CONVERGENCE OF CHOQUET INTEGRAL

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Abstract. In this paper, we consider various types of convergence theorems of Choquet integral. We also show that the autocontinuity of finite fuzzy measure is equivalent to a convergence theorem with respect to convergence in measure.

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1. Introduction

In 1994, Denneberg[2] introduced the concept of convergence in distribution and proved a dominated convergence theorem for the Choquet integral with respect to the fuzzy measure in the sense of Sugeno[5], i.e., non-additive monotone set function.

In 1997, Murofushi et al.[4] proved that autocontinuity is the necessary and sufficient condition. They also showed that the autocontinuity of finite fuzzy measure is equivalent to two convergence theorem with respect to convergence in measure as an application of Denneberg’s dominated convergence theorem. In this paper, we study further about convergence of Choquet integral. Indeed, we consider monotone convergence theorem, a version of Fatou’s lemma, uniform integrability and its related convergence theorem. And, applying this results, we also show that the autocontinuity of finite fuzzy measure is equivalent to a convergence in measure.

2. Preliminaries

We use the same notation as Murofushi et al.[4] used.
Definition 2.1. A fuzzy measure on a measurable shape \((X, \mathcal{F})\) is a real-valued set function \(\mu : \mathcal{F} \to \bar{\mathbb{R}}^+\) satisfying
\[
\begin{align*}
(1) \quad & \mu(\emptyset) = 0, \\
(2) \quad & \mu(A) \leq \mu(B) \quad \text{whenever} \quad A \subseteq B, \quad B \in \mathcal{F},
\end{align*}
\]
where \(\bar{\mathbb{R}}^+ = [0, \infty]\), the set of nonnegative extended real numbers. If \(\mu(X) < \infty\), the fuzzy measure \(\mu\) is said to be finite.

Note that in this paper fuzzy measures are not assumed to be continuous. Throughout the paper we assume that \(\mu\) is a fuzzy measure on a measurable space \((X, \mathcal{F})\).

Definition 2.2. (1) [3] A fuzzy measure \(\mu\) is said to be subadditively continuous from above [resp. below] if for every \(A \in \mathcal{F}\) and \(\epsilon > 0\) there exists a \(\delta > 0\) such that \(\mu\left(A \cup B\right) \leq \mu(A) + \epsilon\) [resp. \(\mu\left(A \setminus B\right) \leq \mu(A) - \epsilon\)] whenever \(B \in \mathcal{F}\) and \(\mu(B) < \delta\).

If \(\mu\) is subadditively continuous both from above and from below, it is said to be subadditively continuous.

(2) [6] A fuzzy measure \(\mu\) is said to be autocontinuous from above [resp. below] if \(\mu\left(A \cup B_n\right) \rightharpoonup \mu(A)\) [resp. \(\mu\left(A \setminus B_n\right) \rightharpoonup \mu(A)\)] whenever \(A \in \mathcal{F}\), \(\{B_n\} \subset \mathcal{F}\), and \(\mu(B_n) \to 0\).

If \(\mu\) is autocontinuous both from above and from below, it is said to be autocontinuous.

The subadditive continuity from above is equivalent to the autocontinuity from above. The subadditive continuity from below implies the autocontinuity from below while the converse does not always hold; it holds when \(\mu\) is finite.

Definition 2.3. The Choquet integral of a measurable function \(f\) with respect to \(\mu\) is defined by
\[
(C) \quad \int f\,d\mu = \int_0^\infty G_f(r)\,dr + \int_{r_f}^0 \left[G_f(r) - \mu(X)\right]\,dr
\]
whenever the right-hand side is not the indeterminate from \(\infty + (-\infty)\), where \(G_f\) is the (decreasing) distribution function \([1]\) of \(f\) defined by
\[
G_f(r) = \mu\left\{x | f(x) > r\right\},
\]
and
\[
r_f = 0 \wedge \inf\left\{r | G_f(r) < \infty\right\}.
\]
A measurable function \(f\) is called integrable if the Choquet integral of \(f\) can be defined and its value is finite.
Definition 2.4. A sequence \( \{f_n\} \) of measurable functions is said to converge to \( f \) in measure, in symbols \( f_n \rightarrow^\mu f \), if, for every \( \epsilon > 0 \),
\[
\lim_{n \to \infty} \mu \left( \left\{ x \mid |f_n(x) - f(x)| > \epsilon \right\} \right) = 0.
\]

Definition 2.5 (Denneberg [2]). A sequence \( \{f_n\} \) of measurable functions is said to converge to \( f \) in distribution, in symbols \( f_n \rightarrow^D f \), if
\[
\lim_{n \to \infty} G_{f_n}(r) = G_f(r) \text{ e.c.},
\]
where “e.c.” stands for “except at most countably many values of \( r \)”. The following is Denneberg’ dominated convergence theorem.

