}) = \phi$ for all $i = 1, 2, \cdots, n$.

Hence we have that

$$\phi = F_0 \bigcap \left( \bigcup_{i=1}^{n} pcl(A_{x_i}) \right) = F_0 \cap X,$$

and hence $F_0 = \phi$. This is a contradiction. □

The proof of the following result is similar to that of theorem 4.7 and is

omitted.

Theorem 4.8. For a space $X$, the following are equivalent.

(1) $X$ is $p^*$-closed.

(2) For each family of regular preclosed subsets of $X$ such that $\bigcap_{\alpha \in \Delta} A_\alpha = \phi$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $\bigcap_{\alpha \in \Delta_0} A_\alpha = \phi$.

(3) For each family $\{A_\alpha : \alpha \in \Delta\}$ of regular preclosed subsets of $X$, if $\bigcap_{\alpha \in \Delta} A_\alpha \neq \phi$ for every finite subset $\Delta_0$ of $\Delta$, then $\bigcap_{\alpha \in \Delta} A_\alpha \neq \phi$.

(4) Every filterbase $F$ in $X$ rp-accumulates to $x_0 \in X$.

(5) Every maximal filterbase $F$ in $X$ rp-converges to $x_0 \in X$.

From Proposition 4.6, we get the following corollary.

Corollary 4.9. If $X$ is an extremally predisconnected space, then the statements in theorem 4.8 and 4.9 are equivalent.

A net $(x_i)_{i \in D}$ in a space $X$ is said to be $p\theta^*$-converges[1](resp. rp-converges) to $x \in X$ if for each preopen set(resp. regular preopen set) $U$ with $x \in U$, there exists $i_0$ such that $x_i \in pcl(U)$ (resp. $x_i \in U$) for all $i \geq i_0$, where $D$ is a directed set.

A net $(x_i)_{i \in D}$ in a space $X$ is said to be $p\theta^*$-accumulates (resp. rp-accumulates) to $x \in X$ if for each preopen set (resp. regular preopen set) $U$ with $x \in U$ and each $i$, $x_i \in pcl(U)$ (resp. $x_i \in U$), where $D$ is a directed set.

It is routine to prove the following propositions.

Proposition 4.10. Let $(x_i)_{i \in D}$ be a net in $X$. For the filterbase $F((x_i)_{i \in D}) = \{\{x_i : i \geq j\} : j \in D\}$ in $X$, 
(1) $F \left( (x_i)_{i \in D} \right)$ $p^\theta$-converges (resp. $rp$-converges) to $x$ if and only if $(x_i)_{i \in D}$ $p^\theta$-converges (resp. $rp$-converges) to $x$.

(2) $F \left( (x_i)_{i \in D} \right)$ $p^\theta$-accumulates (resp. $rp$-accumulates) to $x$ if and only if $(x_i)_{i \in D} p^\theta$-accumulates (resp. $rp$-accumulates) to $x$.

**Proposition 4.11.** Every filter base $F$ in $X$ determines a net $(x_i)_{i \in D}$ in $X$ such that

(1) $F$ $p^\theta$-converges (resp. $rp$-converges) to $x$ if and only if $(x_i)_{i \in D}$ $p^\theta$-converges (resp. $rp$-converges) to $x$.

(2) $F$ $p^\theta$-accumulates (resp. $rp$-accumulates) to $x$ if and only if $(x_i)_{i \in D} p^\theta$-accumulates (resp. $rp$-accumulates) to $x$.

From Proposition 4.10 and 4.11, filterbases and nets are equivalent in the sense of $p^\theta$-converges (resp. $rp$-converges) and $p^\theta$-accumulates (resp. $rp$-accumulates). Therefore, we have the following results.

**Theorem 4.12.** For a space $X$, the following are equivalent.

(1) $X$ is $p$-closed.

(2) Each net $(x_i)_{i \in D}$ in $X$ has a $p^\theta$-accumulation point.

(3) Each universal net in $X$ $p^\theta$-converges.

**Theorem 4.13.** For a space $X$, the following are equivalent.

(1) $X$ is $p^*$-closed.

(2) Each net $(x_i)_{i \in D}$ in $X$ has an $rp$-accumulation point.

(3) Each universal net $rp$-converges.

A space $X$ is called $p$-regular [3] if for each open set $G$ in $X$ with $x \in G$, there exists a preopen set $U$ in $X$ such that $x \in U \subset pcl(U) \subset G$.

**Theorem 4.14.** If $X$ is $p$-regular and $p$-closed, then $X$ is compact.

*Proof.* Suppose $X$ is $p$-regular and $p$-closed, and let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of $X$. Then for each $x \in X$, there exists an $\alpha_x \in \Delta$ such that $x \in G_{\alpha_x}$. So there exists a preopen set $U_x$ in $X$ such that

$$x \in U_x \subset pcl(U_x) \subset G_{\alpha_x},$$

and so $X = \bigcup U_x$. Since $X$ is $p$-closed, there exists a finitely many $x_k$’s such that

$$X = \bigcup pcl(U_{x_k}) \subset \bigcup G_{\alpha_{x_k}},$$
and hence $X$ is compact. □

**Corollary 4.15.** Let $X$ be an extremally predisconnected and $p$-regular space. If $X$ is a $p^*$-closed space, then $X$ is compact.

