SPACES OF CONJUGATION-EQUIVARIANT FULL HOLOMORPHIC MAPS

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Abstract. Let $\text{Rat}_k(\mathbb{C}P^n)$ denote the space of basepoint-preserving conjugation-equivariant holomorphic maps of degree $k$ from $S^2$ to $\mathbb{C}P^n$. A map $f : S^2 \to \mathbb{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbb{C}P^n$. Let $\text{RF}_k(\mathbb{C}P^n)$ denote the subspace of $\text{Rat}_k(\mathbb{C}P^n)$ consisting of full maps. In this paper we determine $H_*(\text{RF}_k(\mathbb{C}P^2); \mathbb{Z}/p)$ for all primes $p$.

1. Introduction

Let $\text{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree $k$ from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^n$. Since $\text{PSL}(n+1, \mathbb{C})$ acts on $\mathbb{C}P^n$ transitively, we can choose the basepoint condition as $f(\infty) = [1, 0, \ldots, 0]$. Such holomorphic maps are given by rational functions:

\begin{equation}
\text{Rat}_k(\mathbb{C}P^n) = \{(p_0(z), \ldots, p_n(z)) : \text{the following (i) and (ii) hold}\}
\end{equation}

(i) Each $p_i(z)$ ($0 \leq i \leq n$) has the form

\begin{align*}
p_0(z) &= z^k + a_{0,1}z^{k-1} + \cdots + a_{0,k} \\
p_i(z) &= a_{i,1}z^{k-1} + \cdots + a_{i,k} (1 \leq i \leq n),
\end{align*}

where $a_{i,j} \in \mathbb{C}$.

(ii) There are no roots common to all $p_i(z)$ for $0 \leq i \leq n$.

A map $f : S^2 \to \mathbb{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbb{C}P^n$. If $f$ is given by a rational function in (1.1), then $f$ is full if and only if the polynomials $p_i(z)$ ($0 \leq i \leq n$) are linearly independent in $\mathbb{C}[z]$. Let $F_k(\mathbb{C}P^n)$ be the subspace of $\text{Rat}_k(\mathbb{C}P^n)$ consisting of full maps. (The motivation for studying $F_k(\mathbb{C}P^n)$ is explained in [3].) We have the following sequence of inclusions:

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We already know the following results about these inclusions:

1. About $\text{Rat}_k(CP^n)$ we have
   
   (i) It is proved in [7] that the inclusion $\text{Rat}_k(CP^n) \hookrightarrow \Omega^2 CP^n \simeq \Omega^2 S^{2n+1}$ is a homotopy equivalence up to dimension $k(2n-1)$. (Throughout this paper, to say that a map $f : X \to Y$ is a homotopy equivalence up to dimension $d$ is intended to mean that $f$ induces isomorphisms in homotopy groups in dimensions less than $d$, and an epimorphism in dimension $d$.)
   
   (ii) The stable homotopy type of $\text{Rat}_k(CP^n)$ was described in [2] as follows. Let $\Omega^2 S^{2n+1} \simeq \bigvee_{s=1}^q \mathbb{D}_q(S^{2n-1})$ be Snaith’s stable splitting of $\Omega^2 S^{2n+1}$. Then

   $\text{Rat}_k(CP^n) \simeq \bigvee_{s=1}^k \mathbb{D}_q(S^{2n-1})$.

2. About $F_k(CP^n)$ we have from [3] that
   
   (i) Particular examples are: $F_k(CP^1) = \text{Rat}_k(CP^1)$ for $1 \leq k$; $F_k(CP^n) = \emptyset$ for $k < n$; and $F_n(CP^n) \cong \mathbb{C}^n \times GL(n, \mathbb{C})$.
   
   (ii) The inclusion $F_k(CP^n) \hookrightarrow \text{Rat}_k(CP^n)$ is a homotopy equivalence up to dimension $2(k-n) + 1$.
   
   (iii) $H_*(F_k(CP^2); \mathbb{Z}/p)$ was determined for all primes $p$.

