ON A CONFORMAL KILLING VECTOR FIELD IN
A COMPACT ALMOST KÄHLERIAN MANIFOLD

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Abstract. In this paper, we will prove that in a compact almost Kählerian manifold $M^n$, any conformal Killing vector field is Killing if $n \geq 4$.

1. Introduction

Let $M$ be an $n$-dimensional Riemannian manifold. We denote respectively by $g_{ij}$ and $\nabla_j$ the metric and the covariant derivative in terms of local coordinates $\{x^i\}$, where Latin indices run over the range $\{1, 2, \cdots, n\}$. A conformal Killing vector field $u^i$ in $M$ is given by

\[ \nabla_k u_j + \nabla_j u_k = 2\rho g_{kj}, \]

where $u_i = g_{ir}u^r$ and $\rho$ is a scalar function, called the associated scalar of $w'$. If $\rho$ vanishes identically, then the vector field is called Killing.

Also, a conformal Killing vector field is Killing in a compact Kählerian manifold [3]. In a Sasakian manifold, any conformal Killing vector field is uniquely decomposed into the summation of Killing and closed conformal Killing [2].

In [1], Y. Ogawa has studied differential operators in an almost Kählerian manifold. Using the operators of the almost Kählerian manifold, we prove the following theorem:

THEOREM. In a compact almost Kählerian manifold $M^n$, any conformal Killing vector field ($n \geq 4$) is Killing.

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2. Preliminaries

We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential \( p \)-form \( u \), the coefficients of its exterior differential \( du \) and the exterior codifferential \( \delta u \) are given by

\[
(du)_{i_1 \ldots i_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_{i_a} u_{i_1 \ldots \widehat{i_a} \ldots i_{p+1}} \quad \text{and} \quad (\delta u)_{i_2 \ldots i_p} = -\nabla^h u_{h i_2 \ldots i_p},
\]

where \( \nabla^h = g^{h j} \nabla_j \) and \( \widehat{i_a} \) means \( i_a \) to be deleted.

We consider an almost Hermitian manifold \( M^n (n = 2m) \) with positive definite metric \( g_{j i} \) and almost complex structure \( \phi_{j i} \). An almost Hermitian manifold is called almost Kählerian if the 2-form \( \phi_{j i} \) is closed. We want to recall some operators for differential forms in the almost Kählerian manifold. Denote by \( \mathcal{F}^p \) the set of all \( p \)-forms. The operators \( \Gamma, \gamma, C, c, \varpi, \) and \( \Phi \) are defined respectively by

\[
(\Gamma u)_{i_1 \ldots i_p} = \sum_{a=0}^{p} (-1)^a \phi_{i_a}^{r} \nabla_r u_{i_0 \ldots \widehat{i_a} \ldots i_p},
\]

\[
(\gamma u)_{i_1 \ldots i_p} = \sum_{a \neq b} (-1)^a \nabla_{i_a} \phi_{i_b}^{r} \cdot u_{i_0 \ldots \widehat{i_a} \ldots \widehat{i_b} \ldots i_p},
\]

\[
(Cu)_{i_2 \ldots i_p} = \phi^r s \nabla_r u_{s i_2 \ldots i_p},
\]

\[
(cu)_{i_2 \ldots i_p} = \sum_{a=2}^{p} \nabla^r \phi_{i_a}^{s} \cdot u_{r i_2 \ldots i_p},
\]

\[
(\varpi u)_{i_2 \ldots i_p} = \sum_{a=2}^{p} \phi_{i_a}^{r} \cdot u_{r i_2 \ldots i_p},
\]

\[
(\Phi u)_{i_1 \ldots i_p} = \sum_{a=1}^{p} \phi_{i_a}^{r} u_{i_1 \ldots r \ldots i_p}
\]

for any \( p \)-form \( u \), where we put \( \phi^{ji} = g^{j i} \phi_{ji} \). For any 0-form \( u_0 \) and 1-form \( u_1 \), we define \( \gamma u_0 = Cu_0 = cu_0 = \varpi u_0 = \Phi u_0 = 0 \) and \( cu_1 = \varpi u_1 = 0 \). In the almost Kählerian manifold, we know \( *\Gamma = -C, *\gamma = -c \) and \( *\Phi = (-1)^p \Phi \) for any \( p \)-form, where \( * \) means the dual mapping [1].

We denote by \( L \) (resp. \( \Lambda \)) the exterior (resp. interior) product with the associated 2-form \( \phi \), then the operators \( L : \mathcal{F}^p \to \mathcal{F}^{p+2} \) and \( \Lambda : \mathcal{F}^p \to \mathcal{F}^{p-2} \) are defined respectively by

\[
Lu_{i_1 \ldots i_p} = \sum_{a=1}^{p} \phi_{i_a}^{r} u_{r i_1 \ldots i_p},
\]

\[
\Lambda u_{i_1 \ldots i_p} = \sum_{a=1}^{p} \phi_{i_a}^{r} u_{i_1 \ldots \widehat{i_a} \ldots i_p}
\]

for any \( p \)-form \( u \), where \( \phi^{ji} = g^{j i} \phi_{ji} \).
\[ \mathcal{F}^p \rightarrow \mathcal{F}^{p-2} \] are written by \( Lu = \phi \wedge u \) and \( \Lambda u = (-1)^p \ast L \ast u \) for any \( p \)-form \( u \). \( \Lambda \) is trivial on 0 and 1-forms. These local expressions are defined by

\[
(Lu)_{kji_1\ldots i_p} = \phi_{kj} u_{i_1i_2\ldots i_p} - \sum_{a=1}^p \phi_{i_aj} u_{i_1\ldots i_{a-1}k\ldots i_p} - \sum_{b=1}^p \phi_{kib} u_{i_1\ldots j\ldots i_p} + \sum_{a<b} \phi_{i_ajb} u_{i_1\ldots k\ldots j\ldots i_p},
\]

\[
(\Lambda u)_{k_3\ldots i_p} = \frac{1}{2} \phi^{rs} u_{rsi_3\ldots i_p}.
\]

For the operators above, we find from [1]:

1. \( (d\Lambda - \Lambda d)u = -(C + c)u \),
2. \( (dL - Ld)u = 0 \),
3. \( (\Gamma u, v) = (u, Cv) \),
4. \( (\gamma u, v) = (u, cv) \),

where \( (\ , \ ) \) denotes the global inner product.

3. Proof of theorem

From (1.1), we find \( \delta u = -n \rho \). Operating \( \phi^k_h \) to (1.1), we obtain

\[
(\Gamma u)_{hj} - (d\Phi u)_{hj} + (\gamma u)_{hj} = 4(L\rho)_{hj}.
\]

Moreover in a compact almost Kählerian manifold, it follows from [1] that for a \( p \)-form \( u \) and a \((p+1)\)-form \( v \)

\[
(\Gamma u, v) = (u, C v), \quad (\gamma u, v) = (u, c v),
\]

where \( (\ , \ ) \) denotes the global inner product.
Substituting (3.2) into (3.1) and owing to (2.6) and (3.2), we get $n\Phi - 4\delta L u = 0$. Applying $\delta$ to this, we find $\delta d \Phi u = 0$, namely $d \Phi u = 0$ $(n \geq 4)$. From (3.1) we obtain $L \delta u = 0$, which means that $\delta u = 0$ $(n \geq 4)$, that is $\rho = 0$ $(n \geq 4)$. Consequently, we complete the proof of Theorem.

References