A function $f : X \rightarrow Y$ is said to be $r$-precontinuous (resp. strongly $\theta$-precontinuous [7]) if for each $x \in X$ and each open set $V$ with $f(x) \in V$, there exists a regular preopen (resp. preopen) set $U$ with $x \in U$ such that $f(U) \subset V$ (resp. $f(pcl(U)) \subset V$).

A function $f : X \rightarrow Y$ is said to be $p$-continuous if for each $x \in X$ and each preopen set $V$ with $f(x) \in V$, there exists an open set $U$ with $x \in U$ such that $f(U) \subset pcl(V)$.

**Theorem 4.16.** Let $X$ be an extremally predisconnected space. Then $f : X \rightarrow Y$ is $r$-precontinuous if and only if $f$ is strongly $\theta$-precontinuous.

*Proof.* It follows from theorem 3.2. □

**Theorem 4.17.** If $f : X \rightarrow Y$ is a strongly $\theta$-precontinuous surjection and $X$ is $p$-closed, then $Y$ is compact.

*Proof.* Let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of $Y$. Then for each $x \in X$, there exists $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$, and so there exists a preopen set $U_x$ with $x \in U_x$ such that $f\left(pcl(U_x)\right) \subset V_{\alpha_x}$. So $\{U_x : x \in X\}$ is a preopen cover of $X$, and hence there exists finitely many $x_i$'s such that $X = \bigcup pcl(U_{x_i})$ and so

$$Y = f\left(\bigcup pcl(U_{x_i})\right) \supseteq \bigcup f\left(pcl(U_{x_i})\right) \subset \bigcup V_{\alpha_{x_i}},$$

and hence $Y$ is compact. □

The following theorems can be proved similarly.

**Theorem 4.18.** If $f : X \rightarrow Y$ is a $r$-precontinuous surjection and $X$ is $p^*$-closed, then $Y$ is compact.

**Theorem 4.19.** If $f : X \rightarrow Y$ is a $p$-continuous surjection and $X$ is compact, then $Y$ is $p^*$-closed.
A space $X$ is said to be $\text{pre-}T_2[8]$ if for each pair of distinct points $x$ and $y$ in $X$, there exist preopen sets $U$ with $x \in U$ and $V$ with $y \in V$ such that $U \cap V = \phi$.

Lemma 4.20. A space $X$ is $\text{pre-}T_2$ if and only if for each pair of distinct points $x$ and $y$ in $X$, there exists a preopen set $U$ with $x \in U$ such that $y \notin \text{pcl}(U)$.

Proposition 4.21. Let $X$ be an extremally predisconnected and $\text{p-closed}$ space, and let $Y$ be a regular preclosed subset of $X$. Then $Y$ is $\text{p-closed}$.

Proof. Let $\{U_i : i \in \Delta\}$ be a family of preopen susets of $X$ such that $Y \subset \bigcup_{i \in \Delta} U_i$. Then

$$X = \left( \bigcup_{i \in \Delta} U_i \right) \bigcup (X - Y)$$

and $X - Y$ is a preopen set, and so there exists a finite subset $\Delta_0$ of $\Delta$ such that

$$X = \left( \bigcup_{i \in \Delta_0} \text{pcl}(U_i) \right) \bigcup \text{pcl}(X - Y).$$

From theorem 3.2,

$$X = \left( \bigcup_{i \in \Delta_0} \text{pcl}(U_i) \right) \bigcup (X - Y)$$

and so $Y \subset \bigcup_{i \in \Delta_0} \text{pcl}(U_i)$, and hence $Y$ is $\text{p-closed}$. □

A function $f : X \to Y$ has a $\text{p-closed graph}$ if for each $(x, y) \notin G(f)$, there exist open set $U$ with $x \in U$ and preopen set $V$ with $y \in V$ such that

$$\left( U \times \text{pcl}(V) \right) \cap G(f) = \phi.$$

Theorem 4.22. Let $X$ be a space, and let $Y$ be a $\text{pre-}T_2$, $\text{p-closed}$ and extremally predisconnected space. If $f : X \to Y$ has a $\text{p-closed graph}$, then $f$ is $\text{p-continuous}$.

Proof. Let $x \in X$ and $V$ be a preopen set with $f(x) \in V$, and let $y \in Y - \text{pcl}(V)$. Then $(x, y) \notin G(f)$, and so there exist open set $U_y(x)$ with $x \in U_y(x)$ and preopen set $V(y)$ with $y \in V(y)$ such that

$$\left( U_y(x) \times \text{pcl}(V(y)) \right) \cap G(f) = \phi.$$
Since $Y$ is pre-$T_2$, we can choose $V(y)$ such that $f(x) \notin pcl(V(y))$. Then the family $\{V(y) : y \in Y - pcl(V)\}$ is a preopen cover of the regular preclosed set $Y - pcl(V)$. From proposition 4.20, there is finitely many $y_i$’s such that
\[ Y - pcl(V) \subset \bigcup_{i=1}^{n} pcl(V(y_i)). \]

Let $U = \bigcap_{i=1}^{n} U_{y_i}(x)$. Then $U$ is an open set with $x \in U$ and contains no point $u \in U$ such that
\[ f(u) \in \bigcup_{i=1}^{n} pcl(V(y_i)). \]

Thus $f(U) \subset pcl(V)$ and hence $f$ is $p$-continuous.  

REFERENCES


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On regular preopen sets and $p^*$-closed spaces

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