In [4] and [5] a conjugation-equivariant version of these results was studied. Let $\text{Map}_T^k(CP^1, CP^n)$ denote the space of continuous basepoint-preserving conjugation-equivariant maps of degree $k$ from $CP^1$ to $CP^n$. We set $\text{RRat}_k(CP^n) = \text{Map}_T^k(CP^1, CP^n) \cap \text{Rat}_k(CP^n)$. An element $(p_0(z), \ldots, p_n(z)) \in \text{Rat}_k(CP^n)$ belongs to $\text{RRat}_k(CP^n)$ if and only if each $p_i(z)$ has real coefficients. Finally we set $\text{RF}_k(CP^n) = \text{RRat}_k(CP^n) \cap F_k(CP^n)$. Then we have the following sequence of inclusions:

$\text{RF}_k(CP^n) \hookrightarrow \text{RRat}_k(CP^n) \hookrightarrow \text{Map}_T^k(CP^1, CP^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$.

Similarly to the above (1) and (2), we know the following results:

3. About $\text{RRat}_k(CP^n)$ we have
   
   (i) The inclusion $\text{RRat}_k(CP^n) \hookrightarrow \text{Map}_T^k(CP^1, CP^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$ is a homotopy equivalence up to dimension $(k+1)(n-1) - 1$. 

When \( n = 1 \), Brockett ([1]) and Segal ([7]) showed that

\[
\text{RRat}_k(\mathbb{C}P^1) \simeq \prod_{i=0}^{k} \text{Rat}_{\min(i,k-i)}(\mathbb{C}P^1).
\]

(2) shows that the most interesting part of the homology of \( F_k(\mathbb{C}P^n) \) is the classes in dimensions \( \geq 2(k-n) + 1 \). Then, in (2)(iii), the homology \( H_*(F_k(\mathbb{C}P^n); \mathbb{Z}/p) \) was determined completely. The result shows that the inclusion \( F_k(\mathbb{C}P^n) \hookrightarrow \text{Rat}_k(\mathbb{C}P^n) \) has a nontrivial kernel in homology in dimensions \( \geq 2(k-n) + 1 \). Similarly, it is interesting to study the homology of \( \text{RF}_k(\mathbb{C}P^n) \) in dimensions \( \geq k-n \). In connection with this, the following result was proved in [5]:

(5) Let \( SO(k)/SO(k-n) \) be the Stiefel manifold of orthonormal \( n \)-frames in \( \mathbb{R}^k \). (When \( k = n \), we understand this as \( O(n) \).) Then there is a map \( \alpha_{k,n} : \text{RF}_k(\mathbb{C}P^n) \to SO(k)/SO(k-n) \) so that \( \alpha_{k,n} \) is a homotopy equivalence up to dimension \( n-1 \).

But the result corresponding to (2)(iii) is left unknown. Hence, the purpose of this paper is to determine \( H_*(\text{RF}_k(\mathbb{C}P^2); \mathbb{Z}/p) \) completely.

In order to state our main result, we define the notation

\[
\frac{S^1 \times \text{Rat}_l(\mathbb{C}P^1)}{S^1 \land D_l(S^1)}
\]

as follows: From (1.2), \( D_l(S^1) \) is a stable summand in \( \text{Rat}_l(\mathbb{C}P^1) \). Since \( \Sigma(K \times L) \simeq \Sigma(K \land L) \lor \Sigma K \lor \Sigma L, \ S^1 \land D_l(S^1) \) is a stable summand in \( S^1 \times \text{Rat}_l(\mathbb{C}P^1) \). Then (1.4) is defined to be the identification space obtained from \( S^1 \times \text{Rat}_l(\mathbb{C}P^1) \) by collapsing \( S^1 \land D_l(S^1) \) to a point.

Then our main result is as follows.

**Theorem A.** Let \( p \) be a prime. Then, we have the following isomorphisms of vector spaces:
\( (i) \) When \( k = 2l \).

\[
H_*(RF_{2l}(\mathbb{C}P^2); \mathbb{Z}/p) \cong H_*(RRat_{2l-2}(\mathbb{C}P^2); \mathbb{Z}/p)
\]

\( \oplus \bigoplus_{i=0}^{l-2} H_*(S^1 \times \text{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p) \)

\( \oplus H_*(S^1 \wedge D(S^1); \mathbb{Z}/p) \).