Theorem 2.1 (Denneberg [2]). If \( \{f_n\} \) is a sequence of measurable functions that converges to \( f \) in distribution, and if \( g \) and \( h \) are integrable functions such that \( G_h \leq G_{f_n} \leq G_g \) e.c., then \( f \) is integrable and
\[
\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.
\]

Definition 2.6 (Murofushi et al.[4]). A measurable function \( f \) is said to be essentially bounded if there is a positive number \( a \) such that \( G_f(-a) = \mu(X) \) and \( G_f(a) = 0 \). A family \( \{F_\lambda\}_{\lambda \in \Lambda} \) of measurable functions is said to be uniformly essentially bounded if there is a positive number \( a \) such that \( G_{f_\lambda}(-a) = \mu(X) \) and \( G_{f_\lambda}(a) = 0 \) for all \( \lambda \in \Lambda \).

Using Theorem 2.1, Murofushi et al.[4] characterized the autocontinuity of finite measure in terms of the Choquet integral.

Theorem 2.2. If \( \mu \) is a finite, then the following three conditions are equivalent to one another.

1. \( \mu \) is autocontinuous.
2. If \( \{f_n\} \) is a sequence of measurable functions that converges to \( f \) in measure, and if \( g \) and \( h \) are integrable functions such that \( G_h \leq G_{f_n} \leq G_g \) e.c., then
\[
\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.
\]
3. If \( \{f_n\} \) is a uniformly essentially bounded sequence of measurable functions that converges to \( f \) in measure, then
\[
\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.
\]
We first consider Monotone Convergence Theorem.

**Theorem 3.1.** Let \( \{ f_n \} \) be a sequence of measurable function.

1. If \( f_n \uparrow \) and \( f_n \rightarrow_D f \), and there exists a Choquet integrable function \( \phi \), such that \( f_n \geq \phi \) for all \( n \), then \( \lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu \).

2. If \( f_n \downarrow \) and \( f_n \rightarrow_D f \), and there exists a Choquet integrable function \( \phi \), such that \( f_n \leq \phi \) for all \( n \), then \( \lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu \).

**Proof.** (1) By the assumption, we see that \( G \phi \leq G f_n(r) \uparrow G(r) \) e.c., hence by the classical monotone convergence theorem, we have
\[
(C) \int f_n d\mu = \int G f_n(r) dr \uparrow \int G f(r) dr = (C) \int f d\mu.
\]
The proof of (2) is same.

**Note.** We also note that the condition \( 0 \leq f_n(x) \uparrow f(x) \) for all \( x \in X \), in general, does not imply that \( f_n \) converges to \( f \) in distribution.

**Theorem 3.2 (Fatou’s Lemma).** Let \( \{ f_n \} \) be a sequence of measurable function. If there exists Choquet integrable function \( \phi \) such that \( f_n \geq \phi \) for all \( n \) and suppose \( g_n = \inf_{k \geq n} f_k \) converges to \( g = \liminf f_n \) in distribution. Then we have
\[
(C) \int \liminf f_n d\mu \leq \liminf (C) \int f_n d\mu.
\]

**Proof.** Since \( g_n \leq g_{n+1}, \quad n = 1, 2, \cdots, \) by Theorem 3.1, we have
\[
\lim(C) \int g_n d\mu = (C) \int g d\mu = (C) \int \liminf f_n d\mu.
\]
Now, we see \( (C) \int f_n d\mu \geq (C) \int g_n d\mu \) for all \( n \), hence we have
\[
\liminf(C) \int f_n d\mu \geq \liminf(C) \int g_n d\mu = (C) \int \liminf f_n d\mu.
\]

**Corollary 3.3.** Let \( \{ f_n \} \) be a sequence of measurable function. If there exists Choquet integrable function \( \phi \) such that \( f_n \leq \phi \) for all \( n \) and suppose \( g_n = \sup_{k \geq n} f_k \) converges to \( g = \liminf f_n \) in distribution. Then we have
\[
(C) \int \liminf f_n d\mu \leq \liminf(C) \int f_n d\mu.
\]

**Note.** In general, \( f_k \rightarrow_D f \) does not imply \( g_n \rightarrow_D g \).
Definition 3.1. A sequence \( \{f_n\} \) is uniformly integrable if
\[
\lim_{\alpha \to \infty} \sup_n (C) \int \left| f_n \right| 1_{|f_n| > \alpha} d\mu = 0.
\]

We note that
\[
G_{|f_n| 1_{|f_n| > \alpha}}(r) = \begin{cases} 
\mu(X) & \text{if } r < 0 \\
G_{|f_n|}(\alpha) & \text{if } 0 \leq r < \alpha \\
G_{|f_n|}(r) & \text{if } \alpha \leq r,
\end{cases}
\]
and hence
\[
(C) \int |f_n| 1_{|f_n| > \alpha} d\mu = \int_0^\alpha G_{|f_n|}(\alpha) d\alpha + \int_\alpha^\infty G_{|f_n|}(r) dr = \alpha G_{|f_n|}(\alpha) + \int_\alpha^\infty G_{|f_n|}(r) dr.
\] (1)

Note. It is well known that any family that is dominated by an integrable function is uniformly integrable ([1], Theorem 9.5E). But the converse is not true. Let \( X = [0, 1] \) and \( \mu \) be Lebesgue measure. Let \( f_n(x) = \frac{1}{n} \) on \((\frac{1}{n+1}, \frac{1}{n}]\) and 0, otherwise. Then we can easily check that this family is uniformly integrable but cannot be dominated by any integrable function since \( \int_{[0,1]} \frac{1}{n} dx = \infty \). Hence the following result is a generalization of the result Theorem 2.1 (Denneberg [2]).

Theorem 3.4. If \( \mu \) is finite and \( \{f_n\} \) converges to \( f \) in distribution for uniformly integrable \( \{f_n\} \), then \( f \) is integrable and
\[
\lim (C) \int f_n d\mu = (C) \int f d\mu.
\]

Proof. Since
\[
\int |f_n| d\mu \leq \alpha \mu(X) + (C) \int |f_n| 1_{|f_n| > \alpha} d\mu,
\]
it follows by the assumption of uniformly integrability of \( \{f_n\} \) that \( (C) \int |f_n| d\mu \) is bounded. By Theorem 3.2 \( f \) is thus integrable. Then by (1), for given \( \epsilon > 0 \), there exist \( \alpha_0 > 0 \) such that for \( \alpha \geq \alpha_0 \),
\[
H(\alpha) = \sup_n \left( \int_\alpha^\infty G_{|f|}(r) + \int_\alpha^\infty G_{|f_n|}(r) dr \right) < \frac{\epsilon}{2}.
\]
On the other hand, by bounded convergence theorem, there exist \( N > 0 \) such that for \( n \geq N \),
\[
\left| \int_{-\alpha_0}^{\alpha_0} G_{|f_n|}(r) dr - \int_{-\alpha_0}^{\alpha_0} G_f(r) dr \right| < \frac{\epsilon}{2}.
\]
Therefore, for \( n \geq N \),
\[
\left| \int G_{|f_n|}(r) dr - \int G_f(r) dr \right| \leq \left| \int_{-\alpha_0}^{\alpha_0} G_{|f_n|}(r) dr - \int_{-\alpha_0}^{\alpha_0} G_f(r) dr \right| + H(\alpha_0) \leq \epsilon,
\]
which completes the proof. \( \Box \)

Using Theorem 3.3, we can similarly characterize the autocontinuity of finite fuzzy measure in terms of the Choquet integral as Murofushi et al.[4] did in their Theorem 3.3.

**Theorem 3.5.** If \( \mu \) is finite, then the following two conditions are equivalent to one another.

1. \( \mu \) is autocontinuous.
2. If \( \{f_n\} \) is a uniformly integrable sequence of measurable functions that converges to \( f \) in measure, then
   \[
   \lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.
   \]

**Proof.** (1) \( \Rightarrow \) (2). This is direct from Theorem 3.1[4] and Theorem 3.4.

(2) \( \Rightarrow \) (1). Let \( A \in \mathcal{F} \), \( \{B_n\} \subset \mathcal{F} \), and \( \mu(B_n) \to 0 \). Since \( \{1_{A \cup B_n}\} \) is a uniformly integrable sequence that converges to \( 1_A \) in measure, it follows that
\[
\mu(A \cup B_n) = (C) \int 1_{A \cup B_n} d\mu \to (C) \int 1_A d\mu = \mu(A).
\]

It follows similarly that \( \mu(A \setminus B_n) \to \mu(A) \). \( \Box \)

**References**


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