\( (ii) \) When \( k = 2l + 1 \).

\[
H_*(RF_{2l+1}(\mathbb{C}P^2); \mathbb{Z}/p) \cong H_*(RRat_{2l}(\mathbb{C}P^2); \mathbb{Z}/p)
\]

\( \oplus \bigoplus_{i=0}^{l-1} H_*(S^1 \times \text{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p) \).

2. Proof of theorem A

In what follows, we consider \( RF_k(\mathbb{C}P^n) \) only for \( n = 2 \) and set

\[
X_k = RRat_k(\mathbb{C}P^2) - RF_k(\mathbb{C}P^2).
\]

**Lemma 2.1.** There is a homeomorphism

\[
X_k \cong S^1 \times_{\mathbb{Z}/2} RRat_k(\mathbb{C}P^1).
\]

Here \( \mathbb{Z}/2 \) acts on \( S^1 \) by antipodal and acts on \( RRat_k(\mathbb{C}P^1) \) by

\[
(-1) \cdot (v_0(z), v_1(z)) = (v_0(z), -v_1(z)),
\]

where \((v_0(z), v_1(z)) \in RRat_k(\mathbb{C}P^1)\) is given in the form (1.1) with \( a_{i,j} \in \mathbb{R} \).

**Proof.** We write an element \((u_0(z), u_1(z), u_2(z)) \in X_k\) in the form (1.1) with \( a_{i,j} \in \mathbb{R} \). Since \( u_1(z) \) and \( u_2(z) \) are linearly dependent but not both are zero, there exist \( \xi \in (\mathbb{R}^2)^* \) and a real polynomial \( \phi(z) \) so that \((u_1(z), u_2(z)) = \xi \phi(z)\). Let \( \mathbb{R}^* \) act on \( RRat_k(\mathbb{C}P^1) \) by

\[
r \cdot (v_0(z), v_1(z)) = (v_0(z), \frac{1}{r} v_1(z)),
\]

where \( r \in \mathbb{R}^* \). Then a homeomorphism \( X_k \cong (\mathbb{R}^2)^* \times_{\mathbb{R}^*} RRat_k(\mathbb{C}P^1) \) is given by \((u_0(z), u_1(z), u_2(z)) \mapsto (\xi, (u_0(z), \phi(z)))\). Since \((\mathbb{R}^2)^* \times_{\mathbb{R}^*} RRat_k(\mathbb{C}P^1) \cong S^1 \times_{\mathbb{Z}/2} RRat_k(\mathbb{C}P^1)\), the result follows. \( \square \)
**Proposition 2.2.** We have the following long exact sequence:

\[
\cdots \to H_*(RF_k(\mathbb{C}P^2); \mathbb{Z}/p) \to H_*(RRat_k(\mathbb{C}P^2); \mathbb{Z}/p) \\
J \bigoplus_{i=0}^k H_{*-i}(S^1 \times Rat_i(\mathbb{C}P^1); \mathbb{Z}/p) \to H_{*-1}(RF_k(\mathbb{C}P^2); \mathbb{Z}/p) \to \cdots,
\]

where the homomorphism \(J\) will be specified later.

**Proof.** Consider the homology sequence of the pair \((RRat_k(\mathbb{C}P^2), RF_k(\mathbb{C}P^2))\). We first prove the following:

**Lemma 2.3.** There is an isomorphism

\[
H_*(RRat_k(\mathbb{C}P^2), RF_k(\mathbb{C}P^2); \mathbb{Z}/p) \cong H_{*-k}(S^1 \times \mathbb{Z}/2 RRat_k(\mathbb{C}P^1); \mathbb{Z}/p).
\]

**Proof.** The lemma is a real version of [3, (5.2)] and a proof is given as follows. Let \(D^{k-1} \to \nu \to X_k\) be the closed normal disk bundle of \(X_k\) in \(RRat_k(\mathbb{C}P^2)\), where \(X_k\) is defined in (2.1). By excision (see [6], Corollary 11.2), we have an isomorphism

\[
H_*(RRat_k(\mathbb{C}P^2), RF_k(\mathbb{C}P^2); \mathbb{Z}/p) \cong H_*(\nu, \partial \nu; \mathbb{Z}/p).
\]

By the Thom isomorphism (see [6], Theorem 10.4), we have an isomorphism

\[
H_*(\nu, \partial \nu; \mathbb{Z}/p) \cong H_{*-k}(X_k; \mathbb{Z}/p).
\]

Now the lemma follows from Lemma 2.1.

Next we study \(S^1 \times \mathbb{Z}/2 RRat_k(\mathbb{C}P^1)\). It is easy to see that the \(\mathbb{Z}/2\)-action on the right-hand side of (1.3) is given as follows: When \(k = 2m\), each \(Rat_i(\mathbb{C}P^1)\) \((0 \leq i \leq m - 1)\) appears twice and \(Rat_m(\mathbb{C}P^1)\) once in \(RRat_{2m}(\mathbb{C}P^1)\). Then \(\mathbb{Z}/2\) exchanges two copies of \(Rat_i(\mathbb{C}P^1)\) \((0 \leq i \leq m - 1)\) and acts on \(Rat_m(\mathbb{C}P^1)\) by the involution \(T : Rat_m(\mathbb{C}P^1) \to Rat_m(\mathbb{C}P^1)\) defined by \(T(p_0(z), p_1(z)) = (p_0(z), -p_1(z))\). When \(k = 2m + 1\), the \(\mathbb{Z}/2\)-action is given similarly.

Now when \(k = 2m + 1\), Proposition 2.2 follows from Lemma 2.3. On the other hand, when \(k = 2m\), we need the following:

**Lemma 2.4.** We have an isomorphism

\[
H_*(S^1 \times_T Rat_m(\mathbb{C}P^1); \mathbb{Z}/p) \cong H_*(S^1 \times Rat_m(\mathbb{C}P^1); \mathbb{Z}/p).
\]
Proof. Since $T_* : H_*(\text{Rat}_m(\mathbb{C}P^1); \mathbb{Z}/p) \to H_*(\text{Rat}_m(\mathbb{C}P^1); \mathbb{Z}/p)$ is the identity mapping, the local system for the fibration $\text{Rat}_m(\mathbb{C}P^1) \to S^1 \times_T \text{Rat}_m(\mathbb{C}P^1) \to S^1$ is simple. Hence Lemma 2.4 holds. This completes the proof of Proposition 2.2.

We study the homomorphism $J$ in Proposition 2.2. Let us introduce the following homomorphisms $\varphi$ and $\psi$:

$$H_*(\Omega^2 S^5; \mathbb{Z}/p) \xrightarrow{\varphi} H_*(\Omega^2 S^3; \mathbb{Z}/p) \xrightarrow{\psi} \bigoplus_{0 \leq i} H_*(\text{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p).$$

For that purpose, we recall the structure of $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$. There is a (torsion free) generator $\iota_{2n-1} \in H_{2n-1}(\Omega^2 S^{2n+1}; \mathbb{Z}/p) \cong \mathbb{Z}/p$, and the following hold.

(i) For $p = 2$,

$$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\iota_{2n-1}, Q_1(\iota_{2n-1}), \ldots, Q_1 \cdots Q_1(\iota_{2n-1}), \ldots].$$

(ii) For an odd prime $p$,

$$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p) \cong \bigwedge(\iota_{2n-1}, Q_1(\iota_{2n-1}), \ldots, Q_1 \cdots Q_1(\iota_{2n-1}), \ldots) \otimes \mathbb{Z}/p[\beta Q_1(\iota_{2n-1}), \ldots, \beta Q_1 \cdots Q_1(\iota_{2n-1}), \ldots].$$

In (i) and (ii), $Q_1$ is the first Dyer-Lashof operation (it takes a class of dimension $d$ to a class of dimension $dp + p - 1$) and $\beta$ is the mod $p$ Bockstein operation.

Now we define the homomorphisms $\varphi$ and $\psi$ as follows:

1. Let $x \in H_*(\Omega^2 S^3; \mathbb{Z}/p)$ be a monomial. We define $\varphi(x)$ to be the element obtained from $x$ by changing all $\iota_1$ to $\iota_3$. Note that if $\mathcal{W}$ is the usual weight on $H_*(\Omega^2 S^3; \mathbb{Z}/p)$, then

$$\deg \varphi(x) = \deg x + 2\mathcal{W}(x).$$

2. Let $x \in H_*(\Omega^2 S^3; \mathbb{Z}/p)$ be a monomial. Then, from (1.2), we can regard that $x \in H_*(\text{Rat}_{\mathcal{W}(x)}(\mathbb{C}P^1); \mathbb{Z}/p)$. We set

$$\psi(x) = (0, \ldots, 0, x, 0, \ldots) \in \bigoplus_{0 \leq i} H_*(\text{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p),$$

where the element $x$ in the right-hand side belongs to the direct summand indexed by $i = \mathcal{W}(x)$.

3. In the situation of (2), using the generators $1 \in H_0(S^1; \mathbb{Z}/p)$ and $e_1 \in H_1(S^1; \mathbb{Z}/p)$, we construct elements $1 \otimes \psi(x)$ and $e_1 \otimes \psi(x)$ of
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\[ \bigoplus_{0 \leq i} H_\ast(S^1 \times \text{Rat}_i(\mathbb{CP}^1); \mathbb{Z}/p) \] belonging to the direct summand indexed by \( i = \mathbb{w}(x) \).

The following lemma describes the homomorphism

\[ J : H_\ast(\text{R} \text{Rat}_k(\mathbb{CP}^2); \mathbb{Z}/p) \to \bigoplus_{i=0}^k H_\ast-(k-1)(S^1 \times \text{Rat}_i(\mathbb{CP}^1); \mathbb{Z}/p). \]

**Lemma 2.5.** Let us denote by \( \mathbb{w} \) the weight on \( H_\ast(\text{R} \text{Rat}_k(\mathbb{CP}^2); \mathbb{Z}/p) \) defined in Section 1 (3)(iii). Let \( \alpha \in H_\ast(\text{R} \text{Rat}_k(\mathbb{CP}^2); \mathbb{Z}/p) \) be a monomial. We write \( \alpha = u_1 \otimes \varphi(x) \), where \( u_1 \in H_1(\Omega S^2; \mathbb{Z}/p) \) is the generator. (Recall that \( H_\ast(\Omega S^2; \mathbb{Z}/p) \cong \mathbb{Z}/p \{u_1\} \).) Then

(i) When \( \mathbb{w}(\alpha) \leq k - 2 \), we have \( J(\alpha) = 0 \).

(ii) When \( \mathbb{w}(\alpha) = k - 1 \), we have \( J(\alpha) = 1 \otimes \psi(x) \). In particular, \( J(u_1^{k-1}) = 1 \otimes 1 \).

(iii) When \( \mathbb{w}(\alpha) = k \), we have \( J(\alpha) = e_1 \otimes \psi(x) \). In particular, \( J(u_1^k) = e_1 \otimes 1 \).

**Proof.** (i), (ii) and (iii) are modifications of Theorem 4.11, Lemmas 5.5 and 5.4 in [3], respectively, and are proved similarly. We check the degree shift. From the above (1), we have

\[ \deg J(\alpha) = \mathbb{w}(\alpha) + \deg x - k + 1. \]

Hence, if \( \mathbb{w}(\alpha) = k - 1 + \epsilon \) (\( \epsilon = 0, 1 \)), then \( \deg J(\alpha) = \deg x + \epsilon. \) This completes the proof of Lemma 2.5.

Now it is easy to prove Theorem A from Proposition 2.2 and Lemma 2.5.

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**References**


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